TEORIE DELLA GRAVITAZIONE

54 ORE 9 CREDITI
36 ORE 6 CREDITI

LUN	F1	14-16
MAR	V1	9-11
GIO	G1	14-16

- Geometria differenziale
- Relatività generale
- Gravità quantistica

www.df.unipi.it/~anselmi

- Appunti del corso di geometria differenziale, G.P. Pirola
  www-dimat.unipv.it/~pirola/corso-intero.pdf
- Differential forms, Weintraub
- Natural operations in differential geometry, Kolar, Michor, Slovak

R. Wald, General Relativity

S. Carroll, Lecture notes on general relativity,
arXiv:gr-qc/9712019

P. Menotti: Lectures on gravitation, arXiv:1703.05155 [gr-qc]

Spazio topologico $\{X, \mathcal{V}\}$

- insieme $X \neq \phi$
- una famiglia $\mathcal{V}$ di sottoinsiemi di X, detti aperti, tale che
  - $X \in \mathcal{V}$, $\phi \in \mathcal{V}$
  - $A \in \mathcal{V}$, $B \in \mathcal{V} \Rightarrow A \cup B \in \mathcal{V}$, $A \cap B \in \mathcal{V}$
  - anche l'unione di un numero infinito di $A_i \in \mathcal{V}$ appartiene a $\mathcal{V}$

Un insieme si dice chiuso se è il complementare di un aperto.

Uno spazio topologico si dice separato (o di Hausdorff) se

$\forall\, p, q \quad p \neq q \quad p \in X,\ q \in X$

$\exists$ aperti U_p e U_q della famiglia V

tali che $p \in U_p$, $q \in U_q$ e

$U_p \cap U_q$.

Un qualunque aperto che contiene p è

detto intorno di p.

Un ricoprimento di X è una famiglia $\mathcal{U}$ di aperti $\mathcal{U}_i \in \mathcal{V}$, tale che

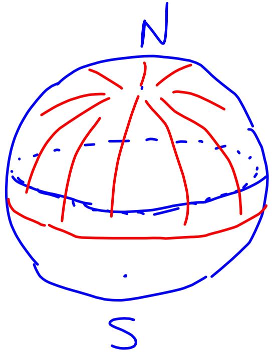
$$X = \bigcup_{i \in I} \mathcal{U}_i$$

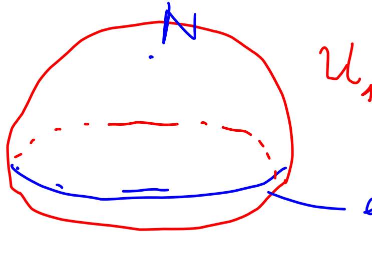
Una varietà topologica è uno spazio topologico X separato che ammette un ricoprimento $\mathcal{U}$ dove tutti gli $\mathcal{U}_i$ sono omeomorfi ad aperti di $\mathbb{R}^n$.

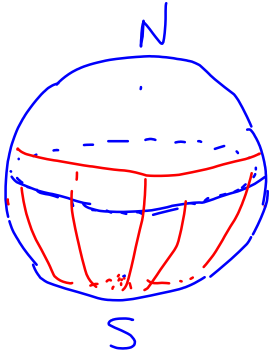
Un omeomorfismo è una funzione continua tra spazi topologici, biunivoca e con inversa continua.

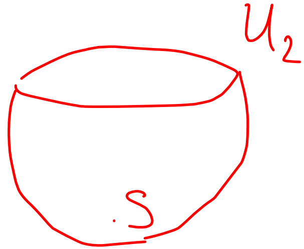
Una funzione $f: X \to Y$ tra due spazi topologici X e Y si dice continua se $f^{-1}(A)$ è un aperto di X ogni volta che A è un aperto di Y.

Sfera









$$\mathcal{U}_1 \cup \mathcal{U}_2 = \text{Sfera}$$

$$S^n = \{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \dots x_{n+1}^2 = 1 \}$$

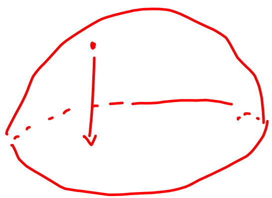
$$\mathcal{U}_i^{\pm} = \{ (x_1, \dots, x_{n+1}) \in S^n : x_i \gtrless 0 \}$$

$2 \times (n+1)$ semisfere

Gli $\mathcal{U}_i^{\pm}$ ricoprono S^n:

$$S^n = \mathcal{U}_1^+ \cup \mathcal{U}_1^- \cup \mathcal{U}_2^+ \cup \mathcal{U}_2^- \cdots \cup \mathcal{U}_{n+1}^+ \cup \mathcal{U}_{n+1}^-$$

Omeomorfismi $\varphi_i^{\pm} : \mathcal{U}_i^{\pm} \longrightarrow \mathbb{R}^n$



$$\varphi_i^{\pm}(x_1, \dots, x_{n+1}) = (x_1, \dots x_{i-1}, x_{i+1}, \dots x_{n+1})$$

proiezione sul piano dell'equatore

Ciascuna coppia $(\mathcal{U}_i^{\pm}, \varphi_i^{\pm})$ si dice carta

La famiglia delle carte si dice atlante.

Altri esempi

- $\mathbb{R}^n$ è una varietà topologica, così come ogni aperto di $\mathbb{R}^n$, così come S^n
- Se X e Y sono varietà topologiche, anche $X \times Y$ è una varietà topologica

Cambi di coordinate

$\mathcal{U}_i$ $\qquad \varphi_i : \mathcal{U}_i \longrightarrow B_i \subset \mathbb{R}^n \qquad B_i = \varphi(\mathcal{U}_i)$ aperto

$\mathcal{U}_j$ $\qquad \varphi_j : \mathcal{U}_j \longrightarrow C_j \subset \mathbb{R}^n$

Se $U_{ij} \equiv U_i \cap U_j \neq \emptyset$

allora in U_{ij} ha senso

$\varphi_i : U_{ij} \to B_{ij} \subset \mathbb{R}^n$

$\varphi_j : U_{ij} \to C_{ij} \subset \mathbb{R}^n$

Cambio di coordinate in U_{ij}

$\varphi_i^{-1} : B_{ij} \to U_{ij}$

$\varphi_j^{-1} : C_{ij} \to U_{ij}$

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \underbrace{C_{ij}}_{\text{aperto di } \mathbb{R}^n} \longrightarrow U_{ij} \longrightarrow \underbrace{B_{ij}}_{\text{aperto di } \mathbb{R}^n}$$

$$\varphi_{ij} : C_{ij} \longrightarrow B_{ij}$$

$$\varphi_{ij}(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$$

La varietà si dice differenziale di classe $\mathcal{C}^k$ se tutti i φ_{ij} sono differenziabili k volte

$$\exists = \varphi_{ij}^{-1} = \varphi_j \circ \varphi_i^{-1} : B_{ij} \longrightarrow C_{ij}$$

Una varietà topologica si dice di classe $\mathcal{C}^0$.

Se $k>0$ le mappe φ_{ij} si dicono diffeomorfismi.

Curve

Sia M = varietà differenziale di classe C^k,

$k \geq 1$

I = intervallo della retta reale

Una curva è una mappa $\gamma : I \longrightarrow M$ di classe C^k

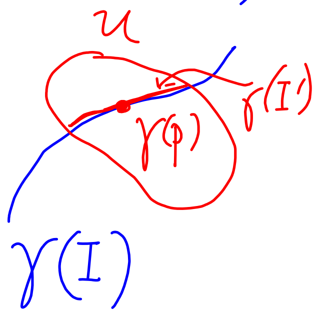
La sua tangente si definisce come segue

Sia $(\mathcal{U}, \varphi)$ una carta (o aperto coordinato)

e $p \in I$ tale che $\gamma(p) \in \mathcal{U}$

$\varphi : \mathcal{U} \longrightarrow \mathbb{R}^n$ $\qquad \gamma : I \longrightarrow M$

$\exists\, I'$ intorno di p tale che $\gamma : I' \to \mathcal{U}$



$$\gamma : I' \longrightarrow \mathcal{U} \qquad \varphi : \mathcal{U} \longrightarrow \mathbb{R}^n$$

$$\varphi \circ \gamma : I' \longrightarrow \mathbb{R}^n \qquad (x_1(t), x_2(t), \dots, x_n(t))$$

$$t \qquad (x_1, \dots, x_n)$$

γ di classe $\mathcal{C}^k$ vuol dire che tutte le $(x_1(t), \dots, x_n(t))$ sono di classe $\mathcal{C}^k$ (per ogni $\mathcal{U}$)

Il vettore tangente a γ in p è

$$(\dot{x}_1(p), \dot{x}_2(p), \dots, \dot{x}_n(p))$$

È un vettore di $\mathbb{R}^n$.

Due curve γ : $(x_1(t), \dots, x_n(t))$

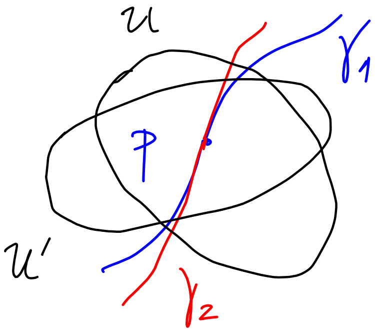
γ' : $(y_1(t), \dots, y_n(t))$

stesse coordinate ($\mathcal{U}$ è lo stesso) calcolate su curve diverse

Sono tangenti in p se

$$(\dot{x}_1(p), \dots \dot{x}_n(p)) = (\dot{y}_1(p), \dots, \dot{y}_n(p))$$

Questa proprietà non dipende dalle coordinate né dalla carta.



$\varphi \circ \gamma_1 : (x_1(t), \dots, x_n(t))$ nelle coordinate di $\mathcal{U}$

$\varphi \circ \gamma_2 : (y_1(t), \dots, y_n(t))$

Considero un'altra carta $\mathcal{U}', \varphi'$

nella quale le curve siano

$\gamma_1' \qquad \varphi' \circ \gamma_1 : \quad (x_1'(t), \dots, x_n'(t))$

$\gamma_2' \qquad \varphi' \circ \gamma_2 : \quad (y_1'(t), \dots, y_n'(t))$

Vogliamo far vedere che

$$\frac{dx_i}{dt}(p) = \frac{dy_i}{dt}(p) \implies \frac{dx_i'}{dt}(p) = \frac{dy_i'}{dt}(p)$$

In un intorno di p contenuto sia in $\mathcal{U}$ che in $\mathcal{U}'$

$$\varphi' \circ \gamma_1 = \underbrace{\varphi' \circ \varphi^{-1}}_{\text{cambio di coordinate}} \circ \underbrace{\varphi \circ \gamma_1}_{(x_1(t), \dots, x_n(t))}$$

$$\varphi' \circ \gamma_1 \longrightarrow (x_1'(t), \dots, x_n'(t))$$

$$\left(x_1'(x(t)), \dots, x_n'(x(t)) \right)$$

similmente

$$\frac{dx_i'}{dt} = \frac{dx^j}{dt} \frac{\partial x_i'}{\partial x_j} \qquad \frac{dy_i'}{dt} = \frac{dy_j}{dt} \frac{\partial y_i'}{\partial y_j} \qquad \frac{\partial \varphi'}{\partial \varphi}$$

$$\left.\frac{dx_i'}{dt}\right|_p - \left.\frac{dy_i'}{dt}\right|_p = \left.\frac{dx^j}{dt}\right|_p \left.\frac{\partial x_i'}{\partial x_j}\right|_p - \left.\frac{dy_j}{dt}\right|_p \left.\frac{\partial y_i'}{\partial y_j}\right|_p = 0$$

Ciò permette di definire una relazione di equivalenza $\gamma \sim \gamma'$ tra curve γ e γ' che passano per lo stesso punto p.

Lo spazio tangente alla varietà M in p, indicato con T_p o $T_{M,p}$, è il quoziente tra l'insieme $\Omega(p)$ di tutte le curve che passano per p e la relazione di equivalenza definita dalla tangenzialità.

Più semplicemente, se associamo a un $v \in T_p$ (cioè $v =$ classe di equivalenza di curve che passano per p) il vettore $(\dot{x}_1(p), \ldots, \dot{x}_n(p)) \in \mathbb{R}^n$ tangente in p (che non dipende dalla carta e dalle coordinate), allora otteniamo un isomorfismo tra T_p e $\mathbb{R}^n$, che dà a T_p una struttura di spazio vettoriale.

Sia M una varietà $\mathcal{C}^\infty$ e $p \in M$

Si dice funzione liscia in un intorno U di p la coppia (f, U) dove $f: U \longrightarrow \mathbb{R}$ è una funzione $\mathcal{C}^\infty(U)$

Due funzioni lisce f e g sono equivalenti $f \sim g$ se esiste un intorno W di p tale che $f|_W = g|_W$. Il quoziente $\mathcal{G}_p$ delle funzioni lisce rispetto a $\sim$, si chiama spazio dei germi delle funzioni lisce in p

Una derivazione in p è un'applicazione

$X : \mathcal{G}_p \longrightarrow \mathbb{R}$ tale che

$$X(f+g) = X(f) + X(g)$$

$$X(\lambda) = 0 \quad \text{se } \lambda = \text{costante (reale)}$$

$$X(fg) = f \cdot X(g) + X(f) \cdot g$$

in p

In particolare $X(\lambda f) = \cancel{X(\lambda) f} + \lambda X(f) =$
$= \lambda X(f)$

Lo spazio D_p delle derivazioni in p è uno spazio vettoriale

Nota. Sia U un aperto di $\mathbb{R}^n$, $p \in U$ e $f \in C^\infty(U)$. Allora esistono $g_i \in C^\infty(U)$ tali che $\forall x \in U$

$$f(x) - f(p) = \sum_i (x_i - x_i(p))\, g_i(x) \quad e \quad g_i(p) = \frac{\partial f}{\partial x^i}(p)$$

Supponiamo $x_i(p) = 0$

$$f(x) - f(0) = \int_0^1 dt \, \frac{df}{dt}(tx_1, \dots, tx_n) =$$

$$= \sum_i \int_0^1 dt \, \frac{\partial f}{\partial x_i}(tx)\, x_i = \sum_{i=1}^n x_i \, g_i(x)$$

$$g_i(x) = \int_0^1 dt \, \frac{\partial f}{\partial x_i}(tx) \qquad g_i(0) = \frac{\partial f}{\partial x^i}(0)$$

Se U è aperto di $\mathbb{R}^n$, $p \in U$,

$X_i = \left.\frac{\partial}{\partial x_i}\right|_p$ è una base canonica dello spazio D_p delle derivazioni in p

Se $f : U \to \mathbb{R}$ $\quad X_i(f) = \left.\frac{\partial f}{\partial x_i}\right|_p$

Se $f = x_j$ $\quad X_i(x_j) = \delta_{ij}$

Le X_i generano D_p

Sia $X \in D_p$, chiamiamo $X(x_i) = a_i$

Allora $X = \sum_{i=1}^{n} a_i X_i$

Chiamiamo $Y = X - \sum_{i=1}^{n} a_i X_i$

Vogliamo mostrare che $Y(f) = 0 \quad \forall f \in \mathcal{G}_p$

$$Y(x_i) = X(x_i) - \sum_{j=1}^{n} a_j X_j(x_i) = a_i - a_i = 0$$

$$f(x) = f(p) + \sum_{i=1}^{n} (x_i - x_i(p))\, g_i(x)$$

$$g_i(p) = \frac{\partial f}{\partial x_i}(p)$$

$$Y(f(x)) = \underbrace{Y(f(p))}_{=0} + \sum_{i=1}^{n} \underbrace{Y(x_i - x_i(p))}_{=0}\, g_i(x) +$$

$$+ \sum_{i=1}^{n} (x_i - x_i(p))\, Y(g_i(x))$$

$$Y(f(x)) = 0 \quad \text{in } p$$

Lo spazio tangente T_p in p è isomorfo allo spazio D_p delle derivazioni in p

$$\zeta : T_p \longrightarrow D_p$$

Sia $v = [\gamma(t)] \in T_p$ e $\gamma(0) = p$

Definiamo $\zeta(v)(f) = \dfrac{d}{dt} f(\gamma(t)) \Big|_{t=0}$

Consideriamo $v_i = [t e_i]$ $\quad e_i$ = base cartesiana

$$\jmath(v_i)(f) = \frac{df}{dt}(\underbrace{t e_i}_{})\Big|_{t=0} = \frac{\partial f}{\partial x_i}(0)$$

$$t e_i = (0, 0, \dots, \underset{i}{t}, 0, \dots)$$

$$\jmath(v_i) = \frac{\partial}{\partial x_i}\Big|_p$$

Questa relazione definisce l'isomorfismo $T_p \to D_p$

Campo vettoriale : funzione lineare

$V : \mathcal{C}^k(M) \longrightarrow \mathcal{C}^{k-1}(M)$ tale che

$$V(fg) = f\,V(g) + g\,V(f)$$

Localmente (cioè in un aperto coordinato)

$$X_i = \frac{\partial}{\partial x_i} \qquad X = a^i(x) \frac{\partial}{\partial x^i}$$

Nell'intersezione tra due aperti coordinati

abbiamo coordinate x e y $\quad y(x) \quad x(y)$

$$X = b^i(y) \frac{\partial}{\partial y^i} = b^i(y(x)) \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j} =$$

$$= a^j(x) \frac{\partial}{\partial x^j} \qquad a^i(x) = b^j(y(x)) \frac{\partial x^i}{\partial y^j}$$

$\mathfrak{X}^{\infty}(M)$ = spazio dei campi vettoriali C^{∞} di una varietà M anch'essa C^{∞}

Se X e Y sono campi XY non è campo, perché non soddisfa la regola di Leibniz:

$$X(Y(fg)) = X(Y(f)\cdot g + f\cdot Y(g)) =$$

$$= \underbrace{XY(f)\cdot g}_{} + \underbrace{Y(f)\cdot X(g) + X(f)\cdot Y(g)}_{\text{violano Leibniz}} + \underbrace{f\cdot XY(g)}_{}$$

$$[X,Y] = XY - YX \text{ è un campo}$$

$$[X,Y](fg) = [X,Y](f)\cdot g + f\,[X,Y](g)$$

Localmente $X = a_i(x)\dfrac{\partial}{\partial x^i}$ $\quad Y = b_i(x)\dfrac{\partial}{\partial x^i}$

$$Z = [X,Y] = c_i(x)\frac{\partial}{\partial x^i} \qquad c_i(x) = a_j\frac{\partial b_i}{\partial x^j} - b_j\frac{\partial a_i}{\partial x^j}$$

$$Z(f) = c_i(x)\frac{\partial f}{\partial x^i} = X(Y(f)) - Y(X(f)) =$$

$$= a_i\frac{\partial}{\partial x^i}\left(b_j\frac{\partial f}{\partial x_j}\right) - b_i\frac{\partial}{\partial x_i}\left(a_j\frac{\partial f}{\partial x^i}\right) =$$

$$= \left(a_i\frac{\partial b_j}{\partial x^i} - b_i\frac{\partial a_j}{\partial x_i}\right)\frac{\partial f}{\partial x^j}$$

Valgono le proprietà

- $[X, Y] = -[Y, X]$

- $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

identità di Jacobi

- $[fX, Y] = f[X, Y] - Y(f) \cdot X$

$$[fX, Y](g) = fXYg - Y(fXg) =$$

$$= f[X, Y](g) + \cancel{fY(X(g))} - Y(f)X(g) \cancel{- fY(X(g))}$$

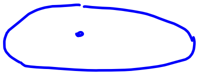
Se $\mathcal{A}$ è un'algebra, una derivazione D di $\mathcal{A}$ è un'applicazione lineare $D : \mathcal{A} \to \mathcal{A}$ tale che $D(ab) = a\,D(b) + b\,D(a) \quad \forall a, b \in \mathcal{A}$

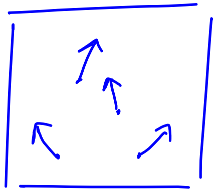
Se X è un campo di $\mathfrak{X}^\infty(M)$, allora $f \to Xf$, $f \in \mathcal{C}^\infty(M)$ $(\equiv \mathcal{A})$ è una derivazione dell'algebra $\mathcal{C}^\infty(M)$

Viceversa, ogni derivazione di $\mathcal{C}^\infty(M)$ è associata a uno e un solo campo di $\mathfrak{X}^\infty(M)$

Fibrato tangente $T_M = \bigcup_{p \in M} T_p$

Se U è un aperto di $\mathbb{R}^n$, abbiamo delle coordinate $(x_1, \dots, x_n)$ in U.

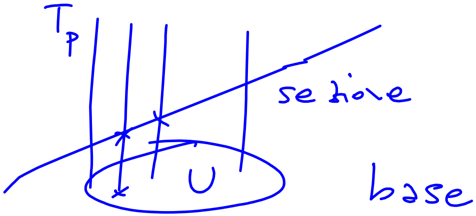




$X = a^i \dfrac{\partial}{\partial x_i}$ sono derivazioni

Le coordinate di T_U sono

$$(\underbrace{x_1, \dots, x_n}_{\text{viaggiamo su } U}, \underbrace{a_1, \dots, a_n}_{\text{viaggiano su } \mathbb{R}^n}) \qquad \underline{\underline{U \times \mathbb{R}^n}}$$



T_p : fibra

Sezione del fibrato tangente :

$$\left(x_1, \dots, x_n, a_1(x), \dots a_n(x) \right)$$

$$X = a_i(x) \frac{\partial}{\partial x^i}.$$

Un campo è una sezione del fibrato tangente

Differenziale di una funzione

$f: M \longrightarrow \mathbb{R} \qquad f \in \mathcal{C}^k(M) \quad k \geq 1$

Sia $p \in M$

$df(p): T_p \xrightarrow[\text{lineare}]{} \mathbb{R} \qquad df(p) \in T_p^*$

Sia $X \in T_p$. Interpretiamo X come derivazione

Allora df è il funzionale lineare definito

dalla relazione $\quad df(X) = X(f)$

OSS. $\quad df(gX) = gX(f) = g\, df(X) \qquad g \in \mathcal{C}^\infty(M)$

In coordinate locali $X = a^i(x) \dfrac{\partial}{\partial x^i}$

$$X(f) = a^i(x)\Big|_p \frac{\partial f}{\partial x^i}\Big|_p = df(X)$$

Prendiamo $f = x^i$

$$dx^i(X) = a^j \frac{\partial x^i}{\partial x^j} = a^i$$

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^{ij}$$

$$df(X) = \frac{\partial f}{\partial x^i} dx^i(X)$$

$$df = \frac{\partial f}{\partial x^i} dx^i$$

Questa relazione dimostra che i dx^i sono una base di T_p^*

Cambio di coordinate

$$x^i \longrightarrow y^i(x)$$

$$\delta^{ij} = dy^i\left(\frac{\partial}{\partial y^j}\right) = dx^i\left(\frac{\partial}{\partial x^j}\right) =$$

$$= dx^i\left(\underbrace{\frac{\partial y^k}{\partial x^j}}\,\frac{\partial}{\partial y^k}\right) = \frac{\partial y^k}{\partial x^j}\,\underset{\stackrel{=}{f}}{dx^i}\left(\frac{\partial}{\partial y^k}\right) =$$

$$X \qquad \underline{\underline{X(f)}}$$

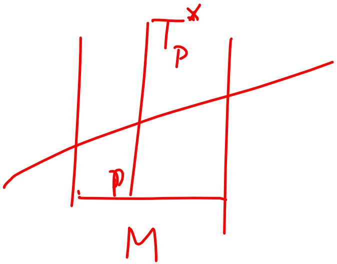
$$= \frac{\partial y^k}{\partial x^j}\,\frac{\partial x^i}{\partial y^k} = \frac{\partial x^i}{\partial y^k}\,dy^k\left(\frac{\partial}{\partial x^j}\right)$$

$$dx^i = \frac{\partial x^i}{\partial y^j}\,dy^j$$

$$dy^i = \frac{\partial y^i}{\partial x^j} dx^j$$

$T^*_M = \bigcup_{p \in M} T^*_p$ è il fibrato cotangente

Localmente $(x_1, \dots, x_n, \omega_1, \dots, \omega_n)$



Sezione $(x_1, \dots, x_n, \omega_1(x), \dots, \omega_n(x))$

Una forma differenziale ω (di grado 1) è una sezione del fibrato cotangente

Localmente $\omega = \omega_i(x)\, dx^i$

$$\omega(X) = \omega_i(x)\, dx^i\left(a^j(x)\frac{\partial}{\partial x^j}\right) = \omega_i(x)\, a^i(x)$$

$$X = a^i(x)\frac{\partial}{\partial x^i}$$

$df = \dfrac{\partial f}{\partial x^i}\, dx^i$ Particolare esempio di forma differenziale (esatta, vedi sotto)

Si definisce $\Lambda^s T^*_M$ come lo spazio delle forme antisimmetriche di grado s (Λ = wedge)

$s=2$ $\quad \omega_2 = \dfrac{\omega_{ij}(x)}{2}\, dx^i \wedge dx^j$

In ogni p:

$\omega_2 : T_p \times T_p \longrightarrow \mathbb{R}$

$\omega_{ij} = -\omega_{ji}$

$X = a^i \dfrac{\partial}{\partial x^i} \quad Y = b^i \dfrac{\partial}{\partial x^i}$

$$dx^i \wedge dx^j (X, Y) = \det \begin{pmatrix} dx^i(X) & dx^i(Y) \\ dx^j(X) & dx^j(Y) \end{pmatrix} =$$

$$= \det \begin{pmatrix} a^i & b^i \\ a^j & b^j \end{pmatrix} = a^i b^j - a^j b^i$$

$$\omega_2(X,Y) = \omega_{ij}\, a^i b^j$$

s qualunque $\qquad \omega_{i_1 \dots i_s}$ totalmente antisimmetriche

$$\omega_s = \frac{1}{s!}\,\omega_{i_1 \dots i_s}(x)\, dx^{i_1} \wedge \dots \wedge dx^{i_s}$$

$$dx^{i_1} \wedge \dots \wedge dx^{i_s}(X_1, \dots X_s) = \det \begin{pmatrix} dx^{i_1}(X_1) & \dots & dx^{i_1}(X_s) \\ \vdots & & \\ dx^{i_s}(X_1) & \dots & dx^{i_s}(X_s) \end{pmatrix}$$

$(T^*_M)^{\otimes s}$: spazio delle forme simmetriche di grado s

Localmente: $\omega_s = \frac{1}{s!}\,\omega_{i_1 \dots i_s}(x)\, dx^{i_1} \otimes \dots \otimes dx^{i_s}$

$\omega_{i_1 \dots i_s}$ totalmente simmetriche

$s=2 \qquad dx^i \otimes dx^j (X, Y) = a^i b^j + a^j b^i$

$g = g_{ij} \dfrac{dx^i \otimes dx^j}{2} \qquad g(X, Y) = g_{ij} a^i b^j$

Derivata esterna : $d : \Lambda^m T_M^* \longrightarrow \Lambda^{m+1} T_M^*$

Se $\omega = \omega_{i_1 \dots i_m} dx^{i_1} \wedge \dots \wedge dx^{i_m}$

$d\omega = \partial_j \omega_{i_1 \dots i_m} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_m}$

$m = 0 \qquad \omega = f \qquad d\omega = \partial_j f \, dx^j = df$

$$d^2 = 0$$

$$d(d\omega) = \partial_k \partial_j \omega_{i_1 \dots i_m} dx^k \wedge dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_m} = 0$$

Una forma differenziale ω si dice <u>chiusa</u> se $d\omega = 0$ cioè localmente

$$\partial_{[j} \omega_{i_1 \dots i_m]} = 0$$

Esempio: $\omega = \omega_i dx^i$

$$d\omega = \partial_j \omega_i \, dx^j \wedge dx^i$$

$$d\omega = 0 \iff \frac{\partial \omega^i}{\partial x^j} = \frac{\partial \omega^j}{\partial x^i}$$

Se $\omega_i = E_i$ campo elettrico

$d\omega = 0$ vuol dire $\operatorname{rot} \vec{E} = 0$

che $\Rightarrow$ $\exists V$ $\vec{E} = -\vec{\nabla} V$ $E_i = -\partial_i V$

$$\omega = \omega_i dx^i = -\partial_i V dx^i = -dV$$

In questo caso $d\omega = 0 \Rightarrow \exists \alpha$ $(\alpha = -V)$

tale che $\omega = d\alpha$

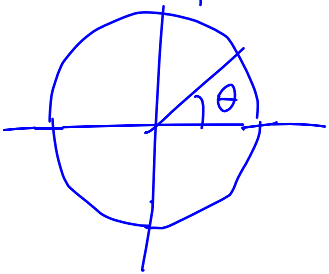
Una forma ω di grado $k \geq 1$ si dice esatta se $\exists\ \alpha$ forma di grado $k-1$ tale che

$$\omega = d\alpha$$

Una forma esatta è sempre chiusa: infatti se $\omega = d\alpha$ $\quad d^2 = 0 \Rightarrow d\omega = dd\alpha = 0$

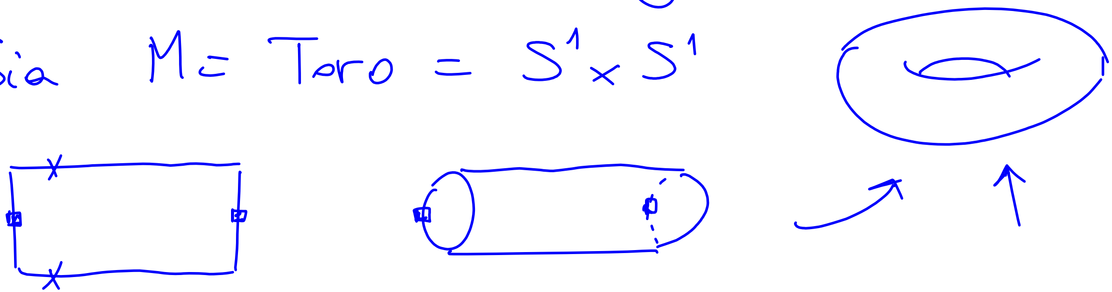
Invece, non tutte le forme chiuse sono esatte. Tuttavia, lo sono su $\mathbb{R}^n$ o aperti di $\mathbb{R}^n$

Esempio. Sia $M = S^1$ (circonferenza)



La forma $\omega = d\theta$ è chiusa ($d\omega = 0$), ma <u>non</u> è esatta globalmente, perché θ non è una funzione definita globalmente su S^1

Sia $M =$ Toro $= S^1 \times S^1$



$dx, dy, dx \wedge dy$ sono chiuse ma non esatte

$M = S^2 \qquad \omega_2 = \operatorname{sen}\theta \, d\theta \wedge d\varphi = d\Omega$

Teorema di Stokes

Sia V un aperto, ∂V il bordo di V

Sia ω una forma differenziale. Allora

$$\int_V d\omega = \int_{\partial V} \omega$$

Se $\omega = \operatorname{sen}\theta \, d\theta \wedge d\varphi$ fosse esatta, $\omega = d\sigma$,

$$4\pi = \int_{S^2} \omega = \int_{S^2} d\sigma \underset{\text{Stokes}}{=} \int_{\partial S^2 = 0} \sigma = 0$$

Elettromagnetismo

$$A = A_\mu \, dx^\mu$$

$$F = dA = \partial_\nu A_\mu \, dx^\nu \wedge dx^\mu =$$

$$= \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\int_{S^2} dA = \int_{S^2} F$$

$$dx \wedge dy \rightarrow d^2x$$

$$\int f(x,y) \frac{dx \wedge dy}{2} = \frac{1}{2} \int f(x,y)\, dx\,dy$$

$$dx^\mu \wedge dx^\nu = \varepsilon^{\mu\nu} d^2x \qquad dx^\mu dx^\nu dx^\rho dx^\sigma = \varepsilon^{\mu\nu\rho\sigma} d^4x$$

L'operatore di bordo ∂ è pure nilpotente

$\partial^2 = 0$

Due forme chiuse ω_1 e ω_2 si dicono equivalenti

se $\omega_1 - \omega_2$ è esatta

Una forma esatta $\omega = d\alpha$ è equivalente

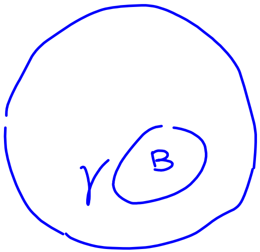
a zero

Le classi di equivalenza delle forme

differenziali rispetto a questa relazione di equivalenza danno la coomologia delle forme differenziali.

Si può definire un' "omologia" basata sulla nilpotenza di ∂.

S^2 :



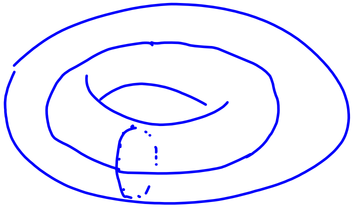
$\omega_0 = 1 \qquad \omega_2 = \operatorname{sen}\theta \, d\theta \wedge d\varphi$

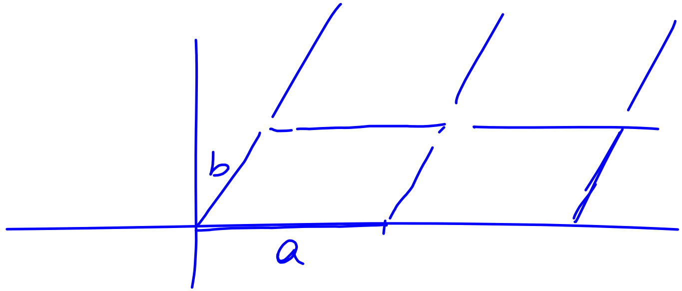
$\exists \, \omega_1$ non banale? NO

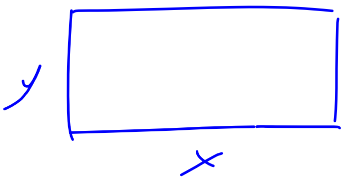
Omologia: $\gamma \quad \partial\gamma = 0 \quad \forall$ tale $\gamma \ \exists \, B \,/\, \gamma = \partial B$

ω_2 è duale S_2, ω_0 è duale del punto

T_2



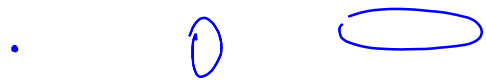




coomologia

$1, \quad dx, \quad dy, \quad dx \wedge dy$

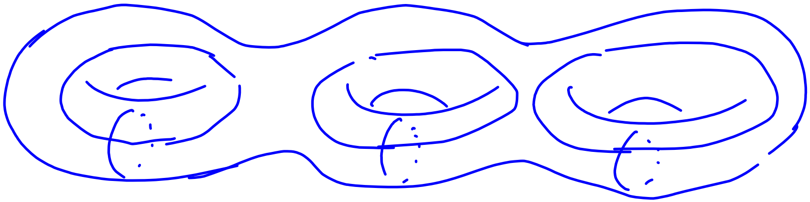
omologia



T_2

toro

Superfici di Riemann

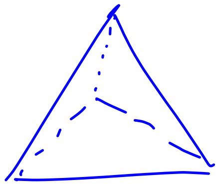


genere g

$g = \#$ di "manici"

punto	$2g$ curve	superficie	
$\downarrow$	$\downarrow$	$\downarrow$	
b^0	b^1	b^2	numeri di Betti

$$b^0 - b^1 + b^2 = 2 - 2g = \text{caratterisica di Eulero}$$



$b^0 = \#$ vertici $= 4$

$b^1 = \#$ spigoli $= 6$

$b^2 = \#$ facce $= 4$

$4 - 6 + 4 = 2$

Prodotto esterno di forme differenziali

$\omega \wedge \Omega$

$$\omega = \omega_{i_1 \dots i_m} dx^{i_1} \wedge \dots \wedge dx^{i_m}$$

$$\Omega = \Omega_{i_1 \dots i_n} dx^{i_1} \wedge \dots \wedge dx^{i_n}$$

$$\omega \wedge \Omega = \omega_{i_1 \dots i_m} \Omega_{i_{m+1} \dots i_{m+n}} dx^{i_1} \wedge \dots \wedge dx^{i_n}$$

$$\omega \wedge \Omega = (-1)^{\text{grado }\omega \cdot \text{grado }\Omega}\ \Omega \wedge \omega$$

$$dx \wedge dy = -dy \wedge dx$$

Derivata esterna del prodotto esterno di due forme

$$d(\omega \wedge \Omega) = d\omega \wedge \Omega + (-1)^{\text{grado }\omega}\ \omega \wedge d\Omega$$

Esercizio

Derivazioni di campi vettoriali

M varietà liscia, $\mathcal{X}(M)$ spazio dei campi lisci

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$

$$\nabla_X Y = Z$$

∇ si dice anche connessione (lineare)

∇ deve soddisfare le seguenti proprietà

a) $\nabla_{fX+gY} Z = f \nabla_X Z + g \nabla_Y Z$

$\forall f, g \in \mathcal{C}^\infty(M)$

b) $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$

c) $\nabla_X (fY) = f\nabla_X Y + X(f)\, Y \qquad \forall f \in C^\infty(M)$

Consideriamo le derivazioni $\dfrac{\partial}{\partial x_i}$ della base canonica in coordinate locali

$$\nabla_{\frac{\partial}{\partial x^i}} \left(\frac{\partial}{\partial x^j} \right) = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k}$$

Γ_{ij}^k sono detti simboli di Christoffel

Per una base generica X_i $\quad \nabla_{X_i} X_j = \sum_k b_{ij}^k X_k$

Sia $\gamma(t)$ una curva $\gamma(t) = (a_1(t), \dots a_n(t))$

e considero $\displaystyle \frac{d}{dt} = \dot{a}_i(t) \frac{\partial}{\partial x_i}$ lo immaginiamo come una derivazione lungo γ

Sia $X = u^i \dfrac{\partial}{\partial x^i}$ un campo

$$\nabla_{\frac{d}{dt}} X = \nabla_{d/dt}\left(u^i \frac{\partial}{\partial x^i}\right) = u^i \nabla_{\frac{d}{dt}}\left(\frac{\partial}{\partial x^i}\right) +$$

$$+ \frac{du^i}{dt} \frac{\partial}{\partial x^i} = u^i \nabla_{\dot{a}_j \frac{\partial}{\partial x^j}}\left(\frac{\partial}{\partial x^i}\right) + \frac{du^i}{dt}\frac{\partial}{\partial x^i} =$$

$$= \frac{du^i}{dt}\frac{\partial}{\partial x^i} + u^i \dot{a}_j \nabla_{\frac{\partial}{\partial x_j}}\left(\frac{\partial}{\partial x^i}\right) =$$

$$= \frac{du^i}{dt}\frac{\partial}{\partial x^i} + u^i \dot{a}_j \Gamma^k_{ji} \frac{\partial}{\partial x^k} =$$

$$= \left[\frac{du^i}{dt} + \Gamma^i_{jk} \dot{a}^j u^k\right] \frac{\partial}{\partial x^i}$$

X si dice parallelo a γ se $\nabla_{\frac{d}{dt}} X = 0$

lungo γ

Dato un punto $p \in M$ e un vettore $V \in T_p$

e una curva γ che passa per p, $\gamma(0) = p$, allora $\exists$ campo X definito lungo γ che soddisfa

$$\nabla_{\frac{d}{dt}} X = 0 \text{ lungo } \gamma \quad \text{e} \quad X(0) = V$$

(Problema di Cauchy) La soluzione definisce il trasporto parallelo del vettore V lungo la curva γ

Curvatura

Sia M una varietà liscia, $\mathcal{X}(M)$ lo spazio dei campi lisci su M, ∇ una connessione lineare.

La curvatura $R_\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ della connessione ∇

è l'applicazione che a tre campi X, Y, Z associa il campo

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

[Si può anche scrivere per brevità

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$]

$R(X,Y)Z$ è lineare in X, Y, Z

Se $f, g, h \in \mathcal{C}^\infty(M)$, allora

$$R(fX, gY)(hZ) = fgh\, R(X,Y)Z$$

Esercizio

$R(X,Y)$ è antisimmetrica in X e Y

Esempio $\quad X = \dfrac{\partial}{\partial x^i} \qquad Y = \dfrac{\partial}{\partial x^j} \qquad Z = \dfrac{\partial}{\partial x^k}$

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = \nabla_{\partial_i} \underbrace{\nabla_{\partial_j} \partial_k} - \nabla_{\partial_j} \nabla_{\partial_i} \partial_k \, +$$

$$- \cancel{\nabla_{[\partial_i, \partial_j]}} \partial_k = \nabla_{\partial_i}\left(\Gamma^l_{jk} \partial_l\right) - \nabla_{\partial_j}\left(\Gamma^l_{ik} \partial_l\right) =$$

$$= \Gamma^l_{jk} \nabla_{\partial_i}(\partial_l) + \partial_i \Gamma^l_{jk} \partial_l - \Gamma^l_{ik} \nabla_{\partial_j}(\partial_l) \, +$$

$$- \partial_j \Gamma^l_{ik} \partial_l = \Gamma^l_{jk} \Gamma^m_{il} \partial_m + \partial_i \Gamma^l_{jk} \partial_l \, +$$

$$- \Gamma^l_{ik} \Gamma^m_{jl} \partial_m - \partial_j \Gamma^l_{ik} \partial_l =$$

$$= \left(\partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{jk} \Gamma^l_{im} - \Gamma^m_{ik} \Gamma^l_{jm}\right) \partial_l$$

$$= R_{ij}{}^{\ell}{}_{k}\, \partial_\ell$$

Torsione

$$T : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

$$T(fX, gY) = \nabla_{fX}(gY) - \nabla_{gY}(fX) +$$

$$- [fX, gY] = f\left(g \nabla_X Y + X(g) Y\right) +$$

$$- g\left(f \nabla_Y X + Y(f) X\right) - fXgY + gYfX =$$

$$= fg\left(\nabla_X Y - \nabla_Y X - [X,Y]\right)$$

$$+ \cancel{f X(g)} Y - g Y(f) X - \cancel{f X(g)} Y +$$

$$+ g Y(f) X = fg\, T(X,Y)$$

$$T(\partial_i, \partial_j) = \nabla_{\partial_i} \partial_j - \nabla_{\partial_j} \partial_i - \cancel{[\partial_i, \partial_j]} =$$

$$= \left(\Gamma_{ij}^k - \Gamma_{ji}^k\right) \partial_k$$

La torsione è zero $\Longleftrightarrow$ i simboli di Christoffel Γ_{ij}^k nella base canonica sono simmetrici

Varietà Riemanniana

(M, g) $\quad M =$ varietà

$g =$ metrica $=$ sezione di $(T^*_M)^{\otimes 2}$ simmetrica e definita positiva

Base $dx^i \otimes dx^j$

In coordinate locali $g = g_{ij}\, dx^i dx^j$ dove

g_{ij} deve essere simmetrica e definita positiva
(proprietà indipendenti dalle coordinate usate)

$g(V, W) = g_{ij} V^i W^j$ $\qquad dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j$

V, W campi vettoriali, $V = V^i \frac{\partial}{\partial x^i}$, $W = W^j \frac{\partial}{\partial x^j}$.

Possiamo diagonalizzare la metrica, anzi ortonormalizzarla

Siano E^i campi vettoriali tali che

$$g(E^i, E^j) = \delta^{ij}$$

$$X = a^i E^i \qquad X \in \mathcal{X}(M)$$

$$g(X, E^i) = g(a^j E^j, E^i) =$$

$$= a^j \, g(E^j, E^i) = a^i$$

$$X = g(X, E^i) E^i$$

Sia ∇ una connessione

Definisco la funzione

$$\nabla_g : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{C}^\infty(M)$$

$$\nabla_g(X,Y,Z) = X\left(g(Y,Z)\right) - g\left(\nabla_X Y, Z\right) + $$
$$- g\left(Y, \nabla_X Z\right)$$

$$\nabla_g(fX, hY, kZ) = fhk\,\nabla_g(X,Y,Z)$$

$f, h, k \in \mathcal{C}^\infty(M)$ Esercizio

La connessione ∇ si dice compatibile colla metrica g se $\nabla g \equiv 0$

In una varietà Riemanniana esiste una ed una sola connessione ∇ compatibile con una metruca data g e priva di torsione

Dobbiamo risolvere $\nabla g = 0$ $\quad T = 0$

Lavoriamo nella base $\frac{\partial}{\partial x^i}$ $\quad dx^j(\partial_i) = \delta^j_i$

Sappiamo che $T = 0$ vuol dire $\Gamma_{ij}^{k} = \Gamma_{ji}^{k}$

$$\nabla_g(\partial_i, \partial_j, \partial_k) = \partial_i \big(g(\partial_j, \partial_k) \big) - g\left(\nabla_{\partial_i} \partial_j, \partial_k \right) +$$

$$- g\left(\partial_j, \nabla_{\partial_i} \partial_k \right) =$$

$$g(\partial_i, \partial_j) = g_{mn}\, dx^m\, dx^n \, (\partial_i, \partial_j) =$$

$$= g_{mn}\, dx^m(\partial_i)\, dx^n(\partial_j) = g_{ij}$$

$$\nabla_g(\partial_i, \partial_j, \partial_k) = \partial_i g_{jk} - g\left(\Gamma_{ij}^m \partial_m, \partial_k \right) +$$

$$- g\left(\partial_j, \Gamma_{ik}^m \partial_m \right) = \partial_i g_{jk} - \Gamma_{ij}^m g_{mk} - \Gamma_{ik}^m g_{mj} =$$

$$= 0$$

$$\partial_i g_{jk} - \Gamma_{ij}^{\ \ m} g_{mk} - \Gamma_{ik}^{\ \ m} g_{mj} = 0 \qquad (1)$$

$$(i \leftrightarrow j) \qquad\qquad +$$

$$\partial_j g_{ik} - \Gamma_{ij}^{\ \ m} g_{mk} - \Gamma_{jk}^{\ \ m} g_{mi} = 0 \qquad (2)$$

$$(k \leftrightarrow j) \qquad\qquad -$$

$$\partial_k g_{ij} - \Gamma_{ik}^{\ \ m} g_{mj} - \Gamma_{jk}^{\ \ m} g_{mi} = 0 \qquad (3)$$

$$(1) + (2) - (3) \Rightarrow 0 = \partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} +$$

$$- 2\, \Gamma_{ij}^{\ \ m} g_{mk} \qquad \Rightarrow \qquad [\, g^{km} g_{mn} = \delta^k_n \,]$$

$$\Gamma_{ij}^{\ \ k} = \frac{1}{2} g^{km} \left(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij} \right)$$

Se invece lavoriamo nella base E^i

$$g(E^i, E^j) = \delta^{ij} \qquad \nabla_{E_i} E_j = b_{ij}^{\ \ k} E_k$$

$$[E_i, E_j] = a_{ij}^{\ \ k} E_k$$

$$T = 0 \qquad \nabla g = 0$$

$$T(E_i, E_j) = \nabla_{E_i} E_j - \nabla_{E_j} E_i - [E_i, E_j] =$$

$$= \left(b_{ij}^{\ \ k} - b_{ji}^{\ \ k} - a_{ij}^{\ \ k} \right) E_k$$

$$\boxed{a_{ij}^{\ \ k} = b_{ij}^{\ \ k} - b_{ji}^{\ \ k}}$$

$$\nabla g(E_i, E_j, E_k) = E_i(\cancel{g(E_j, E_k)}) - g(\nabla_{E_i} E_j, E_k) +$$

δ_{jk}

$$- g(E_j, \nabla_{E_i} E_k) =$$

$$= - g(b_{ij}^{m} E_m, E_k) - g(E_j, b_{ik}^{m} E_m) =$$

$$= - b_{ij}^{m} \delta_{mk} - b_{ik}^{m} \delta_{jm}$$

Tensore di Riemann

$$R : \mathcal{X}(M)^4 \longrightarrow \mathcal{C}^\infty(M)$$

$$R(X, Y, Z, W) = g(R(X,Y)Z, W)$$

$$\{ R(X,Y) = \nabla_X Y - \nabla_Y X - \nabla_{[X,Y]} \}$$

Proprietà

$$R(X,Y,Z,W) = -R(Y,X,Z,W) =$$

$$= -R(X,Y,W,Z) =$$

$$= R(Z,W,X,Y)$$

Esercizio

Identità di Bianchi: $\nabla R = 0$ dove

$$\nabla R(X,Y,Z,T,W) = X(R(Y,Z,T,W)) +$$

$$- R(\nabla_X Y, Z, T, W) - R(Y, \nabla_X Z, T, W) +$$

$$- R(Y,Z,\nabla_X T, W) - R(Y,Z,T,\nabla_X W)$$

Esercizio

Identità algebrica di Bianchi :

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

Esercizio

(segue dal fatto che ∇ ha torsione nulla ?)

Trovare quali ipotesi ($T = 0$? , $\nabla g = 0$?)

sono necessarie alla validità di ciascuna

Tensore di Ricci

$$Ric(X,Y) = \sum_i R(X, E_i, Y, E_i)$$

Curvatura scalare

$$R = \sum_j Ric(E_j, E_j)$$

Prodotto interno o contrazione

V = vettore ω = k-forma

$i_V \omega$ è una $k-1$ forma.

$$(i_V \omega)(v_1, \dots, v_{k-1}) = \omega(V, v_1, \dots, v_{k-1})$$

In coordinate locali

$$\omega = \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$$i_V \omega = k\, V^i \omega_{i i_1 \dots i_{k-1}} dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}}$$

Derivata di Lie, V campo vettoriale, ω forma

$$\mathcal{L}_V \omega = d i_V \omega + i_V d\omega$$

$$\mathcal{L}_V = d i_V + i_V d$$

ω = 0 forma (funzione)

$$\mathcal{L}_V f = i_V df = V^i \frac{\partial f}{\partial x^i}$$

$\omega = 1$ forma $\qquad \omega = \omega_i dx^i$

$i_v \omega = v^i \omega_i \qquad d\omega = \partial_i \omega_j \, dx^i \wedge dx^j$

$$d i_v \omega = d(v^i \omega_i) = \partial_j v^i dx^j \omega_i + v^i \partial_j \omega_i dx^j$$

$$i_v d\omega = v^i \partial_i \omega_j \, dx^j - v^j \partial_i \omega_j \, dx^i$$

$$\mathcal{L}_v \omega = dx^j \left[\partial_j v^i \omega_i + \cancel{v^i \partial_j \omega_i} + v^i \partial_i \omega_j + \right.$$

$$\left. - \cancel{v^i \partial_j \omega_i} \right] = \left(v^i \partial_i \omega_j + \partial_j v^i \omega_i \right) dx^j$$

Proprietà $\quad \mathcal{L}_v (\omega_1 \wedge \omega_2) = \mathcal{L}_v \omega_1 \wedge \omega_2 +$

$+ \omega_1 \wedge \mathcal{L}_v \omega_2$ $\quad$ Esercizio

$$\omega = \omega_{i_1 \dots i_n} \, dx^{i_1} \wedge \dots \wedge dx^{i_n}$$

$$\mathcal{L}_V \omega = \left(n \, \partial_{i_1} V^i \, \omega_{i i_2 \dots i_n} + V^i \partial_i \omega_{i_1 \dots i_n} \right) dx^{i_1} \dots dx^{i_n}$$

Derivata di Lie di un campo vettoriale

Se X e Y sono campi $\quad \mathcal{L}_X Y = [X, Y] = -\mathcal{L}_Y X$

Flusso di un campo vettoriale

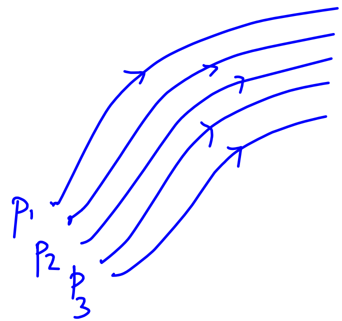
Sia $\quad X = a^i(x) \dfrac{\partial}{\partial x^i} \quad$ un campo vettoriale

A partire da un qualunque punto $(x^1, \dots, x^n) \in M$

costruiamo la curva $\gamma : [0,1] \longrightarrow M$

che soddisfa $\qquad (\phi^1(t), \dots \phi^n(t))$

$$\begin{cases} \dfrac{d\phi^i(t)}{dt} = a^i(\phi(t)) \\ \phi^i(0) = x^i \end{cases}$$



Possiamo scrivere le soluzioni come

$$\phi^i(x,t) \qquad \begin{cases} \dfrac{\partial \phi^i(x,t)}{\partial t} = a^i(\phi(x,t)) \\ \phi^i(x,0) = x^i \end{cases}$$

$$\left. \frac{\partial \phi^i}{\partial x^j} \right|_{t=0} = \delta^i_j$$

Sia $f : M \to \mathbb{R}$

$$\mathcal{L}_X f = \lim_{t \to 0} \frac{1}{t}\left[f(\phi(x,t)) - f(x) \right] = \qquad \underset{\shortparallel}{f(\phi(x,0))}$$

$$= \frac{\partial f}{\partial x^i}(\phi(x,t)) \frac{\partial \phi^i}{\partial t}\Bigg|_{t=0} = a^i(x) \frac{\partial f}{\partial x^i}$$

Sia ω una 1 forma $\omega = \omega_i(x)\, dx^i$

$$x \longrightarrow x' = \phi(x,t)$$

$$\omega_i(x')\, dx'^i = \omega_i(\phi(x,t)) \frac{\partial \phi^i}{\partial x^j}(\phi(x,t))\, dx^j$$

$$\mathcal{L}_X \omega = \frac{\partial}{\partial t}\left[\omega_i(\phi(x,t)) \frac{\partial \phi^i}{\partial x^j}(\phi(x,t))\, dx^j \right]\Bigg|_{t=0} =$$

$$= a^k \partial_k \omega^i \, dx^i + \omega^i \partial_j a^i \, dx^j = \mathcal{L}_X \omega$$

ok

$$\frac{\partial}{\partial t} \frac{\partial \phi^j}{\partial x^j} \Bigg|_{t=0} = \frac{\partial}{\partial x^j} \frac{\partial \phi^i}{\partial t} \Bigg|_{t=0} =$$

$$= \frac{\partial a^i(x)}{\partial x^j}$$

Per un vettore $\quad Y = b^i(x) \dfrac{\partial}{\partial x^i} \qquad x' = \phi(x,t)$

$$b^i(\phi(x,t)) \frac{\partial}{\partial \phi^i} = b^i(\phi(x,t)) \frac{\partial x^j}{\partial \phi^i}(\phi(x,t)) \frac{\partial}{\partial x^j}$$

$$\mathcal{L}_X Y = \frac{d}{dt} \left(b^i(\phi(x,t)) \frac{\partial x^j}{\partial \phi^i}(\phi(x,t)) \frac{\partial}{\partial x^j} \right) \Bigg|_{t=0} =$$

$$= \frac{\partial \phi^k}{\partial t} \frac{\partial b^i}{\partial \phi^k}(\phi(x,t)) \frac{\partial x^j}{\partial \phi^i}(\phi(x,t)) \frac{\partial}{\partial x_j}\bigg|_{t=0} +$$

$$+ b^i(\phi(x,t)) \frac{\partial}{\partial t}\left(\frac{\partial x^j}{\partial \phi^i}\right)(\phi(x,t)) \frac{\partial}{\partial x_j}\bigg|_{t=0} =$$

$$= a^k(x)\,\partial_k b^i \frac{\partial}{\partial x^i} - b^i(x)\,\partial_i a^j \frac{\partial}{\partial x^j} = [X, Y]$$

$$M^i{}_j \equiv \frac{\partial \phi^i}{\partial x^j} \qquad M\Big|_{t=0} = \delta^i_j \qquad \dot{M}^i{}_j = \frac{\partial \phi^i}{\partial x^j \partial t}$$

$$\dot{M}^{-1} = - M^{-1}\dot{M} M^{-1} \qquad \dot{M}^{-1}\Big|_{t=0} = -\dot{M}\Big|_{t=0}$$

$$\dot{M}^i{}_j\Big|_{t=0} = \partial_j a^i \qquad X = a^i \partial_i \quad Y = b^j \partial_j$$

Derivata di Lie della metrica

$$g_{ij}\, dx^i dx^j \longrightarrow \frac{d}{dt} g_{ij}(\phi(x,t)) \frac{\partial \phi^i}{\partial x^k} \frac{\partial \phi^j}{\partial x^l} dx^k dx^l \Big|_{t=0} =$$

$$= a^m \partial_m g_{ij}\, dx^i dx^j + g_{ij}\, \partial_m a^i\, dx^m dx^j +$$

$$+ g_{ij}\, dx^i\, \partial_l a^j\, dx^l =$$

$$= \left[a^m \partial_m g_{ij} + \partial_i a^m g_{mj} + \partial_j a^m g_{mi} \right] dx^i dx^j$$

$$\equiv \mathcal{L}_X g \qquad \mathcal{L}_X g_{ij} = a^m \partial_m g_{ij} + \partial_i a^m g_{mj} + \partial_j a^m g_{mi}$$

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x) \qquad g'_{\mu\nu}(x') = g_{\rho\sigma}(x)\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}$$

$$= g'_{\mu\nu}(x+\xi) \simeq$$

$$\simeq g'_{\mu\nu}(x) + \xi^{\rho}\partial_{\rho} g_{\mu\nu}$$

(al 1° ordine in ξ)

$$\frac{\partial x'^{\mu}}{\partial x^{\rho}} = \delta^{\mu}_{\rho} + \partial_{\rho}\xi^{\mu} \qquad \frac{\partial x^{\rho}}{\partial x'^{\nu}} \simeq \delta^{\rho}_{\nu} - \partial_{\nu}\xi^{\rho}$$

$$g'_{\mu\nu}(x) = -\xi^{\rho}\partial_{\rho} g_{\mu\nu} + g_{\rho\sigma}\left(\delta^{\rho}_{\mu} - \partial_{\mu}\xi^{\rho}\right)\left(\delta^{\sigma}_{\nu} - \partial_{\nu}\xi^{\sigma}\right) =$$

$$= -\xi^{\rho}\partial_{\rho} g_{\mu\nu} - \partial_{\mu}\xi^{\rho} g_{\rho\nu} - \partial_{\nu}\xi_{\rho}\, g^{\rho}_{\mu} + g_{\mu\nu}$$

$$g'_{\mu\nu}(x) - g_{\mu\nu}(x) = -\xi^{\rho}\partial_{\rho} g_{\mu\nu} - \partial_{\mu}\xi^{\rho} g_{\rho\nu} - \partial_{\nu}\xi_{\rho}\, g^{\rho}_{\mu}$$

Scalare : $\varphi'(x') = \varphi(x)$

$$\parallel$$

$$\varphi'(x+\xi) = \varphi'(x) + \xi^\rho \partial_\rho \varphi + \dots$$

$$\varphi'(x) - \varphi(x) \equiv \delta\varphi(x) = \xi^\rho \partial_\rho \varphi$$

Varietà Lorentziana

Si assume l'esistenza di una metrica invertibile e con segnatura $(1, -1, -1, -1)$

$$x^\mu \qquad \frac{\partial}{\partial x^\mu} = \partial_\mu \qquad dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu_\nu$$

$$\nabla_{\partial_\mu}(\partial_\nu) = \Gamma_{\mu\nu}^{\ \ \rho}\, \partial_\rho$$

$$g(E_i, E_j) = \delta_{ij} \quad \longrightarrow \quad g(e_a, e_b) = \eta_{ab} =$$

$$= \mathrm{diag}(1, -1, -1, -1)$$

$$e_a = e_a^{\ \mu} \, \partial_\mu \qquad \nabla_{e_a}(e_b) = \gamma_{ab}^{\ \ c} \, e_c$$

$a, b, c, \dots$ indici piatti

$\mu, \nu, \dots$ indici di spaziotempo

Introduciamo delle forme

$$dx^\mu \left(\frac{\partial}{\partial x^\nu} \right) = \delta^\mu_\nu \qquad e^a \equiv H^a_\mu \, dx^\mu \quad \text{tali che}$$

$$e^a(e_b) = \delta^a_b$$

$$H^a_\mu \, dx^\mu \left(e_b^{\ \nu} \frac{\partial}{\partial x^\nu} \right) = \delta^a_b = H^a_\mu \, e_b^{\ \nu} \, \delta^\mu_\nu = H^a_\mu \, e_b^{\ \mu}$$

H^a_μ è la matrice inversa di $\underline{\underline{e^\mu_a}}$ vettori

e si indica con $\underline{\underline{e^a_\mu}}$ forme

$$\boxed{\delta^a_b = e^a_\mu \, e^\mu_b}$$

$$AB = 1 = BA = 1$$

$$\eta_{ab} = g(e_a, e_b) = g(e^\mu_a \partial_\mu, e^\nu_b \partial_\nu) =$$

$$= e^\mu_a e^\nu_b \, g(\partial_\mu, \partial_\nu) = e^\mu_a e^\nu_b g_{\mu\nu}$$

$$\boxed{\eta_{ab} = e^\mu_a \, g_{\mu\nu} \, e^\nu_b}$$

$$\delta^a{}_b = e^a{}_\mu \, e^\mu{}_b \qquad AB = 1 = BA = 1$$

$$A = (e^a{}_\mu) \qquad B = (e^\mu{}_a)$$

$$A \cdot B = e^a{}_\mu \, e^\mu{}_b = \delta^a{}_b$$

$$B \cdot A = e^\mu{}_a \, e^a{}_\nu = \delta^\mu{}_\nu$$

$$\boxed{e^\mu{}_a \, e^a{}_\nu = \delta^\mu{}_\nu} \qquad \eta_{ab} = e^\mu{}_a \, g_{\mu\nu} \, e^\nu{}_b$$

$$\delta^\rho_\mu \qquad \delta^\sigma_\nu$$

$$\eta_{ab} \, e^a{}_\mu \, e^b{}_\nu = e^\rho{}_a \, g_{\rho\sigma} \, e^\sigma{}_b \, e^a{}_\mu \, e^b{}_\nu = g_{\mu\nu}$$

$$\boxed{g_{\mu\nu} = \eta_{ab} \, e^a{}_\mu \, e^b{}_\nu}$$

$$e_c^{\ \mu} g_{\mu\nu} = \eta_{ab} \underbrace{e^a_{\ \mu} e^b_{\ \nu} e_c^{\ \mu}}_{\delta^a_c} = \eta_{cb} e^b_{\ \nu}$$

$$\underset{\text{vettori}}{\underline{\underline{e_a^{\ \mu}}}} g_{\mu\nu} = \eta_{ab} \underset{\text{forme}}{\underline{\underline{e^b_{\ \nu}}}}$$

μ, ν si alzano e abbassano con $g_{\mu\nu}$

a, b con η_{ab}

$$\nabla_{e_a} e_b = \gamma_{ab}^{\ \ c} e_c$$

Compatibilità metrica

$$\nabla_g(X, Y, Z) = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0$$

$$0 = \nabla_g(e_a, e_b, e_c) = \cancel{e_a(\eta_{bc})} - g(\gamma_{ab}^{\ \ d} e_d, e_c) +$$

$$- g(e_b, \gamma_{ac}^{\ \ d} e_d) =$$

$$= -\gamma_{ab}^{\ \ d}\,\eta_{dc} - \gamma_{ac}^{\ \ d}\,\eta_{db} = 0 \qquad (*)$$

Definiamo $\omega^a{}_b \equiv \gamma^a{}_{cb}\, e^c \equiv \omega_\mu{}^a{}_b\, dx^\mu$

$\omega_\mu{}^a{}_b$ connessione di spin

$$\omega^{ab} = \omega^a{}_c\, \eta^{cb} = \omega_\mu^{ab}\, dx^\mu$$

$(*) \Longrightarrow$ (moltiplico per e^a)

$$0 = -\omega^d{}_b\, \eta_{dc} - \omega^d{}_c\, \eta_{db}$$

$$\Longrightarrow \quad \omega^{ab} + \omega^{ba} = 0$$

Torsione

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$T(e_a, e_b) = \nabla_{e_a}(e_b) - \nabla_{e_b}(e_a) - [e_a, e_b] = T^c_{ab}\, e_c =$$

$$= \gamma^c_{ab}\, e_c - \gamma^c_{ba}\, e_c - [e^\mu_a \partial_\mu, e^\nu_b \partial_\nu] =$$

$$= \underbrace{\gamma^c_{ab}}\, e_c - \underbrace{\gamma^c_{ba}\, e_c} - e^\mu_a \partial_\mu e^\nu_b \partial_\nu +$$

$$+ e^\nu_b (\partial_\nu e^\mu_a) \partial_\mu$$

Moltiplichiamo questa espressione per $\frac{1}{2} e^a \wedge e^b$

$$\frac{1}{2} e^a \wedge e^b \, T(e_a , e_b) = e^a \wedge e^b \, \gamma_{ab}{}^c \, e_c +$$

$$\omega^c{}_b$$

$$- e^a \wedge e^b \, e_a^\mu \partial_\mu e_b^\nu \, \partial_\nu =$$

$$= \omega^c{}_b \wedge e^b \, e_c - e^a_\rho \, dx^\rho \wedge e^b \, e_a^\mu (\partial_\mu e_b^\nu) \partial_\nu =$$

$$\delta^\rho_\mu$$

$$= \omega^c{}_b \wedge e^b \, e_c - de_b^\nu \wedge e^b \, \partial_\nu =$$

$$= \left[\omega^c{}_b \, e^b - de_b^\nu \wedge e^b \, e^c_\nu \right] e_c$$

$$\delta^c_b$$

$$\partial_\nu = e^c_\nu \, e_c^\mu \partial_\mu = e^c_\nu \, e_c$$

$$e^c_\nu \, de_b^\nu = e^c_\nu \, dx^\mu \partial_\mu e_b^\nu = dx^\mu \partial_\mu \left(e^c_\nu \, e_b^\nu \right) +$$

$$-dx^\mu \partial_\mu e^c_\nu \, e^\nu_b = -de^c_\nu \, e^\nu_b$$

$$\frac{1}{2} e^a \wedge e^b \, T(e_a, e_b) =$$

$$= \left[\omega^c{}_b e^b + de^c_\nu \underbrace{e^\nu_b \, e^b_\rho}_{\delta^\nu_\rho} dx^\rho\right] e_c$$

$$= \left[\omega^c{}_b e^b + de^c_\nu \, dx^\nu\right] e_c$$

$$= \left[de^c + \omega^c{}_b e^b\right] e_c \equiv T^c e_c$$

$$T^a \equiv de^a + \omega^a{}_b \wedge e^b \equiv \nabla e^a$$

$$d e^c_\nu \, dx^\nu = dx^\mu \partial_\mu e^c_\nu \, dx^\nu \equiv d e^c$$

$$e^c \equiv e^c_\nu \, dx^\nu$$

$$[e_a, e_b] = a^c_{ab} \, e_c$$

Curvatura

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$$

$$R(e_a, e_b) e_c = R_{ab}{}^d{}_c \, e_d = \nabla_{e_a} \nabla_{e_b} e_c +$$

$$- \nabla_{e_b} \nabla_{e_a} e_c - \nabla_{[e_a, e_b]} e_c =$$

$$= \nabla_{e_a} (\gamma_{bc}^{d} e_d) - \nabla_{e_b} (\gamma_{ac}^{d} e_d) - a^d_{ab} \gamma_{dc}^{e} e_e =$$

$$= \gamma_{bc}^{\ \ d} \gamma_{ad}^{\ \ e} e_e + e_a(\gamma_{bc}^{\ \ d}) e_d +$$

$$- \gamma_{ac}^{\ \ d} \gamma_{bd}^{\ \ e} e_e - e_b(\gamma_{ac}^{\ \ d}) e_d - \alpha_{ab}^{\ \ d} \gamma_{dc}^{\ \ e} e_e$$

Moltiplico per $\frac{1}{2} e^a \wedge e^b$

$$\frac{1}{2} e^a \wedge e^b R(e_a, e_b) e_c =$$

$$= \omega^e{}_d \wedge \omega^d{}_c e_e + e^a \wedge e^b e_a(\gamma_{bc}^{\ \ d}) e_d +$$

$$- \frac{1}{2} e^a \wedge e^b \alpha_{ab}^{\ \ d} \gamma_{dc}^{\ \ e} e_e$$

$$e^a \wedge e^b \, e_a\left(\gamma^d_{bc}\right) e_d = \underbrace{e^a_\mu}_{\delta^\rho_\mu} dx^\mu \, e^b_\nu dx^\nu \, e_a^\rho \, \partial_\rho \gamma^d_{bc} \, e_d$$

$$\left[e_a = e_a^\rho \frac{\partial}{\partial x^\rho} \right] \qquad = d\underbrace{\gamma^d_{bc} \, e^b}_{\omega^d{}_c} \, e_d =$$

$$= d\omega^d{}_c \, e_d - \gamma^d_{bc} \, de^b \, e_d$$

$$\left[[e_a, e_b] = a^c_{ab} \, e_c \right] \qquad -\frac{1}{2} e^a \wedge e^b \, a^d_{ab} \, e_d =$$

$$= -\frac{1}{2} e^a \wedge e^b \, [e_a, e_b] = -\frac{1}{2} e^a_\mu dx^\mu e^b_\nu dx^\nu \cdot$$

$$\cdot 2 \, e_a^\rho \, \partial_\rho e_b^\sigma \frac{\partial}{\partial x^\sigma} = -de^\sigma_b \, e^b \, \partial_\sigma =$$

$$= -dx^\mu \partial_\mu e_b^\sigma \; e_\nu^b \, dx^\nu \partial_\sigma =$$

$$\left[\partial_\mu (e_b^\sigma e_\nu^b) = \partial_\mu \delta_\nu^\sigma = 0 \right]$$

$$= dx^\mu e_b^\sigma \partial_\mu e_\nu^b \, dx^\nu \partial_\sigma = de^d \; e_d$$

$$\boxed{-\frac{1}{2} e^a \wedge e^b \, a_{ab}^{\;\;d} = de^d}$$

$$\frac{1}{2} e^a \wedge e^b R(e_a, e_b) e_c = \omega^e{}_d \wedge \omega^d{}_c \; e_e$$

$$+ d\omega^d{}_c \, e_d - \cancel{\gamma^d_{bc} \, de^b \, e_d} + \cancel{de^d \, \gamma^e_{dc}} \, e_e =$$

$$= \left[\omega^e{}_d \wedge \omega^d{}_c + d\omega^e{}_c \right] e_e$$

$$\frac{e^a \wedge e^b}{2} R_{ab}{}^{c}{}_{d}\, e^d = R^c{}_d\, e^d$$

grado come forma differenziale

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c\, \omega^c{}_b$$

Derivata covariante Sia $T^{a_1 \dots a_m}{}_{b_1 \dots b_n} \in \Lambda^k_{(m,n)}$

$$\nabla T^{a_1 \dots a_m}{}_{b_1 \dots b_n} \equiv dT^{a_1 \dots a_m}{}_{b_1 \dots b_n} + \sum_{i=1}^{m} \omega^{a_i}{}_c\, T^{a_1 \dots \overset{c}{\cancel{a_i}} \dots a_m}{}_{b_1 \dots b_n} +$$

$$-(-1)^k \sum_{i=1}^{n} T^{a_1 \dots a_m}{}_{b_1 \dots \underset{c}{\cancel{b_i}} \dots b_n}\, \omega^c{}_{b_i}$$

Esempio: $e^a \in \Lambda^1_{(1,0)}$

$$\nabla e^a = de^a + \omega^a{}_b e^b = T^a$$

Se $W^a \in \Lambda^0_{(1,0)}$ $V_a \in \Lambda^0_{(0,1)}$

$$\begin{cases} \nabla W^a = dW^a + \omega^a{}_b W^b \\ \nabla V_a = dV_a - V_b\, \omega^b{}_a \end{cases}$$

$W^a V_a \in \Lambda^0_{(0,0)}$ $\nabla(W^a V_a) = d(W^a V_a) =$

$$= dW^a\, V_a + W^a\, dV_a = \nabla W^a\, V_a + W^a \nabla V_a =$$

$$= (dW^a + \cancel{\omega^a{}_b W^b})\, V_a + W^a (dV_a - \cancel{V_b\, \omega^b{}_a})$$

In generale

$$\nabla\left(T_k{}^{a_1\dots a_m}{}_{b_1\dots b_n}\wedge S_h{}^{c_1\dots c_r}{}_{d_1\dots d_p}\right)=\nabla T_k{}^{a_1\dots a_m}{}_{b_1\dots b_n}\wedge S_h{}^{c_1\dots c_r}{}_{d_1\dots d_p}+$$

$$+(-1)^k\, T_k{}^{a_1\dots a_m}{}_{b_1\dots b_n}\wedge \nabla S_h{}^{c_1\dots c_r}{}_{d_1\dots d_p}$$

Esercizio

$$\nabla(e^a\wedge e^b)=d(e^a\wedge e^b)+\omega^a{}_c\, e^c\wedge e^b+\omega^b{}_c\, e^a e^c=$$

$$=de^a\wedge e^b-e^a\wedge de^b+\omega^a{}_c\, e^c\wedge e^b+$$

$$-e^a\,\omega^b{}_c\, e^c=(de^a+\omega^a{}_c\, e^c)\wedge e^b+$$

$$-e^a\,(de^b+\omega^b{}_c\, e^c)=\nabla e^a\wedge e^b-e^a\wedge\nabla e^b$$

$$g_{\mu\nu} = e_\mu^{\ a}\, \eta_{ab}\, e_\nu^{\ b} =$$

$$= e_\mu^{\ a\,\prime}\, \eta_{ab}\, e_\nu^{\ b\,\prime}$$

$e_\mu^{\ a}$ tetrade
vierbein
vielbein

$$e_\mu^{\ a\,\prime} = \Lambda^a{}_b(x)\, e_\mu^{\ b} \qquad \Lambda^a{}_c\, \eta_{cd}\, \Lambda^b{}_d = \eta_{ab}$$

Queste si chiamano trasformazioni di Lorentz locali

Cambiamenti di base (anche che non lasciano η_{ab} invariata)

$$\begin{cases} \underline{e^{a\,\prime} = \Omega^a{}_b\, e^b} \\ e_a{}' = \underbrace{(\Omega^{-1})}_{\ }{}^b{}_a\, e_b \end{cases} \qquad \Omega \in GL(4, \mathbb{R})$$

$$e^a(e_b) = e^{a\,\prime}(e_b') = \delta^a_b$$

$$\left[\nabla_{e_a} e_b = \gamma_{ab}^{c}\, e_c\right] \qquad \nabla_{e_a'} e_b' = \underbrace{\gamma_{ab}^{c}{}'}\, e_c' =$$

$$= \nabla_{(\Omega^{-1})^d{}_a e_d} \left(\Omega^{-1\,e}{}_b\, e_e \right) =$$

$$= (\Omega^{-1})^d{}_a\, \nabla_{e_d} \left(\Omega^{-1\,e}{}_b\, e_e \right) =$$

$$= (\Omega^{-1})^d{}_a (\Omega^{-1})^e{}_b\, \nabla_{e_d} e_e \; +$$

$$+ (\Omega^{-1})^d{}_a\, e_d\left(\Omega^{-1\,e}{}_b \right) e_e =$$

e_f

$$= (\Omega^{-1})^d{}_a (\Omega^{-1})^e{}_b\, \gamma_{de}^{f}\, \Omega^g{}_f\, e_g' \; +$$

$$+ (\Omega^{-1})^d{}_a\, e_d\left(\Omega^{-1\,e}{}_b \right) \Omega^g{}_e\, e_g'$$

$$\gamma_{ab}{}^{c\,\prime} = (\Omega^{-1})^d{}_a (\Omega^{-1})^e{}_b \gamma_{de}{}^f \Omega^c{}_f +$$
$$+ (\Omega^{-1})^d{}_a \, e_d(\Omega^{-1\,e}{}_b) \Omega^c{}_e$$

$$\omega^a{}_b{}' = \gamma^a{}_{cb}{}' e^{c\,\prime} =$$
$$= \Big[(\Omega^{-1})^d{}_c (\Omega^{-1})^e{}_b \gamma_{de}{}^f \Omega^a{}_f +$$
$$+ (\Omega^{-1})^d{}_c \, e_d(\Omega^{-1\,e}{}_b) \Omega^a{}_e\Big] \Omega^c{}_g e^g =$$
$$= (\Omega^{-1})^e{}_b \gamma_{de}{}^f \Omega^a{}_f e^d +$$
$$+ e_d(\Omega^{-1\,e}{}_b) \Omega^a{}_e e^d =$$

$$= \Omega^a{}_f \, \omega^f{}_e \, (\Omega^{-1})^e{}_b + \Omega^a{}_e \, d(\Omega^{-1})^e{}_b$$

$$e^d \, e_d(f) = df = e^d_\mu \, dx^\mu \, e^\nu_d \frac{\partial f}{\partial x^\nu} = dx^\mu \frac{\partial f}{\partial x^\mu}$$

$$\omega' = \Omega \omega \Omega^{-1} + \Omega \, d \Omega^{-1}$$

$$T^a = \nabla e^a = de^a + \omega^a{}_b \, e^b$$

$$T = de + \omega e$$

$$T' = de' + \omega' e' = d(\Omega e) +$$

$$+ \left(\Omega \omega \Omega^{-1} + \Omega \, d \Omega^{-1} \right) \Omega e =$$

$$= \cancel{d\Omega \wedge e} + \Omega\, de + \Omega\, \omega\, e +$$

$$+ \cancel{\Omega\, d\Omega^{-1}}\, \Omega\, e = \Omega\, \nabla e = \Omega\, T$$

$$d\Omega + \Omega\, d\Omega^{-1}\, \Omega = 0$$ (derivata della matrice inversa)

$$d\Omega^{-1} = -\Omega^{-1}\, d\Omega\, \Omega^{-1}$$

$$T' = \Omega\, T = \nabla' e' = \nabla' \Omega\, e =$$

$$= \Omega\, \nabla e \qquad \nabla' \Omega = \Omega\, \nabla$$

$$\boxed{\nabla' = \Omega\, \nabla\, \Omega^{-1}}$$

$$\left[R = d\omega + \omega\omega \right] \qquad \omega' = \Omega\omega\Omega^{-1} + \Omega d\Omega^{-1}$$

$$R' = d\omega' + \omega'\omega' = d\left(\Omega\omega\Omega^{-1} + \Omega d\Omega^{-1}\right) +$$

$$+ \left(\Omega\omega\Omega^{-1} + \Omega d\Omega^{-1}\right)\left(\Omega\omega\Omega^{-1} + \Omega d\Omega^{-1}\right) =$$

$$= \cancel{d\Omega\,\omega\,\Omega^{-1}} + \Omega\, d\omega\, \Omega^{-1} - \cancel{\Omega\,\omega\, d\Omega^{-1}} +$$

$$+ \cancel{d\Omega\, d\Omega^{-1}} + \Omega\,\omega\,\omega\,\Omega^{-1} + \cancel{\Omega\,\omega\, d\Omega^{-1}} +$$

$$+ \cancel{\Omega\, d\Omega^{-1}\,\Omega\,\omega\,\Omega^{-1}} + \cancel{\Omega\, d\Omega^{-1}\,\Omega\, d\Omega^{-1}} =$$

$$= \Omega R \Omega^{-1}$$

$$\boxed{R' = \Omega R \Omega^{-1}}$$

Identità di Bianchi

$$T = \nabla e = de + \omega e \qquad\qquad T^a$$

$$\nabla T = \nabla\nabla e = dT + \omega T =$$

$$= d(de + \omega e) + \omega(de + \omega e) =$$

$$= d\omega\, e - \cancel{\omega de} + \cancel{\omega de} + \omega\omega e =$$

$$= R e \qquad\qquad R = d\omega + \omega\omega$$

$$\nabla T^a = R^a{}_b\, e^b$$

Torsione nulla $\Rightarrow$ $R^a{}_b\, e^b = 0$

Scrivo $R^a{}_b$ come una 2-forma

$$R^a{}_b = \frac{1}{2} R^a{}_{b\mu\nu}\, dx^\mu dx^\nu = \frac{1}{2} R^a{}_{bcd}\, e^c \wedge e^d$$

$$T^a = 0 \Rightarrow 0 = R^a{}_b\, e^b = \frac{1}{2} R^a{}_{bcd}\, e^b e^c e^d$$

$$\Rightarrow R^a{}_{bcd} + R^a{}_{dbc} + R^a{}_{cdb} = 0$$

$$\nabla R^a{}_b = dR^a{}_b + \omega^a{}_c R^c{}_b - R^a{}_c\, \omega^c{}_b$$

$$\nabla R = dR + \omega R - R\omega =$$

$$= d(\cancel{d\omega} + \omega\omega) + \omega(d\omega + \cancel{\omega\omega}) +$$

$$- (d\omega + \cancel{\omega\omega})\omega = 0 \qquad \boxed{\nabla R = 0}$$

$$e^a = e^a{}_\mu \, dx^\mu$$

Lo vediamo come un cambio di base

$$e^{a\prime} = \Omega^a{}_b \, e^b$$

$$\omega' = \Omega \omega \Omega^{-1} + \Omega d \Omega^{-1}$$

Base di partenza : $\underbrace{dx^\mu}_{= e^a}$ $\quad \nabla_{\partial_\mu} \partial_\nu = \Gamma_{\mu\nu}{}^\rho \partial_\rho$

$$\Gamma_{\mu\nu}{}^\rho dx^\mu \equiv \Gamma^\rho{}_\nu \quad (= \omega^a{}_b)$$

$$\omega^a{}_b = e^a{}_\rho \Gamma^\rho{}_\nu e^\nu{}_b + e^a{}_\rho \, d e^\rho{}_b$$

$$\boxed{\omega_\mu{}^a{}_b = e^a_\rho \Gamma_{\mu\nu}{}^\rho e^\nu_b + e^a_\rho \partial_\mu e^\rho_b} \qquad (*)$$

Posso definire

$$\nabla T^{a_1 \dots a_m \, \nu_1 \dots \nu_s}{}_{b_1 \dots b_n \, \mu_1 \dots \mu_r} \qquad \text{con } \omega^a{}_b \longrightarrow \Gamma^\rho{}_\nu$$

$$\nabla e^a_\mu = d e^a_\mu + \omega^a{}_b \, e^b_\mu - e^a_\nu \, \Gamma^\nu{}_\mu$$

$$\nabla_\nu e^a_\mu = \partial_\nu e^a_\mu + \omega_\nu{}^a{}_b \, e^b_\mu - e^a_\rho \, \Gamma^\rho_{\nu\mu}$$

Moltiplico per e^μ_c

$$e^\mu_c \nabla_\nu e^a_\mu = e^\mu_c \partial_\nu e^a_\mu + \omega^a{}_{\nu c} - e^a_\rho \Gamma^\rho_{\nu\mu} e^\mu_c$$

$$= - e^a_\mu \partial_\nu e^\mu_c - e^a_\rho \Gamma^\rho_{\nu\mu} e^\mu_c + \omega^a{}_{\nu c} = 0$$

per (*) [Non abbiamo usato la compatibilità metrica]

$$\nabla g_{\mu\nu} = d g_{\mu\nu} - g_{\rho\nu} \Gamma^{\rho}{}_{\mu} - g_{\mu\rho} \Gamma^{\rho}{}_{\nu}$$

$$\parallel$$

$$dx^{\sigma} \nabla_{\sigma} g_{\mu\nu} \qquad \nabla_{\sigma} g_{\mu\nu} = \partial_{\sigma} g_{\mu\nu} - \Gamma_{\sigma\mu}^{\rho} g_{\rho\nu} - \Gamma_{\sigma\nu}^{\rho} g_{\mu\rho}$$

Compatibilità metrica $\iff \nabla_{\sigma} g_{\mu\nu} = 0 \iff$

$$\Gamma_{\mu\nu}^{\rho} = \Gamma_{\mu\nu}^{\rho}(g)$$

Allora $\omega_{\mu}{}^{a}{}_{b} = e^{a}_{\rho} \Gamma_{\mu\nu}^{\rho} e^{\nu}_{b} + e^{a}_{\rho} \partial_{\mu} e^{\rho}_{b}$

$$\Rightarrow \quad \omega_{\mu}{}^{a}{}_{b} = \omega_{\mu}{}^{a}{}_{b}(e) \quad \left[\iff D_{\mu} e^{a}_{\nu} = 0 \right.$$

$$\left. \Gamma_{\mu\nu}^{\rho} = \Gamma_{\mu\nu}^{\rho}(g) \right]$$

$$g_{\mu\nu} = e_\mu^{\ a}\, \eta_{ab}\, e_\nu^{\ b}$$

$$\nabla_\mu e_\nu^{\ a} = 0 \quad \not\Rightarrow \quad \nabla_\rho g_{\mu\nu} = 0$$

a meno che non si richieda anche $\nabla \eta_{ab} = 0$

(che è sempre la compatibilità metrica nella base degli e_a)

Torsione $\quad T^a = \nabla e^a = de^a + \omega^a{}_b\, e^b$

Nella base dx^μ [invece che e^a]

$$\nabla dx^\mu = \cancel{d\, dx^\mu} + \Gamma^\mu{}_\nu\, dx^\nu = dx^\rho\, \Gamma^\mu{}_{\rho\nu}\, dx^\nu$$

Nella base dx^μ la torsione nulla $\nabla dx^\mu = 0$ implica

$$\Gamma^\rho_{[\mu\nu]} = 0, \quad \text{cioè} \quad \Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$$

$$\nabla e^a = \nabla(e^a_\mu dx^\mu) = e^a_\mu \nabla dx^\mu$$

$$\nabla e^a = 0 \iff \nabla dx^\mu = 0$$

Identità di Bianchi

$$\nabla R = 0 \qquad \nabla R^a{}_b = 0$$

$$\nabla\left(R^a{}_{bcd} \frac{e^c \wedge e^d}{2}\right) = 0 =$$

$$= \frac{1}{2} \nabla R^a{}_{bcd}\, e^c \wedge e^d + \frac{1}{2} R^a{}_{bcd} \nabla e^c \wedge e^d - \frac{1}{2} R^a{}_{bcd}\, e^c \wedge \nabla e^d =$$

$$= \frac{1}{2} \nabla R^a{}_{bcd}\, e^c \wedge e^d + R^a{}_{bcd}\, \nabla e^c \wedge e^d$$

Se la torsione è nulla ($\nabla e^a = 0$), allora

$$0 = \nabla R^a{}_{bcd}\, e^c \wedge e^d =$$

$$= dx^\mu \nabla_\mu R^a{}_{bcd}\, e^c \wedge e^d =$$

$$= \nabla_f R^a{}_{bcd}\, e^f \wedge e^c \wedge e^d = 0$$

$$\nabla = dx^\mu \nabla_\mu = e^f \nabla_f = e^f_\mu dx^\mu \nabla_f \qquad \nabla_\mu = e_\mu{}^a \nabla_a =$$

$$= \nabla_a (e_\mu{}^a \dots)$$

Potremmo anche scrivere $\nabla_\mu R^a{}_{b\nu\rho}\, dx^\mu dx^\nu dx^\rho = 0$

$$0 = \nabla_e R^a{}_{bcd} + \nabla_d R^a{}_{bec} + \nabla_c R^a{}_{bde}$$

(solo a torsione nulla)

Versioni contratte di queste identità di Bianchi

$$R^a{}_{bad} = R_{bd} \quad \text{(tensore di Ricci)}$$

$$R^a{}_{bad}\, \eta^{bd} = R \quad \text{(curvatura scalare)}$$

$\underline{a=c}$: $$0 = \nabla_e R_{bd} - \nabla_d R_{be} + \nabla_a R^a{}_{bde}$$

Moltiplico per η^{bd} (e uso $\nabla \eta^{ab} = 0$)

$$0 = \nabla_e R - \nabla^b R_{be} - \nabla_a R^a{}_e$$

$$\nabla_a R^a_{\ b} = \frac{1}{2} \nabla_b R$$

Campo scalare in gravità esterna

$$\varphi(x) \qquad \nabla \varphi = d\varphi = dx^\mu \nabla_\mu \varphi = dx^\mu \partial_\mu \varphi$$

$$\nabla_\mu \varphi = \partial_\mu \varphi$$

accoppiamento non minimale

$$S = \frac{1}{2} \int_M d^4x \sqrt{-g} \Big[\underbrace{g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi + \xi R \varphi^2 - m^2 \varphi^2}_{\mathcal{L}(x)} \Big]$$

$$\mathcal{L}'(x') = \mathcal{L}(x)$$

ξ = costante arbitraria

$$\varphi'(x') = \varphi(x) \qquad R'(x') = R(x)$$

$$\nabla'_\mu \varphi'(x') = \frac{\partial x^\nu}{\partial x^{\mu'}} \nabla_\nu \varphi(x)$$

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x^{\mu'}} \frac{\partial x^\sigma}{\partial x^{\nu'}} g_{\rho\sigma}(x)$$

$$g^{\mu\nu\,'}(x') = \frac{\partial x^{\mu'}}{\partial x^\rho} \frac{\partial x^{\nu'}}{\partial x^\sigma} g^{\rho\sigma}(x)$$

$$d^4x' \sqrt{-g'(x')} = d^4x \sqrt{-g(x)} \qquad g \equiv \det g_{\mu\nu}$$

Si possono aggiungere infiniti termini (di dimensione superiore)

$$\int \sqrt{-g}\, R^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi \, d^4x, \quad \int d^4x \sqrt{-g} \left(\nabla_\mu \varphi\, g^{\mu\nu} \nabla_\nu \varphi\right)^2$$

Spazio piatto : $g_{\mu\nu} = \eta_{\mu\nu}$ $\left(R = 0 , \; R_{\mu\nu} = 0 , \; \Gamma^{\rho}_{\mu\nu} = 0 , \; R^{a}{}_{bcd} = 0 \right)$

Teorie di gauge QED : A_μ

$$F^{\dot a}_{\mu\nu} = \partial_\mu A^{\dot a}_\nu - \partial_\nu A^{\dot a}_\mu + f^{\dot a \dot b \dot c} A^{\dot b}_\mu A^{\dot c}_\nu \qquad (*)$$

$$F^{\dot a} = dA^{\dot a} + \frac{1}{2} f^{\dot a \dot b \dot c} A^{\dot b} \wedge A^{\dot c} = \frac{F^{\dot a}_{\mu\nu}}{2} dx^\mu \wedge dx^\nu$$

$$\boxed{A^{\dot a} = A^{\dot a}_\mu dx^\mu}$$ connessione di gauge

(a torsione nulla)

$$F^{\dot a}_{\mu\nu} = \nabla_\mu A^{\dot a}_\nu - \nabla_\nu A^{\dot a}_\mu + f^{\dot a \dot b \dot c} A^{\dot b}_\mu A^{\dot c}_\nu \quad \leftarrow \quad (**)$$

Infatti

$$\nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \Gamma_{\mu\nu}^{\ \ \rho} A_\rho - \partial_\nu A_\mu + \Gamma_{\nu\mu}^{\ \ \rho} A_\rho$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$S = -\frac{1}{4}\int d^4x \sqrt{-g}\; F_{\mu\nu}^{\ \ a} F_{\rho\sigma}^{\ \ a} g^{\mu\rho} g^{\nu\sigma} +$$

D arbitrario $\quad d^Dx \sqrt{|g|}$ $\qquad + \theta \int d^4x\, F_{\mu\nu}^{\ \ a} F_{\rho\sigma}^{\ \ b} \varepsilon^{\mu\nu\rho\sigma}$

$(1, -1, -1, -1)$ $\qquad \omega' = \Omega\, \omega\, \Omega^{-1} + \Omega\, d\Omega^{-1}$

$$A' = U A U^{-1} + U dU^{-1} \qquad \omega^a{}_b$$

$U(x) \in G$ gruppo di gauge $(SU(N), \ldots)$

QED: $U = e^{i\Lambda} \in U(1)$

$$A' = e^{i\Lambda} A e^{-i\Lambda} + e^{i\Lambda}(-i d\Lambda) e^{-i\Lambda} =$$

$$= A - i d\Lambda$$

$$A_\mu' = A_\mu - i \partial_\mu \Lambda$$

$$\boxed{F_{\mu\nu}' = U F_{\mu\nu} U^{-1}} \hookleftarrow R^a{}_b$$

$$\int d^4x \, F^{\dot{a}}_{\mu\nu} F^{\dot{a}}_{\rho\sigma} \varepsilon^{\mu\nu\rho\sigma} \propto \int_M F^{\dot{a}} \wedge F^{\dot{a}} = \int \text{derivata totale}$$

$$F^{\dot{a}} \wedge F^{\dot{a}} = \frac{1}{4} F^{\dot{a}}_{\mu\nu} F^{\dot{a}}_{\rho\sigma} dx^\mu dx^\nu dx^\rho dx^\sigma$$

A torsione non nulla le espressioni (*) e (**) di $F^{\dot{a}}_{\mu\nu}$ sono entrambe corrette, cioè l'azione

$$-\frac{1}{4}\int \sqrt{-g}\, d^4x\, F^{\dot{a}}_{\mu\nu} F^{\dot{a}}_{\rho\sigma}\, g^{\mu\rho} g^{\nu\sigma}$$ è invariante

sotto diffeomorfismi in entrambi i casi

La ragione è che la torsione trasforma come un tensore $T^a = \nabla e^a$

Infatti $\omega' = \Omega\,\omega\,\Omega^{-1} + \Omega\, d\,\Omega^{-1} \implies$

$$dx^{\lambda'}\Gamma^{\mu}_{\lambda\nu}{}' = \Omega^{\mu}{}_{\rho}\, dx^{\beta}\,\Gamma^{\rho}_{\beta\sigma}(\Omega^{-1})^{\sigma}{}_{\nu}$$

$$+\,\Omega^{\mu}{}_{\rho}\, dx^{\beta}\,\partial_{\beta}\,\Omega^{-1\,\rho}{}_{\nu} \qquad \Omega^{\lambda}{}_{\beta} = \frac{\partial x^{\lambda'}}{\partial x^{\beta}}$$

$$dx^{\lambda'} = \frac{\partial x^{\lambda'}}{\partial x^{\tau}} dx^{\tau} = \Omega^{\lambda}{}_{\tau}\, dx^{\tau}$$

$$\Gamma^{\mu'}_{\lambda\nu} = \Omega^{\mu}{}_{\rho} (\Omega^{-1})^{\beta}{}_{\lambda} \Gamma^{\rho}_{\beta\sigma} (\Omega^{-1})^{\sigma}{}_{\nu} +$$

$$+ \Omega^{\mu}{}_{\rho} (\Omega^{-1})^{\beta}{}_{\lambda}\, \partial_{\beta} (\Omega^{-1})^{\rho}{}_{\nu}$$

$$\Gamma'_{\mu\nu} - \Gamma'_{\nu\mu} = \underbrace{\Omega^{\mu}{}_{\rho} (\Omega^{-1})^{\beta}{}_{\lambda} \left(\Gamma^{\rho}_{\beta\sigma} - \Gamma^{\rho}_{\sigma\beta}\right)(\Omega^{-1})^{\sigma}{}_{\nu}}_{\text{come un tensore}} +$$

$$+ \underbrace{\Omega^{\mu}{}_{\rho} \left[(\Omega^{-1})^{\beta}{}_{\lambda}\, \partial_{\beta} (\Omega^{-1})^{\rho}{}_{\nu} - (\Omega^{-1})^{\beta}{}_{\nu}\, \partial_{\beta} (\Omega^{-1})^{\rho}{}_{\lambda} \right]}_{=0}$$

Infatti è $\displaystyle - \underbrace{\partial_{\beta} \Omega^{\mu}{}_{\rho}}_{= \partial_{\beta}\partial_{\rho} x^{\mu'} \text{ simmetrico in } \beta \leftrightarrow \rho} \underbrace{\left[(\Omega^{-1})^{\beta}{}_{\lambda} (\Omega^{-1})^{\rho}{}_{\nu} - (\Omega^{-1})^{\beta}{}_{\nu} (\Omega^{-1})^{\rho}{}_{\lambda} \right]}_{\text{antisimm. in } \beta \leftrightarrow \rho}$

$$F^2 \sim \vec{E}^2 - \vec{H}^2$$

$$F \wedge F \sim \vec{E} \cdot \vec{H}$$

Fermioni

$$\Psi = \begin{pmatrix} \psi \\ \phi \end{pmatrix}$$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma^\mu = (\sigma_0, \sigma_1, \sigma_2, \sigma_3) = (\sigma_0, \vec{\sigma})$$

$$\tilde{\sigma}^\mu = (\sigma_0, -\vec{\sigma}) \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}$$

$$\tilde{\sigma}^\mu = \sigma_2 \sigma^{\mu *} \sigma_2$$

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \qquad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\gamma^{0\,2} = 1$$

Trasformazioni di Lorentz

$$x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu \qquad \eta_{\mu\nu} = \Lambda^\rho{}_\mu \eta_{\rho\sigma} \Lambda^\sigma{}_\nu$$

$$x'^2 = x^{\mu\prime} \eta_{\mu\nu} x^{\nu\prime} = \Lambda^\mu{}_\alpha x^\beta \eta_{\mu\nu} \Lambda^\nu{}_\beta x^\beta$$

$$\Lambda^\rho{}_\mu \eta_{\rho\sigma} \Lambda^\sigma{}_\nu \eta^{\nu\beta} = \delta_\mu^\beta = \Lambda^\rho{}_\mu \Lambda_\rho{}^\beta$$

$$\iff \delta_\mu^\beta = \Lambda_\mu{}^\rho \Lambda^\beta{}_\rho$$

$$\psi' = A\psi \qquad A = \begin{pmatrix} \tilde{A} & 0 \\ 0 & A \end{pmatrix}$$

$$\bar\psi = \psi^+ \gamma^0 \qquad \psi' = A\psi$$

$$\bar\psi' = \psi^{+\prime}\gamma^0 = \psi^+ A^+ \gamma^0 = \bar\psi\gamma^0 A^+\gamma^0$$

$$\bar\psi'\psi' = \bar\psi\psi = \psi^+ A^+ \gamma^0 A \psi =$$

$$= \bar\psi \gamma^0 A^+ \gamma^0 A \psi$$

$$\boxed{\gamma^0 A^+ \gamma^0 = A^{-1}}$$

Vogliamo anche $\bar\psi'\gamma^\mu\partial'_\mu\psi' = \bar\psi\gamma^\mu\partial_\mu\psi$

$(\gamma^\mu)_{\alpha\beta}$

$$\bar\psi \underbrace{A^{-1}\gamma^\mu A \Lambda_\mu{}^\nu}_{=\gamma^\nu}\partial_\nu\psi$$

$$\partial_\mu' = \frac{\partial}{\partial x^{\mu'}} = \frac{\partial x^\nu}{\partial x^{\mu'}}\frac{\partial}{\partial x^\nu} \qquad x^{\mu'} = \Lambda^\mu{}_\nu\, x^\nu$$

$$\frac{\partial x^{\mu'}}{\partial x^\nu} = \Lambda^\mu{}_\nu \qquad \Lambda_\mu{}^\rho \Lambda^\mu{}_\nu\, x^\nu = x^\rho = \Lambda_\mu{}^\rho x'^\mu$$

$$\frac{\partial x^\nu}{\partial x^{\mu'}} = \Lambda_\mu{}^\nu \qquad \partial_\mu' = \Lambda_\mu{}^\nu \partial_\nu$$

$$A^{-1}\gamma^\mu A\, \Lambda_\mu{}^\nu = \gamma^\nu \qquad A^{-1}\gamma^\mu A = \Lambda^\mu{}_\nu\, \gamma^\nu$$

Considero il gruppo $SL(2,\mathbb{C})$ = matrici complesse invertibili 2×2 con $\det = 1$

$A \in SL(2,\mathbb{C})$ si può scrivere come

$A = a\,\sigma_0 + \vec{b}\cdot\vec{\sigma}$ dove $a, \vec{b}$ sono

complessi e $a^2 - \vec{b}^{\,2} = 1$ $\quad A^{-1} = a\,\sigma_0 - \vec{b}\cdot\vec{\sigma}$

Si dimostra che

$$\underline{A^{+}\sigma^{\mu}A = \Lambda^{\mu}{}_{\nu}\,\sigma^{\nu}}$$

dove $\Lambda^{\mu}{}_{\nu}$ è una trasformazione di Lorentz

$$\gamma^0 A^{+} \gamma^0 = A^{-1} \implies$$

$$\underset{\gamma^0}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \begin{pmatrix} \tilde{A}^{+} & 0 \\ 0 & A^{+} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \tilde{A}^{-1} & 0 \\ 0 & A^{-1} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & A^+ \\ \tilde{A}^+ & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} A^+ & 0 \\ 0 & \tilde{A}^+ \end{pmatrix}$$

$$\tilde{A}^{-1} = A^+ \qquad A^{-1} = \tilde{A}^+$$

$$\tilde{A} = (A^{-1})^+ = a^* \sigma_0 - \vec{b}^* \cdot \vec{\sigma}$$

Vale anche $\underline{\tilde{A}^+ \tilde{\sigma}^\mu \tilde{A} = \Lambda^\mu{}_\nu \tilde{\sigma}^\nu}$

$$\mathcal{A}^{-1} \gamma^\mu \mathcal{A} = \Lambda^\mu{}_\nu \gamma^\nu$$

Infatti

$$\mathcal{A}^{-1} \gamma^\mu \mathcal{A} = \begin{pmatrix} \tilde{A}^{-1} & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} \tilde{A} & 0 \\ 0 & A \end{pmatrix} =$$

$$= \begin{pmatrix} A^+ & 0 \\ 0 & \tilde{A}^+ \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu A \\ \tilde{\sigma}^\mu \tilde{A} & 0 \end{pmatrix} = \begin{pmatrix} 0 & A^+ \sigma^\mu A \\ \tilde{A}^+ \tilde{\sigma}^\mu \tilde{A} & 0 \end{pmatrix} = \Lambda^\mu{}_\nu \gamma^\nu$$

La relazione tra Λ e A è la seguente:

Se scrivo $\Lambda^\mu{}_\nu = (e^\rho)^\mu{}_\nu$ allora $A = e^{\Sigma^\mu{}_\nu \rho^\nu{}_\mu}$

$$\Sigma^\mu{}_\nu = -\frac{1}{8}[\gamma^\mu, \gamma_\nu] \qquad \rho = \rho^\mu{}_\nu$$

Per dimostrare $A^{-1}\gamma^\mu A = \Lambda^\mu{}_\nu \gamma^\nu$ $\qquad A = \exp(\mathrm{tr}[\Sigma\rho])$

$\Lambda = e^\rho$

Si usa l'identità (formula di Campbell-Baker-Hausdorff)

$$e^A B e^{-A} = e^{\mathrm{ad}_A} B = B + [A,B] + \frac{1}{2}[A,[A,B]] + \dots + \frac{1}{n!}\underbrace{[A,[A,\dots[A,B]\dots]]}_{n} + \dots$$

$\mathrm{ad}_A B = [A,B]$

operatore aggiunto

assieme a $\qquad [\Sigma^{\rho\sigma}, \gamma^\mu] = \frac{1}{2}(\eta^{\mu\rho}\gamma^\sigma - \eta^{\mu\sigma}\gamma^\rho)$

$\mu = \rho$: $\quad -\frac{1}{8}\left[[\gamma^\mu, \gamma^0], \gamma^\mu\right] = -\frac{1}{8}\left\{-2\gamma^\mu - 4\gamma^\mu - 4\gamma^\mu - 2\gamma^\mu\right\} =$

$$= \frac{3}{2}\gamma^\mu = \frac{1}{2}\left(4\gamma^\mu - \gamma^\mu\right) \quad \underline{\text{ok}}$$

$$B = \gamma^\mu \qquad A = -\Sigma^\mu{}_\nu \rho^\nu{}_\mu = -\operatorname{tr}[\Sigma\rho]$$

$$\Lambda^{-1}\gamma^\mu\Lambda = e^A B e^{-A} = \gamma^\mu - \left[\operatorname{tr}[\Sigma\rho], \gamma^\mu\right] + \dots =$$

$$= \gamma^\mu - \rho_{\nu\lambda}\left[\Sigma^{\lambda\nu}, \gamma^\mu\right] + \dots$$

$$= \gamma^\mu - \rho_{\nu\lambda}\frac{1}{\cancel{2}}\eta^{\lambda\mu}\gamma^\nu\cancel{2} + \dots =$$

$$= \gamma^\mu - \rho^{\nu\mu}\gamma_\nu + \dots = \left(\delta^\mu_\nu + \rho^\mu{}_\nu + \dots\right)\gamma^\nu =$$

$$\rho_{\mu\nu} = -\rho_{\nu\mu} \qquad = \Lambda^\mu{}_\nu\gamma^\nu \qquad \Lambda = e^\rho$$

Azione dei fermioni in gravità esterna

Da adesso quanto detto va riferito a indici piatti $a, b, \dots$

γ^a, σ^a, $\tilde{\sigma}^a$, $\gamma_a = \eta_{ab}\gamma^b$, $\Sigma^a{}_b = -\frac{1}{8}[\gamma^a, \gamma_b]$, $\rho^a{}_b$, $\Lambda^a{}_b$, etc.

$e^{a'} = \Omega^a{}_b e^b$ $\quad \Omega \in GL(4,\mathbb{R})$ vengono ristrette a trasformazioni di Lorentz locali $\quad e^{a'} = \Lambda^a{}_b e^b$

dove $\quad \Lambda^a{}_b = \Lambda^a{}_b(x)$

Non interessano gli indici di spaziotempo $\mu, \nu, \dots$

ma solo gli indici piatti $a, b, \dots$ e quindi anche quelli spinoriali

$\bar{\psi} \gamma^\mu \partial_\mu \psi$ cosa diventa ?

$$S = \int e^\mu_a \bar{\psi} \gamma^a (\partial_\mu + \tilde{\omega}_\mu) \psi \, e \, d^4x$$

$\tilde{\omega}_\mu = ?$

$$e = \sqrt{-g} = \det(e^a_\mu)$$

$$g = \det(g_{\mu\nu}) = \det(e^a_\mu \eta_{ab} e^b_\nu) = -\det{}^2(e^a_\mu)$$

$$\psi' = A \psi \qquad A = A(x) \qquad \tilde{\omega}_\mu = \Sigma^a{}_b \, \omega_\mu{}^b{}_a$$

Diffeomorfismi:

$$S' = \int e_a^{\mu\,\prime}\,\bar{\psi}'\,\gamma^a(\partial_\mu' + \tilde{\omega}_\mu')\psi'\,e'\,d^4x'$$

$$e'\,d^4x' = e\,d^4x \qquad \psi' = \psi \qquad \bar{\psi}' = \bar{\psi}$$

$$\partial_\mu' = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}, \qquad \tilde{\omega}_\mu' = \frac{\partial x^\nu}{\partial x'^\mu}\,\tilde{\omega}_\nu$$

$$e_a^{\mu\,\prime} = \frac{\partial x'^\mu}{\partial x^\nu}\,e_a^\nu \qquad \Longrightarrow \qquad S' = S$$

Trasformazioni di Lorentz locali $\quad e' = e \quad x' = x$

$$\psi' = A\psi \qquad \bar{\psi}' = \bar{\psi}A^{-1} \qquad e_a^{\mu\,\prime} = \Lambda_a{}^b\,e_b^\mu \quad \partial_\mu' = \partial_\mu$$

Dobbiamo avere

$$\tilde{\omega}' = \mathcal{A}\,\tilde{\omega}\,\mathcal{A}^{-1} + \mathcal{A}\,d\mathcal{A}^{-1}$$

$$S' = \int e\, d^4x\; e^{\mu}_{b}\, \Lambda_a{}^{b}\, \bar{\psi}\, \mathcal{A}^{-1} \gamma^a \left(\partial_\mu + \mathcal{A}\,\tilde{\omega}_\mu\, \mathcal{A}^{-1} + \right.$$

$$\left. + \mathcal{A}\, \partial_\mu \mathcal{A}^{-1} \right) \mathcal{A}\, \psi =$$

$$= \int e\, d^4x\; e^{\mu}_{b}\, \Lambda_a{}^{b}\, \bar{\psi}\, \mathcal{A}^{-1} \gamma^a \left(\cancel{(\partial_\mu \mathcal{A})} + \mathcal{A}\, \partial_\mu + \right.$$

$$\left. + \mathcal{A}\,\tilde{\omega}_\mu + \mathcal{A} \cancel{(\partial_\mu \mathcal{A}^{-1})} \mathcal{A} \right) \psi =$$

$$= \int e\, d^4x\; \bar{\psi}\, e^{\mu}_{b}\, \underbrace{\Lambda_a{}^{b}\, \mathcal{A}^{-1} \gamma^a \mathcal{A}}\, (\partial_\mu + \tilde{\omega}_\mu)\, \psi$$

$$= S \qquad\qquad \gamma^b = \mathcal{A}^{-1} \gamma^a \mathcal{A}\, \Lambda_a{}^{b}$$

Dobbiamo dimostrare

$$\tilde{\omega}' = \mathcal{A}\,\tilde{\omega}\,\mathcal{A}^{-1} + \mathcal{A}\, d\mathcal{A}^{-1}$$

Sappiamo che $\tilde{\omega} = \omega^a{}_b\, \Sigma^b{}_a = \mathrm{tr}[\omega \Sigma]$

e $\omega' = \Lambda \omega \Lambda^{-1} + \Lambda\, d\Lambda^{-1}$

$$\tilde{\omega}' = \mathrm{tr}[\omega' \Sigma] = \mathrm{tr}[(\Lambda \omega \Lambda^{-1} + \Lambda\, d\Lambda^{-1})\Sigma] =$$

$$= \underbrace{\mathrm{tr}[\Lambda^{-1}\Sigma\Lambda\ \omega]}_{= \mathcal{A}\,\tilde{\omega}\,\mathcal{A}^{-1}} + \underbrace{\mathrm{tr}[\Sigma\Lambda\, d\Lambda^{-1}]}_{= \mathcal{A}\, d\mathcal{A}^{-1}}$$

$$\gamma^a \Lambda_a{}^b = \mathcal{A}\,\gamma^b\,\mathcal{A}^{-1} \qquad \gamma_a \Lambda^a{}_b = \mathcal{A}\,\gamma_b\,\mathcal{A}^{-1}$$

$$(\Lambda^{-1}\Sigma\Lambda)^d{}_c = (\Lambda^{-1})^d{}_a \Sigma^a{}_b \Lambda^b{}_c =$$

$$= \Lambda_a{}^d \Sigma^a{}_b \Lambda^b{}_c = \frac{1}{8}\Lambda_a{}^d[\gamma^a, \gamma_b]\Lambda^b{}_c =$$

$$= \mathcal{A}\,\Sigma^d{}_c\,\mathcal{A}^{-1} \qquad \frac{1}{8}[\mathcal{A}\gamma^d\mathcal{A}^{-1}, \mathcal{A}\gamma_c\mathcal{A}^{-1}]$$

$$\mathrm{tr}[\Lambda^{-1}\Sigma\Lambda\,\omega] = \mathcal{A}\,\mathrm{tr}[\Sigma\omega]\,\mathcal{A}^{-1} = \mathcal{A}\,\tilde{\omega}\,\mathcal{A}^{-1}$$

$$\Lambda = e^{\rho} \qquad \mathcal{A} = e^{\mathrm{tr}[\Sigma\rho]}$$

$$\mathrm{tr}[\Sigma\Lambda d\Lambda^{-1}] \sim \mathrm{tr}[\Sigma e^{\rho} e^{-\rho}(-d\rho)] =$$

$$\sim -\mathrm{tr}[\Sigma d\rho]$$

$$\mathcal{A}\,d\mathcal{A}^{-1} \sim e^{\mathrm{tr}[\Sigma\rho]}\,e^{-\mathrm{tr}[\Sigma\rho]}\left(-\mathrm{tr}[\Sigma\rho]\right) \sim -\mathrm{tr}[\Sigma\rho]$$

per ρ infinitesimo

Per ρ qualunque si usa

$$e^{-M} d e^{M} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \underbrace{[M, [M, \ldots [M}_{n}, dM] \ldots]]$$

$$= dM - \frac{1}{2!}[M, dM] + \frac{1}{3!}[M, [M, dM]] + \ldots$$

Dobbiamo far vedere che

$$\mathcal{A} = e^{\mathrm{tr}[\Sigma \rho]}$$

$$\mathcal{A}\, d\mathcal{A}^{-1} = \mathrm{tr}\left[\Sigma \Lambda\, d\Lambda^{-1}\right]$$

$$\Lambda = e^{\rho} \qquad \rho^a{}_b$$

$$\mathrm{tr}\left[\Sigma \Lambda\, d\Lambda^{-1}\right] = \mathrm{tr}\left[\Sigma\, e^{\rho} d\, e^{-\rho}\right] =$$

$$= -\,\mathrm{tr}\left[\Sigma\, d\rho\right] - \frac{1}{2} \mathrm{tr}\left[\Sigma\, [\rho, d\rho]\right] + \ldots$$

$$\mathcal{A}\, d\mathcal{A}^{-1} = -\,\mathrm{tr}\left[\Sigma\, d\rho\right] - \frac{1}{2}\left[\mathrm{tr}[\Sigma \rho], \mathrm{tr}[\Sigma\, d\rho]\right] + \ldots$$

Dobbiamo verificare che

$$\Sigma^{ab}\,[\rho, d\rho]_{ba} = \rho_{ab}\, d\rho_{cd}\, [\Sigma^{ab}, \Sigma^{cd}]$$

$$[\Sigma^{ab}, \Sigma^{cd}] = \frac{1}{2}\left(\eta^{ac}\Sigma^{bd} - \eta^{ad}\Sigma^{bc} - \eta^{bc}\Sigma^{ad} + \eta^{bd}\Sigma^{ac}\right)$$

$$\Sigma^{ab}[\rho, d\rho]_{ba} = \frac{1}{\cancel{2}}\rho_{ab}\, d\rho_{cd}\left(\eta^{ac}\Sigma^{bd}\right)\cdot \overset{2}{\cancel{4}} =$$

$$= -2\,(\rho\, d\rho)_{bd}\,\Sigma^{bd}$$

$$\Sigma^{ab}\left(\rho_{bc}\, d\rho^{c}{}_{a} - d\rho_{bc}\,\rho^{c}{}_{a}\right) = 2\,\Sigma^{ab}(\rho\, d\rho)_{ba}$$

Più velocemente, dobbiamo mostrare

$$\mathrm{tr}\left[\Sigma\,\Lambda\, d\Lambda^{-1}\right] = \mathcal{A}\, d\mathcal{A}^{-1}$$

$$\mathcal{A}^{-1}\gamma^a \mathcal{A} = \Lambda^a{}_b\, \gamma^b \qquad \gamma_a \Lambda^a{}_b = \mathcal{A}\gamma_b \mathcal{A}^{-1}$$

$$\mathrm{tr}\left[\Sigma\,\Lambda\, d\Lambda^{-1}\right] = -\mathrm{tr}\left[\Sigma\, d\Lambda\, \Lambda^{-1}\right] = \frac{1}{8}\left(\gamma^a\gamma_b - \gamma_b\gamma^a\right) d\Lambda^b{}_c \left(\Lambda^{-1}\right)^c{}_a =$$

$$= \frac{1}{8}\left[(\Lambda^{-1})^c{}_a\gamma^a,\; d\left(\gamma_b\Lambda^b{}_c\right)\right] = \frac{1}{8}\left[\mathcal{A}\gamma^c\mathcal{A}^{-1},\; d\mathcal{A}\,\gamma^c\mathcal{A}^{-1} +\right.$$

$$\left. + \mathcal{A}\gamma^c d\mathcal{A}^{-1}\right] = \mathcal{A}\, d\mathcal{A}^{-1} + \frac{1}{4}\cancel{\mathcal{A}\gamma^c\mathcal{A}^{-1} d\mathcal{A}\,\gamma_c\,\mathcal{A}^{-1}}$$

Infatti, $\mathcal{A}^{-1}d\mathcal{A}$ = comb. lin. comm. tra Σ e Σ =

= comb. lin. Σ, ma $\gamma^c\Sigma^{ab}\gamma_c = 0$!

$$M^{ab} = -2i\Sigma^{ab} = \text{generatori del gruppo di Lorentz}$$

$$[M^{ab}, M^{cd}] = i(\eta^{bc}M^{ad} - \eta^{ac}M^{bd} - \eta^{bd}M^{ac} + \eta^{ad}M^{bc})$$

$$A = e^{\Sigma^{ab}\rho_{ba}} = e^{-\frac{i}{2}M^{ab}\rho_{ab}}$$

Altra rappresentazione: $(\bar{M}^{ab})_{cd} = i(\delta^a_c\delta^b_d - \delta^a_d\delta^b_c)$

$$e^{-\frac{i}{2}\bar{M}^{ab}\rho_{ab}} = e^{\rho} = \Lambda$$

$$-\frac{i}{2}(\bar{M}^{ab}\rho_{ab})_{cd} = \frac{1}{2}(\rho_{cd} - \rho_{dc}) = \rho_{cd}$$

Derivata covariante dei fermioni

$$S = \int e^{\mu}_{a} \bar{\psi} \gamma^{a} (\partial_\mu + \tilde{\omega}_\mu) \psi \, e \, d^4x =$$

$$= \int d^4x \, e \, e^{\mu}_{a} \bar{\psi} \gamma^{a} \nabla_\mu \psi$$

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{8} \omega_\mu{}^{a}{}_{b} [\gamma^b, \gamma_a] \psi$$

$$\nabla \psi = dx^\mu \nabla_\mu \psi = d\psi - \frac{1}{8} \omega^{a}{}_{b} [\gamma^b, \gamma_a] \psi$$

Identità di Bianchi

$$\nabla^2 \psi = -\frac{1}{8} R^{a}{}_{b} [\gamma^b, \gamma_a] \psi$$

$$R^a{}_b = \frac{1}{2} R^a{}_{b\mu\nu}\, dx^\mu dx^\nu \qquad R^a{}_{b\mu\nu} R^b{}_a{}^{\mu\nu}\sqrt{-g}$$

$$F^{\dot a} = \frac{1}{2} F^{\dot a}_{\mu\nu}\, dx^\mu dx^\nu \sim F^{\dot a}_{\mu\nu} F^{\dot a\,\mu\nu}\sqrt{-g}$$

$$\nabla\nabla\psi = \left(d - \frac{1}{8}\omega^a{}_b[\gamma^b,\gamma_a]\right)\left(d\psi - \frac{1}{8}\omega^c{}_d[\gamma^d,\gamma_c]\psi\right) =$$

$$= -\frac{1}{8} d\omega^c{}_d[\gamma^d,\gamma_c]\psi \checkmark + \cancel{\frac{1}{8}\omega^c{}_d[\gamma^d,\gamma_c]d\psi} +$$

$$-\cancel{\frac{1}{8}\omega^a{}_b[\gamma^b,\gamma_a]d\psi} + \frac{1}{128}\omega^a{}_b\,\omega^c{}_d\big[[\gamma^b,\gamma_a],[\gamma^d,\gamma_c]\big]\psi =$$

$$= -\frac{1}{8}\left(d\omega^a{}_b \checkmark + \omega^a{}_c\,\omega^c{}_b\right)[\gamma^b,\gamma^a]\psi$$

Deve valere

$$\frac{1}{16}\,\omega^a{}_b\,\omega^c{}_d\,\big[[\gamma^b,\gamma_a],[\gamma^d,\gamma_c]\big] =$$

$$= -\omega^a{}_c\,\omega^c{}_b\,[\gamma^b,\gamma_a]$$

Infatti $\Sigma^a{}_b \propto$ generatori del gruppo di Lorentz $\Longrightarrow$

$$[\Sigma^a{}_b,\Sigma^c{}_d] = \text{comb. lineare di } \Sigma =$$

$$= \underbrace{f^a{}_b{}^c{}_d{}^e{}_f}_{\text{costanti di struttura}}\,\Sigma^f{}_e$$

$$\big[[\gamma^a,\gamma^b],[\gamma^c,\gamma^d]\big] = B\Big\{\eta^{ac}[\gamma^b,\gamma^d]-\eta^{ad}[\gamma^b,\gamma^c]+$$

$$-\eta^{bc}[\gamma^a,\gamma^d]+\eta^{bd}[\gamma^a,\gamma^c]\Big\}$$

$$[\Sigma^{ab},\Sigma^{cd}] = \frac{1}{2}\left(\eta^{ac}\Sigma^{bd}-\eta^{ad}\Sigma^{bc}-\eta^{bc}\Sigma^{ad}+\eta^{bd}\Sigma^{ac}\right)$$

$$B = -4 \qquad \gamma^a \gamma_a = 4 \qquad \gamma^b \gamma^a \gamma_b = -2\gamma^a$$

$$\gamma^b \gamma^a \gamma^d \gamma_b = 4\eta^{ad}$$

$b = c$

$$(\gamma^a\gamma^b - \gamma^b\gamma^a)(\gamma_b\gamma^d - \gamma^d\gamma_b) - (a \leftrightarrow d) =$$

$$= 4\gamma^a\gamma^d + 2\gamma^a\gamma^d + 2\gamma^a\gamma^d + 4\eta^{ad} - (a \leftrightarrow d) =$$

$$= 8[\gamma^a, \gamma^d] = -2B[\gamma^a, \gamma^d] \quad \Rightarrow B = -4$$

$$\frac{1}{\cancel{16}}\, \omega^a{}_b\, \omega^c{}_d\, (-\cancel{4})\, \eta_{ac} [\gamma^b, \gamma^d] \cdot \cancel{4}$$

$$= -\omega^a{}_c\, \omega^c{}_b [\gamma^b, \gamma_a] \qquad \underline{\text{OK!}}$$

$\omega^{ab} = -\omega^{ba}$ (se vale la compatibilità metrica $\nabla \eta^{ab} = 0$)

Trasformazioni infinitesime

φ , ψ , A_μ , $e_\mu^{\ a}$, $g_{\mu\nu}$

4 $16 - 10 = 6$ trasf. di Lorentz

Gauge simmetrica : $e_\mu^{\ a} = \eta_{\mu b} e_\nu^{\ b} \eta^{\nu a}$ $\partial^\mu \omega_\mu^{\ ab} = 0$

$\partial_\mu A^\mu = 0$ non si può risolvere

Diffeomorfismi : $x^{\mu\prime} = x^\mu - \xi^\mu(x)$ $|\xi| \ll 1$

Lorentz locale : $e_\mu^{\ a\prime} = \Lambda^a{}_b \, e_\mu^{\ b}$ $\Lambda^a{}_b = \delta^a_b + \Theta^a{}_b$

$|\Theta| \ll 1$ $\Theta^{ab} = -\Theta^{ba}$

Scalare: $\varphi'(x') = \varphi(x) = \varphi'(x - \xi) =$

$\cong \varphi'(x) - \xi^\mu \partial_\mu \varphi + O(\xi^2)$

$$\delta\varphi = \varphi' - \varphi = \xi^\mu \partial_\mu \varphi$$

Vettore

$$A'_\mu(x') = A_\nu(x)\frac{\partial x^\nu}{\partial x'^\mu} = A'_\mu(x) - \xi^\rho \partial_\rho A_\mu(x) =$$

$$= A_\nu(x)\left(\delta^\nu_\mu + \partial_\mu \xi^\nu\right)$$

$$x^\mu = x'^\mu + \xi^\mu$$

$$\delta A_\mu(x) = A'_\mu(x) - A_\mu(x) = \xi^\rho \partial_\rho A_\mu + A_\nu \partial_\mu \xi^\nu$$

Vierbein

$$\delta e_\nu^{\ a} = \xi^\rho \partial_\rho e_\mu^{\ a} + e_\nu^{\ a} \partial_\mu \xi^\nu + \theta^a{}_b e_\mu^{\ b}$$

$$\delta (g_{\mu\nu}) = \delta (e_\mu^{\ a} \eta_{ab} e_\nu^{\ b}) = \xi^\rho \partial_\rho g_{\mu\nu} +$$

$$+ \underbrace{e_\rho^{\ a} \partial_\mu \xi^\rho \eta_{ab} e_\nu^{\ b}}_{g_{\rho\nu}} + e_\rho^{\ a} \partial_\nu \xi^\rho \eta_{ab} e_\mu^{\ b} =$$

$$= g_{\mu\rho} \partial_\nu \xi^\rho + g_{\nu\rho} \partial_\mu \xi^\rho + \xi^\rho \partial_\rho g_{\mu\nu} =$$

$$= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = g_{\nu\rho} \partial_\mu \xi^\rho + g_{\mu\rho} \partial_\nu \xi^\rho +$$

$$+ \partial_\mu g_{\nu\rho} \xi^\rho + \partial_\nu g_{\mu\rho} \xi^\rho - 2 T_{\mu\nu}^{\ \ \rho} \xi_\rho$$

usando torsione nulla e compatibilità metrica

$$T_{\mu\nu}^{\ \ \rho} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$$

$\xi_\nu = g_{\nu\rho} \xi^\rho$ si chiama vettore di Killing se soddisfa $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$

Fermione

$$\delta\psi = \xi^\mu \partial_\mu \psi - \frac{1}{8}\theta^a{}_b [\gamma^b, \gamma_a]\psi$$

il fermione si comporta come uno scalare sotto diffeomorfismi

Lorentz : $\psi' = \Lambda\psi = e^{tr[\Sigma\theta]}\psi$

$$\delta\psi_{Lorentz} = tr[\Sigma\theta]\psi = -\frac{1}{8}\theta^a{}_b [\gamma^b, \gamma_a]\psi$$

Connessione di spin $\qquad \Lambda \simeq 1+\theta$

$$\omega' = \Lambda\omega\Lambda^{-1} + \Lambda d\Lambda^{-1} \simeq \omega + \theta\omega - \omega\theta - d\theta$$

$$\delta\omega = -\nabla\theta \qquad \nabla\theta^a{}_b = d\theta^a{}_b + \omega^a{}_c\theta^c{}_b + - \theta^a{}_c\omega^c{}_b$$

$$\delta \omega_\mu{}^{ab} = \xi^\rho \partial_\rho \omega_\mu{}^{ab} + \partial_\mu \xi^\rho \omega_\rho{}^{ab} - \nabla_\mu \theta^{ab}$$

$$\left(\nabla \eta^{ab} = 0 \right)$$

Azione gravitazionale (Palatini)

$$S_{Pal} = C \int_M \underbrace{R^{ab} \wedge e^c \wedge e^d} \; \varepsilon_{abcd}$$

$$\varepsilon^{0123} = 1 \qquad \varepsilon_{0123} = -1 \qquad R^{ab} = \frac{1}{2} R_{\mu\nu}{}^{ab} \, dx^\mu \wedge dx^\nu$$

Si assume la compatibilità metrica $\nabla \eta^{ab} = 0$ $\left(\omega^{ab} = -\omega^{ba}\right)$, ma non necessariamente torsione nulla.

Formalismo del prim'ordine: sia e^a che $\omega^a{}_b$ sono indipendenti

$$S_{Pal} = c \int_M \frac{1}{2} R^{ab}_{\mu\nu} \underbrace{dx^\mu dx^\nu e^c_\rho e^d_\sigma dx^\rho dx^\sigma}_{\varepsilon^{\mu\nu\rho\sigma} d^4x} \varepsilon_{abcd} =$$

$$= \frac{c}{2} \int R^{ab}_{\mu\nu} e^c_\rho e^d_\sigma \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{abcd} \, d^4x =$$

$$= \frac{c}{2} \int R^{ab}_{fg} e^f_\mu e^g_\nu e^c_\rho e^d_\sigma \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{abcd} \, d^4x =$$

$$\sqrt{-g} = e = \det e^a_\mu = -\frac{1}{24} \varepsilon^{\mu\nu\rho\sigma} \varepsilon_{abcd} e^a_\mu e^b_\nu e^c_\rho e^d_\sigma$$

$$e^f_\mu e^g_\nu e^c_\rho e^d_\sigma \varepsilon^{\mu\nu\rho\sigma} = A \, \varepsilon^{fgcd}$$

$$A \, \varepsilon^{fgcd} \varepsilon_{fgcd} = -24 A = -24 e \qquad A = e$$

$$S_{Pal} = \frac{c}{2} \int e\, R^{ab}{}_{fg}\, \varepsilon^{fgcd}\, \varepsilon_{abcd} =$$

$$= \frac{c}{2} \int e\, R^{ab}{}_{fg}\, (-2)\, (\delta^f_a\, \delta^g_b)\, 2 =$$

$$= -2c \underbrace{\int e\, R\, d^4x}_{\text{Azione di Hilbert (se scritta per la metrica)}}$$

$$S_H = -\frac{1}{2k^2} \int_M \sqrt{-g}\, R\, d^4x \qquad c = \frac{1}{4k^2}$$

$$S_{Pal} = \frac{1}{4k^2} \int R^{ab} \wedge e^c \wedge e^d\, \varepsilon_{abcd} =$$

$$= \frac{1}{4\kappa^2}\int_M (d\omega + \omega\wedge\omega)^{ab} e^c e^d \varepsilon_{abcd}$$

Variazione rispetto a ω :

$$\delta S_{Pal} = \frac{1}{4\kappa^2}\int_M \Big(\underbrace{d\delta\omega + \delta\omega\wedge\omega + \omega\wedge\delta\omega}_{\nabla\delta\omega \,=\, d\omega + \omega\delta\omega - (-1)^1\delta\omega\,\omega}\Big)^{ab} e^c e^d \varepsilon_{abcd}$$

$$\delta S_{Pal} = \frac{1}{4\kappa^2}\int_M (\nabla\delta\omega^{ab}) e^c e^d \varepsilon_{abcd} =$$

$$= \frac{1}{4\kappa^2}\int_M \Big[\cancel{\underbrace{\nabla(\delta\omega^{ab} e^c e^d \varepsilon_{abcd})}_{= d(\delta\omega^{ab} e^c e^d \varepsilon_{abcd})}} + 2\delta\omega^{ab}\underbrace{\nabla e^c}_{= T^c} e^d \varepsilon_{abcd}\Big]$$

$$T^a = \nabla e^a$$

$\nabla \varepsilon_{abcd} = 0 = \nabla \varepsilon^{abcd}$. Infatti ($d\varepsilon^{abcd} = 0$), per la compatibilità metrica abbiamo:

$$\nabla \varepsilon^{abcd} = \omega^{a}{}_{f}\, \varepsilon^{fbcd} + \omega^{b}{}_{f}\, \varepsilon^{afcd} + \omega^{c}{}_{f}\, \varepsilon^{abfd} + \omega^{d}{}_{f}\, \varepsilon^{abcf} = 0$$

per ogni scelta di indici

0123

0023

ecc.

$$\delta S_{Pal} = \frac{1}{2k^2} \int_M \delta\omega_\mu^{ab}\, T^{c}_{\nu\rho}\, e^{d}_{\sigma}\, \varepsilon_{abcd}\, \varepsilon^{\mu\nu\rho\sigma} d^4x = 0$$

$$\Rightarrow \quad T^{c}_{\nu\rho}\, e^{d}_{\sigma}\, \varepsilon_{abcd}\, \varepsilon^{\mu\nu\rho\sigma} = 0$$

$$0 = T^{c}_{\nu\rho}\, e^{d}_{\sigma}\, \varepsilon_{abcd}\, \varepsilon^{\mu\nu\rho\sigma} \Rightarrow T^{c}_{mn}\, \varepsilon^{fmnd}\, \varepsilon_{abcd} = 0$$

$$0 = T_{mn}{}^{c}\,\varepsilon^{fmnd}\,\varepsilon_{abcd} = -T_{mn}{}^{c}\begin{vmatrix}\delta_a^f & \delta_a^m & \delta_a^n\\ \delta_b^f & \delta_b^m & \delta_b^n\\ \delta_c^f & \delta_c^m & \delta_c^n\end{vmatrix} =$$

$$= -2T_{bc}{}^{c}\,\delta_a^f + 2\left(T_{ac}{}^{c}\,\delta_b^f - T_{ab}{}^{f}\right)$$

$$T_{ab}{}^{d} = T_{ac}{}^{c}\,\delta_b^d - T_{bc}{}^{c}\,\delta_a^d \qquad \text{in dimensione } n$$

$$d = b \qquad T_{ab}{}^{b} = n\,T_{ac}{}^{c} - T_{ac}{}^{c} \Rightarrow T_{ac}{}^{c} = 0$$

se $n \neq 2$

che $\Rightarrow \quad T_{ab}{}^{c} = 0$

Sappiamo che $\nabla e^a = 0 = de^a + \omega^a{}_b\, e^b =$

$$= dx^\mu dx^\nu\left(\partial_\mu e^a_\nu + \omega_\mu{}^a{}_b\, e^b_\nu\right)$$

$$\Rightarrow \quad \partial_\mu e_\nu^{\ a} + \omega_\mu^{\ a}{}_b \, e_\nu^{\ b} = \partial_\nu e_\mu^{\ a} + \omega_\nu^{\ a}{}_b \, e_\nu^{\ b}$$

Sappiamo che $\nabla_\mu e_\nu^{\ a} = \underbrace{\partial_\mu e_\nu^{\ a} + \omega_\mu^{\ a}{}_b \, e_\nu^{\ b}}_{} - \underline{\Gamma_{\mu\nu}^{\ \ \rho} e_\rho^{\ a}} = 0$

$$\Rightarrow \quad \Gamma_{\mu\nu}^{\ \ \rho} e_\rho^{\ a} = \Gamma_{\nu\mu}^{\ \ \rho} e_\rho^{\ a} \quad \Rightarrow \quad \Gamma_{\mu\nu}^{\ \ \rho} = \Gamma_{\nu\mu}^{\ \ \rho}$$

$$\nabla \eta^{ab} = 0 \qquad g_{\mu\nu} = e_\mu^{\ a} \eta_{ab} e_\nu^{\ b} \quad \Rightarrow \quad \nabla g_{\mu\nu} = 0$$

$$\Rightarrow \quad \Gamma_{\mu\nu}^{\ \ \rho} = \Gamma_{\mu\nu}^{\ \ \rho}(g) = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right)$$

$$\left(\text{via } \nabla_\mu e_\nu^{\ a} = 0 \right) \quad \Rightarrow \quad \omega_\mu^{\ ab} = \omega_\mu^{\ ab}(e)$$

$$\omega_\mu^{\ a}{}_b(e) = e_\rho^{\ a} \Gamma_{\mu\nu}^{\ \ \rho}(g) e_b^{\ \nu} + e_\rho^{\ a} \partial_\mu e_b^{\ \rho}$$

$$S_{Pal} \sim \int \frac{1}{2} \omega A(e) \omega + B(e) \omega \equiv S(e,\omega)$$

$$\left.\frac{\delta S}{\delta \omega}\right|_e = A(e)\,\omega + B(e) = 0 \qquad \omega = -A^{-1} B = \omega(e)$$

$$\frac{\delta S(e,\omega)}{\delta e} = \frac{1}{2} \omega \frac{\delta A(e)}{\delta e} \omega + \frac{\delta B(e)}{\delta e} \omega$$

$$\bar{S}(e) \equiv S(e, \omega(e))$$ formalismo del second'ordine

$$\frac{\delta \bar{S}(e)}{\delta e} = \left.\frac{\delta S}{\delta e}\right|_\omega + \cancel{\left.\frac{\delta S}{\delta \omega}\right|_e} \frac{\delta \omega}{\delta e} = \left.\frac{\delta S}{\delta e}\right|_\omega$$

$$\delta S_{Pal} = \frac{1}{2k^2} \int_M R^{ab}\, \delta e^c\, e^d\, \varepsilon_{abcd} =$$

$$= \cancel{2} \frac{1}{4k^2} \int_M \frac{1}{\cancel{2}} R^{ab}{}_{mn}\, e^m e^n A^c{}_p e^p e^d\, \varepsilon_{abcd} =$$

$$\delta e^c = A^c{}_p e^b \qquad = e\, \varepsilon^{mnpd}$$

$$= \frac{1}{4k^2} \int_M R^{ab}{}_{mn} A^c{}_p\, \underbrace{e^m_\mu e^n_\nu e^p_\rho e^d_\sigma\, \varepsilon^{\mu\nu\rho\sigma}}\, d^4x\, \varepsilon_{abcd} =$$

$$= \frac{1}{4k^2} \int_M R^{ab}{}_{mn} A^c{}_p\, \varepsilon^{mnpd}\, \varepsilon_{abcd}\, e =$$

$$= \frac{1}{4k^2} \int_M R^{ab}{}_{mn} A^c{}_p\, e\, (-1) \begin{vmatrix} \delta^m_a & \delta^n_a & \delta^p_a \\ \delta^m_b & \delta^n_b & \delta^p_b \\ \delta^m_c & \delta^n_c & \delta^p_c \end{vmatrix} =$$

$$= -\frac{1}{4\kappa^2}\int_M e\, A^c_p \left(2R\,\delta^p_c - 2R^p_c - 2R^p_c\right) =$$

$$= \frac{1}{\kappa^2}\int_M e\, A^c_p \left(R^p_c - \frac{1}{2} R\,\delta^p_c\right) = 0$$

$$\Rightarrow \quad R^a_b - \frac{1}{2} R\,\delta^a_b = 0 \qquad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

Termine cosmologico

$$\int e^a \wedge e^b \wedge e^c \wedge e^d\, \varepsilon_{abcd} = \int e^a_\mu e^b_\nu e^c_\rho e^d_\sigma\, \varepsilon_{abcd}\, \varepsilon^{\mu\nu\rho\sigma} d^4x =$$

$$= \int e(-24)\, d^4x = -24\int \sqrt{-g}\, d^4x$$

$$S_H = -\frac{1}{2k^2}\int_M \sqrt{-g}\,(R+2\Lambda) = S_{Pal} =$$

$$= \frac{1}{4k^2}\int_M R^{ab} e^c e^d \,\varepsilon_{abcd} + \frac{\Lambda}{24k^2}\int_M e^a e^b e^c e^d \,\varepsilon_{abcd}$$

$$\frac{\delta S_{Pal}}{\delta e^a} = \text{come prima} + \frac{\Lambda}{6k^2}\int_M A^a_p \, e^p e^b e^c e^d \,\varepsilon_{abcd} =$$

$$= \text{come prima} + \frac{\Lambda}{6k^2}\int_M e \, A^a_p \,\varepsilon^{pbcd}\,\varepsilon_{abcd} =$$

$$= \text{come prima} - \frac{\Lambda}{k^2}\int_M e\, A^a_b \,\delta^b_a = 0 \implies$$

$$R^a_b - \frac{1}{2}\delta^a_b R - \Lambda\,\delta^a_b = 0$$

In $d=2$

$$\int_M R^{ab}\,\varepsilon_{ab} = \int_M \left(d\omega^{ab} + \cancel{\omega^a{}_c\,\omega^{cb}}\right)\varepsilon_{ab} =$$

$$= \int_M d\left(\omega^{ab}\,\varepsilon_{ab}\right)$$

In $d=4$ possiamo considerare

$$\chi(M) = -\frac{1}{32\pi^2}\int_M R^{ab}\wedge R^{cd}\,\varepsilon_{abcd}$$ caratteristica di Eulero

$$= -\frac{1}{32\pi^2}\int_M dC$$ C = forma di Chern-Simons

$$\chi(M) = -\frac{1}{32\pi^2}\int_M (d\omega+\omega\omega)^{ab}(d\omega+\omega\omega)^{cd}\,\varepsilon_{abcd} =$$

$$
= -\frac{1}{32\pi^2}\int_M d\left(\omega^{ab}\, d\omega^{cd}\, \varepsilon_{abcd}\right) +
$$

$$
-\frac{1}{32\pi^2}\int_M 2\, d\omega^{ab}\, \omega^{c}{}_{f}\, \omega^{fd}\, \varepsilon_{abcd} +
$$

$$
-\frac{1}{32\pi^2}\int_M \omega^{a}{}_{f}\, \omega^{fb}\, \omega^{c}{}_{g}\, \omega^{gd}\, \varepsilon_{abcd}
$$

$$
\omega^{a}{}_{f}\, \omega^{fb}\, \omega^{c}{}_{g}\, \omega^{gd}\, \varepsilon_{abcd} = -\frac{1}{4}\, \underbrace{\varepsilon^{afmn}}\, \varepsilon_{mnpq}\, \omega^{pq} \times
$$

$$
\times\, \omega_{f}{}^{b}\, \omega^{c}{}_{g}\, \omega^{gd}\, \underbrace{\varepsilon_{abcd}} = +\frac{1}{4}\, \varepsilon_{mnpq}\, \omega^{pq}\, \omega_{f}{}^{b} \times
$$

$$
\times\, \omega^{c}{}_{g}\, \omega^{gd} \begin{vmatrix} \cancel{\delta^{f}_{b}} & \delta^{m}_{b} & \delta^{n}_{b} \\ \delta^{f}_{c} & \delta^{m}_{c} & \delta^{n}_{c} \\ \delta^{f}_{d} & \delta^{m}_{d} & \delta^{n}_{d} \end{vmatrix} = -\frac{1}{2}\, \varepsilon_{bdpq}\, \omega^{pq}\, \omega_{c}{}^{b}\, \omega^{c}{}_{g}\, \omega^{gd} +
$$

$$
+\frac{1}{2}\, \varepsilon_{bcpq}\, \omega^{pq}\, \omega_{d}{}^{b}\, \omega^{c}{}_{g}\, \omega^{gd} =
$$

$$= - \varepsilon_{bdpq}\, \omega^{pq}\, \omega_c{}^b\, \omega^c{}_f\, \omega^{fd} =$$

$$= \varepsilon_{bdpq}\, \omega^{pq}\, \underbrace{\omega^b{}_c\, \omega^c{}_f\, \omega^{fd}}_{(\omega\omega\omega)^{bd}} =$$

$$= - \varepsilon_{bdpq}\, \omega^{pq}\, \omega^{fd}\, \omega^c{}_f\, \omega^b{}_c =$$

$$= \varepsilon_{bdpq}\, \omega^{pq}\, \underbrace{\omega^{df}\, \omega_f{}^c\, \omega_c{}^b}_{(\omega\omega\omega)^{db}} = 0$$

$d = 2$

$$\omega^{\overset{0}{a}\overset{1}{c}}\, \omega_{\underset{1}{c}}{}^{\overset{0}{b}}\, \varepsilon_{\underset{0}{a}\underset{0}{b}} = -\frac{1}{2} \varepsilon^{ac} \varepsilon_{mn}\, \omega^{mn}\, \omega_c{}^b\, \varepsilon_{ab} =$$

$$\varepsilon^{ac} \varepsilon_{mn} = - \delta^a_m \delta^c_n + \delta^a_n \delta^c_m$$

$$= \frac{1}{2} \varepsilon_{mn}\, \omega^{mn}\, \omega_b{}^b = 0$$

$$2\, d\omega^{ab}\, \omega^{c}{}_{g}\, \omega^{gd}\, \varepsilon_{abcd} = (\text{da dimostrare})$$

$$= \frac{2}{3}\, d\left[\omega^{ab}\, \omega^{c}{}_{g}\, \omega^{gd}\, \varepsilon_{abcd}\right] =$$

$$= \frac{2}{3}\, d\omega^{ab}\, \omega^{c}{}_{g}\, \omega^{gd}\, \varepsilon_{abcd} +$$

$$- \frac{4}{3}\, \omega^{ab}\, d\omega^{c}{}_{g}\, \omega^{gd}\, \varepsilon_{abcd} \qquad \text{Infatti:}$$

$$\omega^{ab}\, d\omega^{cg}\, \omega_{g}{}^{d}\, \varepsilon_{abcd} = -\frac{1}{4}\, \omega^{ab}\, \varepsilon^{cgmn}\, \varepsilon_{mnpq} \cdot$$

$$\times\, d\omega^{pq}\, \omega_{g}{}^{d}\, \varepsilon_{abcd} = \frac{1}{4}\, \omega^{ab}\, d\omega^{pq}\, \omega_{g}{}^{d}\, \varepsilon_{mnpq} \cdot$$

$$\times \begin{vmatrix} \delta^{g}_{a} & \delta^{m}_{a} & \delta^{n}_{a} \\ \delta^{g}_{b} & \delta^{m}_{b} & \delta^{n}_{b} \\ \delta^{g}_{d} & \delta^{m}_{d} & \delta^{n}_{d} \end{vmatrix} = \frac{1}{4}\, \omega^{ab}\, d\omega^{pq}\, \omega_{a}{}^{d}\, \varepsilon_{bdpq}\, 2 +$$

$$-\frac{2}{4}\,\omega^{ab}\,d\omega^{pq}\,\omega_b{}^d\,\varepsilon_{adpq} + \cancel{\frac{2}{4}\,\omega^{ab}\,d\omega^{pq}\,\omega_d{}^d\,\varepsilon_{abpq}} =$$

$$= \omega^{ab}\,d\omega^{pq}\,\omega_a{}^d\,\varepsilon_{dbpq} = -\,d\omega^{ab}\,\omega^c{}_g\,\omega^{gd}\,\varepsilon_{abcd}$$

Alla fine

$$\chi(M) = -\frac{1}{32\pi^2}\int_M dC$$

$$C = \varepsilon_{abcd}\,\omega^{ab}\left(d\omega^{cd} + \frac{2}{3}\,\omega^c{}_g\,\omega^{gd}\right)$$

È definita <u>solo</u> carta per carta (del fibrato tangente)

C = forma di Chern-Simons

È invariante sotto diffeomorfismi (è una forma)

Non è invariante sotto trasf. di Lorentz locali: $\delta\omega = -\nabla\theta$

Caso analogo in QED

$$\int F_{\mu\nu} F_{\rho\sigma} \varepsilon^{\mu\nu\rho\sigma} = 4 \int \partial_\mu A_\nu \partial_\rho A_\sigma \varepsilon^{\mu\nu\rho\sigma} =$$

$$= 4 \int \partial_\mu C^\mu \qquad C^\mu = \varepsilon^{\mu\nu\rho\sigma} A_\nu \partial_\rho A_\sigma =$$

$$= \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} A_\nu F_{\rho\sigma}$$

Sotto trasf. di gauge $\qquad \delta C^\mu = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \partial_\nu \Lambda F_{\rho\sigma} =$

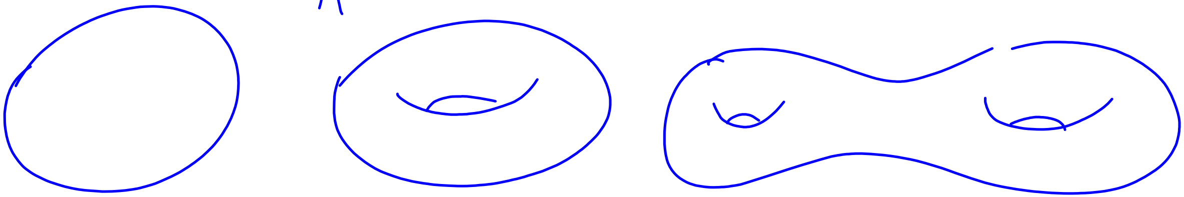
$$= \partial_\nu \left(\frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \Lambda F_{\rho\sigma} \right)$$

Sono derivate totali: non contribuiscono alle eq. del moto

Ma $\Omega \equiv R^{ab} \wedge R^{cd} \varepsilon_{abcd} = dC$ <u>non</u> vuol dire che Ω sia esatta, quindi $\int_M dC \neq 0$ anche se $\partial M = 0$ (Stokes)

Per esempio: in $d=2$

$$\chi(M) \propto \int_M R^{ab}\varepsilon_{ab} = \int_M d(\omega^{ab}\varepsilon_{ab}) = 2-2g$$



$g=0$ $\qquad$ $g=1$ $\qquad$ $g=2$ $\qquad$ ...

$\chi(M)$ è un invariante topologico, cioè non dipende dalla metrica:
$$\delta\chi(M) = -\frac{1}{32\pi^2}\int_M d\,\delta C$$

δC è invariante e Stokes $\Rightarrow \delta\chi = 0 \quad \forall$ variazione

La forma di Chern-Simons ha interesse in dimensione dispari

$d=3$

$$\overset{\circ}{C}{}^{\mu} = \overset{\circ}{\varepsilon}{}^{\mu\nu\rho\sigma} A_\nu \partial_\rho A_\sigma = \frac{1}{2} \overset{\circ}{\varepsilon}{}^{\mu\nu\rho\sigma} A_\nu F_{\rho\sigma}$$

$$\delta \overset{\circ}{C}{}^{\mu} = \frac{1}{2} \overset{\circ}{\varepsilon}{}^{\mu\nu\rho\sigma} \partial_\nu \Lambda F_{\rho\sigma} = \partial_\nu \left(\frac{1}{2} \overset{\circ}{\varepsilon}{}^{\mu\nu\rho\sigma} \Lambda F_{\rho\sigma} \right)$$

$$C_{d=3} = C^{\mu=0}_{d=4} \qquad C = \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

$$\delta C = \partial_\nu \left(\frac{1}{2} \Lambda F_{\rho\sigma} \varepsilon^{\nu\rho\sigma} \right)$$

$\int_M C$ è gauge invariante, perché δC è esatta su M

In $d=4$ abbiamo anche $\int_M R^{ab} \wedge R_{ab}$. Si chiama caratteristica di Pontryagin. Anche

$R^{ab} \wedge R_{ab} = d\Omega$ per certa Ω localmente

Gli altri termini quadratici nelle curvature che possiamo aggiungere all'azione sono

$$\int e\, R^2 d^4x, \quad \int e\, R_{ab} R^{ab}, \quad \int e\, R_{abcd} R^{abcd}$$

$$\int \sqrt{-g}\, R^2, \quad \int \sqrt{-g}\, R_{\mu\nu} R^{\mu\nu}, \quad \int \sqrt{-g}\, R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$$

Ora $R_{ab} = R_{acb}{}^{c}$ tensore di Ricci, $R = R_a{}^a$

Tensore di Weyl = tensore di Riemann a cui ho sottratto tutte le tracce

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{d-2}\left(R_{\mu\sigma} g_{\nu\rho} - R_{\nu\sigma} g_{\mu\rho} - R_{\mu\rho} g_{\nu\sigma} + R_{\nu\rho} g_{\mu\sigma}\right) +$$
$$+ \frac{1}{(d-1)(d-2)} R \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right)$$

Proprietà: $W_{\mu\nu}{}^{\mu}{}_{\rho} = 0 \qquad W_{\mu\nu\rho\sigma} = - W_{\nu\mu\rho\sigma} = W_{\rho\sigma\mu\nu}$

$$W_{\mu\nu\rho\sigma} + W_{\mu\sigma\nu\rho} + W_{\mu\rho\sigma\nu} = 0$$

$$W_{abcd} = -\frac{1}{4} \varepsilon_{abmn} \varepsilon_{cdpq} W^{mnpq}$$

$W^{\mu}{}_{\nu\rho\sigma}$ è invariante ~~sotto~~ trasformazioni di Weyl (trasformazioni conformi locali)

$$g_{\mu\nu}(x) \longrightarrow g_{\mu\nu}(x)\, e^{-2\Omega(x)}$$

L'azione quadratica $\int \underbrace{\sqrt{-g}}_{e^{-4\Omega}} \underbrace{W^{\mu}{}_{\nu\rho\sigma}} \underbrace{W^{\alpha}{}_{\beta\gamma\delta}} \overbrace{g_{\mu\alpha}}^{e^{-2\Omega}} \underbrace{g^{\nu\beta}}_{e^{2\Omega}} \underbrace{g^{\rho\gamma}}_{e^{2\Omega}} \underbrace{g^{\sigma\delta}}_{e^{2\Omega}}$

è Weyl invariante

In $d=4$

$$R_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma} - \frac{1}{2}\left(R_{\mu\sigma} g_{\nu\rho} - R_{\nu\sigma} g_{\mu\rho} - R_{\mu\rho} g_{\nu\sigma} + R_{\nu\rho} g_{\mu\sigma}\right) +$$

$$- \frac{R}{6}\left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}\right)$$

$$\mathrm{Riem}^2 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \mathrm{Weyl}^2 + \frac{1}{4}\Big(\mathrm{Ric}^2\, 4\cdot 4 - 2\,\mathrm{Ric}^2 +$$

$$-2\,\mathrm{Ric}^2 + 2R^2 + 2R^2 - 2\,\mathrm{Ric}^2 - 2\,\mathrm{Ric}^2\Big) +$$

$$+ \frac{1}{36} R^2 \left(4\cdot 4\cdot 2 - 2\cdot 4\right) + \frac{R}{\cancel{6}_{3}}\cancel{2}\left(R - 4R - 4R + R\right) =$$

$$= \mathrm{Weyl}^2 + 2\,\mathrm{Ric}^2 + R^2 + \frac{2}{3}R^2 - 2R =$$

$$= W^2 + 2\,\mathrm{Ric}^2 - \frac{1}{3}R^2$$

$$\Rightarrow \int \sqrt{-g}\, W^2 \doteq \int \sqrt{-g}\left(\mathrm{Riem}^2 - 2\,\mathrm{Ric}^2 + \frac{1}{3}R^2\right)$$

$$\chi(M) = -\frac{1}{32\pi^2}\int_M \frac{R^{ab}_{\ \ \mu\nu}}{2}\,\frac{R^{cd}_{\ \ \rho\sigma}}{2}\,\varepsilon_{abcd}\,\varepsilon^{\mu\nu\rho\sigma}\,d^4x$$

$$R^{ab}_{\ \ mn}R^{cd}_{\ \ pq}\,\varepsilon_{abcd}\,\varepsilon^{mnpq} = -R^{ab}_{\ \ mn}R^{cd}_{\ \ pq}\begin{vmatrix}\delta^a_m & \delta^b_m & \delta^c_m & \delta^d_m\\ \delta^a_n & \delta^b_n & \delta^c_n & \delta^d_n\\ \delta^a_p & \delta^b_p & \delta^c_p & \delta^d_p\\ \delta^a_q & \delta^b_q & \delta^c_q & \delta^d_q\end{vmatrix} =$$

$$= -2R^{ab}_{\ \ an}\left(2\delta^n_b R - 2\cdot 2R^{nd}_{\ \ bd}\right) +$$

$$-2R^{ab}_{\ \ cn}\left(2\delta^n_a R^{cd}_{\ \ bd} - 2R^{cd}_{\ \ ad}\delta^n_b + 2R^{cn}_{\ \ ab}\right) =$$

$$= -4R^2 \overline{+\ 8}\,Ric^2 + 4Ric^2 + 4Ric^2 - 4Riem^2 \Rightarrow$$

$$\chi(M) = \frac{1}{32\pi^2}\int_M \sqrt{-g}\left(Riem^2 - 4Ric^2 + R^2\right)$$

In $d=3$ il tensore di Weyl è identicamente nullo

(che $\Rightarrow$ Riem $\sim$ Ric)

Se $g_{\mu\nu} = \eta_{\mu\nu}\, e^{2\phi(x)}$, $W^{\mu}{}_{\nu\rho\sigma} = 0$

(metriche conformemente piatte)

Se $d>3$ e $W^{\mu}{}_{\nu\rho\sigma} = 0$, allora localmente (carta per carta) la metrica è conformemente piatta

Tensore di Codazzi: qualunque $T_{\mu\nu}$ tale che

$T_{\mu\nu} = T_{\nu\mu}$ e $\nabla_X T(Y,Z) = \nabla_Y T(X,Z)$

$$\nabla_\mu T_{\nu\rho} = \nabla_\nu T_{\mu\rho}$$

Tensore di Schouten $\quad P_{\mu\nu} = \frac{1}{d-2} R_{\mu\nu} - \frac{R}{2(d-1)} g_{\mu\nu}$

$$R_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma} + g_{\mu\rho} P_{\nu\sigma} - g_{\mu\sigma} P_{\nu\rho} - g_{\nu\rho} P_{\mu\sigma} + g_{\nu\sigma} P_{\mu\rho}$$

Tensore di Cotton $\quad C_{\mu\nu\rho} = \nabla_\mu P_{\nu\rho} - \nabla_\nu P_{\mu\rho}$

In $d=3$ una metrica è localmente conformemente piatta, se e solo se il tensore di Schouten è un tensore di Codazzi, cioè si annulla il tensore di Cotton

Azione gravitazionale (Hilbert con il formalismo del 1° ordine - Palatini)

$$S = -\frac{1}{2k^2}\int_M \sqrt{-g}\, R\, d^4x$$

$g_{\mu\nu}$ e $\Gamma^{\rho}_{\mu\nu}$ campi indipendenti, assumendo torsione nulla, ma non compatibilità metrica

$$S(g,\Gamma) = -\frac{1}{2k^2}\int_M \sqrt{-g}\; g^{\mu\nu} R_{\mu\nu}\, d^4x$$

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\rho} + \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\mu}_{\rho\alpha} - \Gamma^{\alpha}_{\rho\nu}\Gamma^{\mu}_{\sigma\alpha}$$

$$R_{\nu\sigma} = \partial_\mu \Gamma^{\mu}_{\nu\sigma} - \partial_\sigma \Gamma^{\mu}_{\nu\mu} + \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\mu}_{\mu\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\mu}_{\sigma\alpha}$$

$$\frac{\delta S}{\delta g^{\mu\nu}} = -\frac{1}{2k^2}\int_M \sqrt{-g}\,\delta g^{\mu\nu}\left(R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R\right)$$

$$\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}}(-1)(-g)\,g_{\mu\nu}\,\delta g^{\mu\nu} = \sqrt{-g}\;\frac{-g_{\mu\nu}}{2}\,\delta g^{\mu\nu}$$

$$\delta g = g\,g^{\alpha\beta}\,\delta g_{\alpha\beta} = -g\,g_{\alpha\beta}\,\delta g^{\alpha\beta}$$

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \implies R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$$

$$S(g,\Gamma) = -\frac{1}{2k^2}\int_M \sqrt{-g}\,g^{\mu\nu}\left(\partial_\lambda\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\mu\lambda} + \Gamma^\alpha_{\mu\nu}\Gamma^\lambda_{\lambda\alpha} - \Gamma^\alpha_{\lambda\mu}\Gamma^\lambda_{\nu\alpha}\right) =$$

$$= -\frac{1}{2k^2}\int_M \partial_\lambda\left[\sqrt{-g}\left(g^{\mu\nu}\Gamma^\lambda_{\mu\nu} - g^{\mu\lambda}\Gamma^\nu_{\mu\nu}\right)\right] +$$

$$-\frac{1}{2k^2}\int_M \Big[-\partial_\lambda\left(\sqrt{-g}\,g^{\mu\nu}\right)\Gamma^\lambda_{\mu\nu} + \partial_\lambda\left(\sqrt{-g}\,g^{\mu\lambda}\right)\Gamma^\nu_{\mu\nu} +$$

$$+\sqrt{-g}\,g^{\mu\nu}\left(\Gamma^\alpha_{\mu\nu}\Gamma^\lambda_{\lambda\alpha} - \Gamma^\alpha_{\lambda\mu}\Gamma^\lambda_{\nu\alpha}\right)\Big]$$

$$\frac{\delta S}{\delta \Gamma^\rho_{\mu\nu}} = 0 \quad \Rightarrow \quad \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) =$$

$$= \Gamma^\rho_{\mu\nu}(g)$$

Esercizio

$$S_H(g) = S\left(g, \Gamma(g)\right)$$

$$\frac{\delta S_H}{\delta g^{\mu\nu}} = \left.\frac{\delta S}{\delta g^{\mu\nu}}\right|_\Gamma + \cancel{\left.\frac{\delta S}{\delta \Gamma^\rho_{\alpha\beta}}\right|_g \frac{\delta \Gamma^\rho_{\alpha\beta}}{\delta g^{\mu\nu}}} = \left.\frac{\delta S}{\delta g^{\mu\nu}}\right|_\Gamma$$

$$S(\chi) = \int \left(\frac{1}{2} \chi M \chi + A \chi + B \right)$$

$$\frac{\delta S}{\delta \chi} = M\chi + A = 0 \qquad \chi = -M^{-1}A$$

$$S(-M^{-1}A) = \int \left(\frac{1}{2} A M^{-1} A - A M^{-1} A + B \right) =$$

$$= \int \left(-\frac{1}{2} A M^{-1} A + B \right)$$

$$S(g, \Gamma(g)) = -\frac{1}{2\kappa^2} \int_M \left[\partial_\lambda w^\lambda - \sqrt{-g}\, g^{\mu\nu} \left(\Gamma^\alpha_{\mu\nu} \Gamma^\lambda_{\lambda\alpha} - \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\nu\alpha} \right) \right]$$

$$w^\lambda = \sqrt{-g} \left(g^{\mu\nu} \Gamma^\lambda_{\mu\nu} - g^{\mu\lambda} \Gamma^\nu_{\mu\nu} \right)$$

In gravità non esistono invarianti locali

$$\varphi(x) \neq \varphi'(x) \qquad \varphi(x) = \varphi'(x')$$

Scalare

$$\delta\varphi = \xi^\rho \partial_\rho \varphi$$

Gli invarianti si ottengono solo integrando, come nel caso della forma di Chern-Simons in $d=3$

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu} = \frac{1}{\cancel{2}}\sqrt{-g}\, g^{\mu\nu}\cancel{2}\nabla_\mu \xi_\nu$$

$$\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

$$\delta\int\sqrt{-g}\,\varphi = \int\sqrt{-g}\left(g^{\mu\nu}\nabla_\mu\xi_\nu\,\varphi + \xi_\mu g^{\mu\nu}\nabla_\nu\varphi\right) =$$

$$\varphi = 1,\ R,\ R_{\mu\nu}R^{\mu\nu}, \dots$$

$$= \int_M \sqrt{-g}\, g^{\mu\nu} \nabla_\nu (\xi_\mu \varphi) = \int_M \sqrt{-g}\, \nabla_\nu (g^{\mu\nu} \xi_\mu \varphi) =$$

$$= \int_M \partial_\nu \left(\sqrt{-g}\, g^{\mu\nu} \xi_\mu \varphi\right) \qquad \sqrt{-g}\, \nabla_\mu J^\mu = \partial_\mu (\sqrt{-g}\, J^\mu)$$

Tensore energia-impulso $\qquad S_m$ = azione della materia (bosonica)

$$S = -\frac{1}{2\kappa^2} \int \sqrt{-g}\,(R + 2\Lambda) + S_m$$

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \qquad T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{\mu\nu}}$$

$$\frac{\delta S}{\delta g^{\mu\nu}} = -\frac{1}{2\kappa^2} \int \sqrt{-g}\, \delta g^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - g_{\mu\nu} \Lambda\right) +$$

$$+ \int \delta g^{\mu\nu} \frac{\delta S_m}{\delta g^{\mu\nu}} =$$

$$= -\frac{1}{2k^2}\int_M \sqrt{-g}\,\delta g^{\mu\nu}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}\Lambda - \kappa^2 T_{\mu\nu}\right] = 0$$

$$\Rightarrow \quad R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}\Lambda = k^2 T_{\mu\nu}$$

Con fermioni: $\quad T^{\mu}_{a} \equiv -\frac{1}{e}\frac{\delta S_m}{\delta e^a_\mu}$

Con soli bosoni $\qquad g_{\rho\sigma} = e^c_\rho \eta_{cd} e^d_\sigma$

$$T^{\mu}_{a} = -\frac{1}{e}\frac{\delta S_m}{\delta e^a_\mu} = -\frac{1}{e}\frac{\delta S_m}{\delta g_{\rho\sigma}}\frac{\delta g_{\rho\sigma}}{\delta e^a_\mu} = \frac{1}{\cancel{2}}T^{\rho\sigma}\delta^\mu_\rho \delta^c_a \eta_{cd} e^d_\sigma \cancel{2} = T^{\mu\sigma}e_{a\sigma}$$

$$T^{\mu\nu} = T^{\mu}_{a} e^{a\nu} \qquad S = S_{Pal} + S_m \qquad \delta e^a_\mu = A^a_b e^b_\mu$$

$$\delta S = \frac{1}{k^2}\int_M e\, A^c_p \left(R^p_c - \frac{1}{2}R\,\delta^p_c - \lambda\,\delta^p_c\right) +$$

$$+ \int_M \frac{\delta S_m}{\delta e^a_\mu} A^a_b \, e^b_\mu =$$

$$= \frac{1}{k^2} \int_M e A^a_b \left[R^b_a - \frac{1}{2} \delta^b_a R - \Lambda \delta^b_a - k^2 \, T^\mu_a e^b_\mu \right] = 0$$

$$\Longrightarrow \quad R_{ab} - \frac{1}{2} \eta_{ab} R - \Lambda \eta_{ab} = k^2 \, T_{ab}$$

$$T_{ab} = -\frac{1}{e} \frac{\delta S_m}{\delta e^a_\mu} e_{\mu b}$$ <u>non</u> è simmetrico

Esiste un tensore energia impulso simmetrico (sulle soluzioni alle equazioni del moto) $\Longleftrightarrow$ la teoria è invariante di Lorentz (globale)

QED $\partial_\mu F^{\mu\nu} = J^\nu \Rightarrow \partial_\nu J^\nu = 0$

$J^\mu = \bar\psi \gamma^\mu \psi \qquad \partial_\mu J^\mu = \bar\psi \not{D} \psi + \bar\psi \overleftarrow{\not{D}} \psi$

$T_{[ab]} = 0$ se le eq. del moto della materia

Facciamo una trasformazione di Lorentz locale

$\delta_\theta e^a_\mu = \Theta^a{}_b \, e^b_\mu \qquad \Theta^{ab} = -\Theta^{ba} \qquad \delta_\theta \chi = \dots$ su S_m

$$\delta_\theta S_m = \int \frac{\delta S_m}{\delta e^a_\mu} \Theta^a{}_b \, e^b_\mu + \sum_\chi \int \frac{\delta S_m}{\delta \chi} \delta_\theta \chi =$$

$$= -\int e \, T_{[ab]} \Theta^{ab} + \sum_\chi \int \frac{\delta S_m}{\delta \chi} \delta \chi = 0$$

sempre (senza usare le eq. del moto, perché è una simmetria)

$$\Longrightarrow \quad T_{[ab]} = \sum_{X} \frac{\delta S_m}{\delta X} A_X$$

La parte antisimm. di T_{ab} è una combinazione lineare delle eq. del moto della materia

Conservazione del tensore energia impulso

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - g_{\mu\nu}\Lambda = k^2 T_{\mu\nu} \quad \Longrightarrow \quad \nabla^{\mu} T_{\mu\nu} = 0$$

per l'identità di Bianchi contratta

$$\nabla^{\mu} R_{\mu\nu} = \frac{1}{2} \nabla_{\nu} R$$

Cos'è $T_{\mu\nu}$ per la gravità? $\frac{\delta S_{tot}}{\delta g_{\mu\nu}} = 0$ sulle sol. alle eq. del moto

Sviluppo attorno al piatto

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa\,\phi_{\mu\nu} \qquad g^{\mu\nu} = \eta^{\mu\nu} - \underbrace{2\kappa\,\phi^{\mu\nu}} + O(\phi^2)$$

Gli indici di ϕ sono alzati e abbassati con η.

$$S_{\Gamma\Gamma} = \frac{1}{2\kappa^2}\int_M \sqrt{-g}\, g^{\mu\nu}\left(\Gamma_{\mu\nu}^{\alpha}\Gamma_{\lambda\alpha}^{\lambda} - \Gamma_{\mu\lambda}^{\alpha}\Gamma_{\nu\alpha}^{\lambda}\right)$$

$$\Gamma_{\mu\nu}^{\rho} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) =$$

$$= \kappa\left(\partial_\mu \phi_\nu^{\rho} + \partial_\nu \phi_\mu^{\rho} - \partial^\rho \phi_{\mu\nu}\right) + O(\phi^2)$$

$$S_{\Gamma\Gamma} = \frac{1}{2\kappa^2}\int\left(\Gamma_{\mu\nu}^{\alpha}\eta^{\mu\nu}\Gamma_{\lambda\alpha}^{\lambda} - \eta^{\mu\nu}\Gamma_{\mu\lambda}^{\alpha}\Gamma_{\nu\alpha}^{\lambda}\right) + O(\kappa\phi^3)$$

$$\Gamma^{\lambda}_{\mu\lambda} = \kappa\left(\partial_\mu\phi + \cancel{\partial^\nu\phi_{\mu\nu}} - \cancel{\partial^\rho\phi_{\mu\rho}}\right) + O(\phi^2) = \kappa\,\partial_\mu\phi + O(\phi^2)$$

$$\phi = \phi_{\mu\nu}\eta^{\mu\nu}$$

$$\eta^{\mu\nu}\Gamma^{\rho}_{\mu\nu} = \kappa\left(2\partial^\mu\phi^\rho_\mu - \partial^\rho\phi\right) + O(\phi^2)$$

$$S_{\Gamma\Gamma} = \frac{1}{2\cancel{\kappa^2}}\int \cancel{\kappa}\left(2\partial^\mu\phi^\rho_\mu - \partial^\rho\phi\right)\cancel{\kappa}\,\partial_\rho\phi +$$

$$- \frac{1}{2\cancel{\kappa^2}}\int \cancel{\kappa^2}\left(\partial_\mu\phi^\alpha_\lambda + \partial_\lambda\phi^\alpha_\mu - \partial^\alpha\phi_{\mu\lambda}\right)\left(\partial^\mu\phi^\lambda_\alpha + \partial_\alpha\phi^{\mu\lambda} - \partial^\lambda\phi^\mu_\alpha\right) +$$

$$+ O(\phi^3) = \frac{1}{2}\int 2\partial^\mu\phi^\nu_\mu\,\partial_\nu\phi - \partial^\nu\phi\,\partial_\nu\phi - \cancel{\partial_\mu\phi^\alpha_\lambda\,\partial^\mu\phi^\lambda_\alpha} +$$

$$- \cancel{\partial_\mu\phi^\alpha_\lambda\,\partial_\alpha\phi^{\mu\lambda}} + \cancel{\partial_\mu\phi^\alpha_\lambda\,\partial^\lambda\phi^\mu_\alpha} - \cancel{\partial_\lambda\phi^\alpha_\mu\,\partial^\mu\phi^\lambda_\alpha} +$$

$$- \partial_\lambda\phi^\alpha_\mu\,\partial_\alpha\phi^{\mu\lambda} + \cancel{\partial_\lambda\phi^\alpha_\mu\,\partial^\lambda\phi^\mu_\alpha} + \cancel{\partial^\alpha\phi_{\mu\lambda}\,\partial^\mu\phi^\lambda_\alpha} +$$

$$+\partial^\alpha\phi_{\mu\lambda}\,\partial_\alpha\phi^{\mu\lambda} - \underline{\partial^\alpha\phi_{\mu\lambda}\,\partial^\lambda\phi^\mu_{\ \alpha}}\Big) + O(\phi^3) =$$

$$= \frac{1}{2}\int\Big(\partial_\rho\phi_{\mu\nu}\,\partial^\rho\phi^{\mu\nu} - 2\,\partial^\mu\phi_{\mu\nu}\,\partial^\rho\phi^\nu_{\ \rho}$$

$$+ 2\,\partial^\rho\phi_{\mu\rho}\,\partial^\mu\phi - \partial_\mu\phi\,\partial^\mu\phi\Big) + O(\phi^3) =$$

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g_{\mu\nu}} \qquad (\text{caso bosonico}) \qquad g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa\phi_{\mu\nu}$$

$$T^{\mu\nu} = -\frac{\cancel{2}}{\sqrt{-g}}\,\frac{1}{\cancel{2}\kappa}\,\frac{\delta S_m}{\delta\phi_{\mu\nu}} \qquad \frac{\delta S_m}{\delta\phi_{\mu\nu}} = -\kappa\sqrt{-g}\,T^{\mu\nu}$$

$$S_m = S_m(\phi) = S_m(0) + \frac{\delta S_m}{\delta\phi_{\mu\nu}}\Big|_{\phi=0}\phi_{\mu\nu} + O(\phi^2) =$$

$$= S_m(0) - \int\kappa\,\underbrace{T^{\mu\nu}\phi_{\mu\nu}} + O(\kappa^2\phi^2) \qquad \text{"}J^\mu A_\mu\text{"}$$

$$S \equiv S_H + S_m = \frac{1}{2}\int \Big(\partial_\rho \phi_{\mu\nu}\, \partial^\rho \phi^{\mu\nu} - 2\, \partial^\mu \phi_{\mu\nu}\, \partial^\rho \phi^\nu_\rho$$

$$+ 2\, \partial^\rho \phi_{\mu\rho}\, \partial^\mu \phi - \partial_\mu \phi\, \partial^\mu \phi \Big) + S_m(0) - \kappa \int \phi_{\mu\nu} T^{\mu\nu}$$

$$+ O(\kappa\phi^3, \kappa^2 \phi^2 \chi^2) \qquad T^{\mu\nu} \sim \chi^2 \qquad \chi = \text{campo di materia}$$

$$\frac{\delta S}{\delta \phi^{\mu\nu}} = -\Box \phi_{\mu\nu} + \partial_\mu \partial_\rho \phi^\rho_\nu + \partial_\nu \partial_\rho \phi^\rho_\mu +$$

$$- \partial_\mu \partial_\nu \phi - \eta_{\mu\nu}\, \partial^\rho \partial^\sigma \phi_{\rho\sigma} + \eta_{\mu\nu} \Box \phi - \kappa T_{\mu\nu} +$$

$$+ O(\kappa \phi^2, \kappa^2 \phi \chi^2) = 0$$

All'ordine più basso ($\kappa = 0$, teoria libera)

$$\big[\, \Box A_\mu - \partial_\mu (\partial \cdot A) \sim J_\mu, \quad \text{gauge-fixing: } \partial \cdot A = 0 \,\big]$$

$$-\Box \phi_{\mu\nu} + \partial_\mu \partial_\rho \phi^\rho_\nu + \partial_\nu \partial_\rho \phi^\rho_\mu - \partial_\mu \partial_\nu \phi$$

$$- \eta_{\mu\nu} \partial^\rho \partial^\sigma \phi_{\rho\sigma} + \eta_{\mu\nu} \Box \phi = 0$$

gauge fixing $\quad \partial^\mu \phi_{\mu\nu} - \frac{1}{2} \partial_\nu \phi = 0$

$$\delta \phi_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + O(\kappa \phi) \qquad \delta \phi = 2 \partial \cdot \xi + O(\kappa \phi)$$

$$\delta \left(\partial^\mu \phi_{\mu\nu} - \frac{1}{2} \partial_\nu \phi \right) = \Box \xi_\nu + \cancel{\partial_\nu \partial \cdot \xi} - \cancel{\partial_\nu \partial \cdot \xi} = \Box \xi_\nu$$

Gauge residua :

$$\delta \phi_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \quad \text{con} \quad \Box \xi_\nu = 0$$

$$-\Box\phi_{\mu\nu} + \cancel{\frac{1}{2}\partial_\mu\partial_\nu\phi} + \cancel{\frac{1}{2}\partial_\mu\partial_\nu\phi} - \cancel{\partial_\mu\partial_\nu\phi} +$$

$$-\eta_{\mu\nu}\frac{1}{2}\Box\phi + \eta_{\mu\nu}\Box\phi = 0$$

$$-\Box\phi_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\Box\phi = 0$$ Traccia: $-\Box\phi + 2\Box\phi = 0$

$$\Rightarrow \Box\phi = 0 \Rightarrow \Box\phi_{\mu\nu} = 0$$

Passiamo allo spazio degli impulsi : $\phi_{\mu\nu}(k)$

$$\Box\phi_{\mu\nu} = 0 \Rightarrow k^2 = 0$$

Scegliamo $k^\mu = (k, 0, 0, k)$

$$\partial^\mu \phi_{\mu\nu} - \frac{1}{2}\partial_\nu \phi = 0 \Rightarrow \qquad k^\mu \phi_{\mu\nu} = \frac{1}{2} k_\nu \phi$$

$$\nu = 0 : \quad \cancel{k}(\phi_{00} + \phi_{30}) = \frac{1}{2}\cancel{k}\phi \qquad \phi = \phi_{00} - \phi_{11} - \phi_{22} - \phi_{33}$$

$$\nu = 1 : \quad \cancel{k}(\phi_{01} + \phi_{31}) = 0$$

$$\nu = 2 : \quad \cancel{k}(\phi_{02} + \phi_{32}) = 0$$

$$\nu = 3 : \quad \cancel{k}(\phi_{03} + \phi_{33}) = -\frac{\cancel{k}}{2}\phi$$

Gauge residua : $\delta\phi_{\mu\nu}(k) = -ik_\mu \xi_\nu(k) - ik_\nu \xi_\mu(k)$

$\partial_\mu \longrightarrow -ik_\mu$ $\qquad k^2 = 0$ vuol dire $\Box \xi_\mu(x) = 0$

$$k_\mu = (k, 0, 0, -k)$$

$$\delta\phi_{00} = -2ik\xi_0 = \phi'_{00} - \phi_{00} \qquad \phi'_{00} = \phi_{00} - 2ik\xi_0$$

$\Rightarrow$ Posso usare ξ_0 per annullare ϕ_{00}'.

Buttando i primi, posso assumere $\phi_{00} = 0$

$\delta\phi_{01} = -ik\xi_1$: uso ξ_1 per avere $\phi_{01} = 0$

$\delta\phi_{02} = -ik\xi_2$: uso ξ_2 per avere $\phi_{02} = 0$

$\delta\phi_{03} = -ik(\xi_3 - \xi_0)$: uso ξ_3 per imporre $\phi_{03} = 0$

$$\begin{cases} \phi_{00} + \phi_{30} = \frac{1}{2}\phi \\ \phi_{01} + \phi_{31} = 0 \\ \phi_{02} + \phi_{32} = 0 \\ \phi_{03} + \phi_{33} = -\frac{1}{2}\phi \end{cases} \qquad \phi_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \alpha\, \mathcal{E}_+ + \beta\, \mathcal{E}_- = \begin{pmatrix} \alpha+\beta & i(\alpha-\beta) \\ i(\alpha-\beta) & -\alpha-\beta \end{pmatrix}$$

$$\mathcal{E}_+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \qquad \mathcal{E}_- = \mathcal{E}_+^*$$

$$R_\theta = \begin{pmatrix} \cos\theta & \operatorname{sen}\theta \\ -\operatorname{sen}\theta & \cos\theta \end{pmatrix}$$

$$R_\theta\, \mathcal{E}_\pm\, R_\theta^{-1} = \begin{pmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{pmatrix} \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix} \begin{pmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{pmatrix} =$$

$$= \begin{pmatrix} e^{\pm i\theta} & \pm i e^{\pm i\theta} \\ \pm i e^{\pm i\theta} & -e^{\pm i\theta} \end{pmatrix} \begin{pmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{pmatrix} =$$

$$= e^{\pm i\theta} \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix} \begin{pmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{pmatrix} = e^{\pm 2i\theta}\, \mathcal{E}_\pm$$

Vettore: $\varepsilon_\pm = \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$ $\quad R_\theta \varepsilon_\pm = e^{\pm i\theta} \varepsilon_\pm$

In dimensione d: $\dfrac{d(d-3)}{2}$ elicità

(per i vettori $d-2$)

In teoria dei campi quantistici

Gauge-fixing: $\quad g_\nu = \partial^\mu \phi_{\mu\nu} - \frac{1}{2} \partial_\nu \phi$

$$S_H \longrightarrow S_H + \frac{1}{2\lambda} \int g_\mu \eta^{\mu\nu} g_\nu + S_{ghost}$$

λ = parametro arbitrario (parametro di gauge-fixing)

QED

$$\Box A_\mu - \partial_\mu (\partial \cdot A) \sim J_\mu \qquad A_\mu = A_\mu(J)$$

$$\delta A_\mu = \partial_\mu \Lambda \qquad \left(-k^2 \eta_{\mu\nu} + k_\mu k_\nu\right) A^\nu(k) \sim J_\mu$$

$k^2 \eta_{\mu\nu} - k_\mu k_\nu$ non è invertibile, perché

$$\left(k^2 \eta_{\mu\nu} - k_\mu k_\nu\right) k^\nu = 0$$

. k^ν è un autovettore nullo e

corrisponde all'invarianza di gauge $\delta A_\mu(k) = -i k_\mu \Lambda(k)$

$$\lambda = \frac{1}{2} \qquad S_H + \int y_\mu \eta^{\mu\nu} y_\nu = \frac{1}{2} \int \Big(\partial_\rho \phi_{\mu\nu} \partial^\rho \phi^{\mu\nu}$$

$$- \cancel{2 \partial^\mu \phi_{\mu\nu} \partial^\rho \phi_\rho^{\ \nu}} + \cancel{2 \partial^\rho \phi_{\mu\rho} \partial^\mu \phi} - \partial_\mu \phi \partial^\mu \phi +$$

$$+ \cancel{2 \partial^\mu \phi_{\mu\nu} \partial^\rho \phi_\rho^{\ \nu}} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \cancel{2 \partial^\rho \phi_\rho^{\ \mu} \partial_\mu \phi} \Big) =$$

$$= \frac{1}{2} \int \left(\partial_\rho \phi_{\mu\nu} \partial^\rho \phi^{\mu\nu} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \right) =$$

$$= \frac{1}{2} \int \phi_{\mu\nu} (-\Box) \frac{\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}}{2} \phi_{\rho\sigma}$$

Spazio degli impulsi : $k^2 \left(\mathbb{1}_{\mu\nu\rho\sigma} - \frac{1}{2} \eta_{\mu\nu} \eta_{\rho\sigma} \right) \equiv Q_{\mu\nu\rho\sigma}$

$\mathbb{1}_{\mu\nu\rho\sigma} = \frac{1}{2} \left(\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} \right)$ è l'identità sui tensori simmetrici a due indici

Cerco $P_{\mu\nu\rho\sigma}$ tale che

$$Q_{\mu\nu\rho\sigma} P^{\rho\sigma}{}_{\alpha\beta} = \mathbb{1}_{\mu\nu\alpha\beta}$$

$$\mathbb{1}\,\eta\eta = \mathbb{1}_{\mu\nu\rho\sigma}\,\eta^{\rho\sigma}\eta_{\mu\nu} = \eta\eta$$

$$P_{\mu\nu\rho\sigma} = \frac{1}{k^2}\left(\mathbb{1}_{\mu\nu\rho\sigma} - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}\right)$$

Infatti: $\left(\mathbb{1} - \frac{1}{2}\eta\eta\right)\left(\mathbb{1} - \frac{1}{2}\eta\eta\right) = \mathbb{1} +$

$$- \cancel{\frac{1}{2}\eta\eta} - \cancel{\frac{1}{2}\eta\eta} + \cancel{\frac{1}{4}}\,\cancel{4}\,\eta\eta = \mathbb{1}$$

$iP_{\mu\nu\rho\sigma}$ si chiama propagatore $\qquad -\frac{1}{2k^2}\int R\sqrt{-g}$

$$\langle \phi_{\mu\nu}(k)\,\phi_{\rho\sigma}(-k)\rangle_0 = iP_{\mu\nu\rho\sigma} \qquad \phi_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\phi^{\mu\nu} P_{\mu\nu\rho\sigma}\,\phi^{\rho\sigma} = \frac{\phi_{\mu\nu}\phi^{\mu\nu}}{k^2} = \frac{2(a^2+b^2)}{k^2} \geq 0$$

$SS^\dagger = 1$ richiede che il residuo a $k^2=0$ sia
positivo

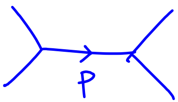
$$\phi \, \Box \left(m^2 + \Box \right) \phi \qquad\qquad R + R_{\mu\nu} R^{\mu\nu} + R^2 + \dots$$

$$\frac{1}{k^2 (k^2 - m^2)} = \left(\frac{1}{k^2} - \frac{1}{k^2 - m^2} \right) \frac{1}{m^2}$$

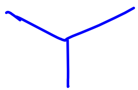
$\uparrow$ ghost

$$\langle \phi(p) \phi(-p) \rangle = \frac{i}{p^2 - m^2 + i\epsilon} \qquad\qquad \frac{1}{x - i\epsilon} = \mathcal{P}\left(\frac{1}{x}\right) + i\pi\,\delta(x)$$

$SS^\dagger = 1 \qquad S = 1 + iT \qquad iT$ = diagrammi di Feynman

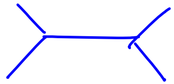


 $= \dfrac{i}{p^2 - m^2 + i\epsilon}$

 $= -i$

$$S = 1 + iT \qquad 1 = SS^\dagger = (1+iT)(1-iT^\dagger) = 1 + iT - iT^\dagger + TT^\dagger$$

$$iT - iT^\dagger = \underline{-TT^\dagger \leq 0}$$

$iT =$  $= \dfrac{-i}{p^2-m^2+i\epsilon} =$

$$= -i\,P\frac{1}{p^2-m^2} - \pi\,\delta(p^2-m^2)$$

$$iT - iT^\dagger = -2\pi\,\delta(p^2-m^2) \leq 0$$

Un propagatore $\dfrac{-i}{p^2-m^2+i\epsilon}$ è incompatibile coll'unitarietà.
Si chiama "ghost"

Gauge-fixing in teoria dei campi quantistici

$$S_H = -\frac{1}{2k^2}\int\sqrt{-g}\,(2\Lambda + R) \qquad \delta g_{\mu\nu} = -g_{\mu\rho}\partial_\nu C^\rho - g_{\nu\rho}\partial_\mu C^\rho - C^\rho\partial_\rho g_{\mu\nu}$$

diffeomorfismi

$$S(\Phi, K) = S_H + \int \left(g_{\mu\rho}\partial_\nu C^\rho + g_{\nu\rho}\partial_\mu C^\rho + C^\rho \partial_\rho g_{\mu\nu} \right) K^{\mu\nu}$$

C^μ sono considerati dei campi e variabili di grassmann

θ = variabile di grassmann soddisfa $\theta^2 = 0$

$\{\theta^i, \theta^j\} = 0$ (come una forma) Anche $K^{\mu\nu}$ sono di grassmann

$\Phi = \{ g_{\mu\nu}, C^\rho, \dots \}$ $K = \{ K^{\mu\nu}, \dots \}$

$$-\frac{\delta_r S}{\delta K^{\mu\nu}} = -g_{\mu\rho}\partial_\nu C^\rho - g_{\nu\rho}\partial_\mu C^\rho - C^\rho \partial_\rho g_{\mu\nu} \equiv \delta_{diff}\, g_{\mu\nu}$$

$$\frac{\delta_r f(\theta)}{\delta\theta}\delta\theta = \delta f = \delta\theta \frac{\delta_\ell f(\theta)}{\delta\theta}$$

$$\int \frac{\delta_\ell S_H}{\delta g_{\mu\nu}} \frac{\delta_r S}{\delta K^{\mu\nu}} = 0 = \int \frac{\delta S_H}{\delta g_{\mu\nu}(x)} \delta_{diff}\, g_{\mu\nu}(x)$$

Formalismo canonico di gauge

$\Phi^\alpha = \{ g_{\mu\nu}, C^\rho, \bar{C}^\sigma, B^\mu \}$ $\qquad K_\alpha = (K^{\mu\nu}, K^C_\rho, K^{\bar{C}}_\sigma, K^B_\mu)$

Siano $X(\Phi, K)$ e $Y(\Phi, K)$ funzionali locali di Φ, K.

Antiparentesi $\quad (X,Y) = \int dx \left(\frac{\delta_R X}{\delta \Phi^\alpha(x)} \frac{\delta_L Y}{\delta K_\alpha(x)} - \frac{\delta_R X}{\delta K_\alpha(x)} \frac{\delta_L Y}{\delta \Phi^\alpha(x)} \right)$

Sia $\epsilon_X = 0$ se X è bosonico, $\epsilon_X = 1$ se è fermionico
e $\epsilon_K = \epsilon_\Phi + 1$

(a) $(Y,X) = -(-1)^{(\epsilon_X+1)(\epsilon_Y+1)} (X,Y)$ $\qquad$ (c) $\epsilon_{(X,Y)} = \epsilon_X + \epsilon_Y + 1 \mod 2$

(b) $(-1)^{(\epsilon_X+1)(\epsilon_Z+1)} (X,(Y,Z)) +$ permutazioni cicliche $= 0$

Esempio : $X = S$, azione, $\epsilon_S = 0$ $\qquad (S,S) = -(-1)(S,S)$

Se $\epsilon_F = 1$ $\quad (F,F) = 0$

(b) con $X = Y = S$ e $\epsilon_Z = 0$

$$(S,(S,Z)) + (Z,(S,S)) + (S,(Z,S)) = 0$$

$$(S,(S,Z)) = -\frac{1}{2}(Z,(S,S))$$

Se $(S,S)=0$, allora l'operatore ad_S $\left(\mathrm{ad}_S X = (S,X)\right)$ è nilpotente: $\mathrm{ad}_S\, \mathrm{ad}_S X = (S,(S,X)) = 0 \quad \forall X$

Le antiparentesi con un tale S definiscono una coomologia

X è chiuso se $(S,X)=0$, X è esatto se $\exists Y \mid (S,Y)=X$
e definisco $X \sim Y$ se $X-Y$ è esatta

Cercare un'estensione S di S_H che soddisfi $(S,S)=0$
(master equation)

La soluzione è

$$S(\Phi,K) = S_H + S_K \qquad S_H = -\frac{1}{2\kappa^2}\int\sqrt{-g}\,(2\Lambda + R) \text{ o qualunque}$$

azione invariante sotto diffeomorfismi

$$S_K = \int\left(g_{\mu\rho}\partial_\nu C^\rho + g_{\nu\rho}\partial_\mu C^\rho + C^\rho\partial_\rho g_{\mu\nu}\right)K^{\mu\nu} + \int C^\rho\partial_\rho C^\sigma K^C_\sigma$$

$$- \int B^\rho K^{\bar{C}}_\rho$$

C^μ si chiamano ghost di Faddeev-Popov

$\bar{C}^\mu$ si chiamano antighost

$(S,S)=0$ contiene sia l'invarianza di S_H sotto diffeomorfismi, che la chiusura dell'algebra (il commutatore di due diffeomorfismi è un diffeomorfismo)

$$(S, S_H) = \int \left(\frac{\delta_r S}{\delta \phi^\alpha} \cancel{\frac{\delta_\ell S_H}{\delta K_\alpha}} - \frac{\delta_r S}{\delta K_\alpha} \frac{\delta S_H}{\delta \phi^\alpha} \right) = - \int \frac{\delta_r S}{\delta K^{\mu\nu}} \frac{\delta S_H}{\delta g_{\mu\nu}} =$$

$$= \int \delta_{diff} g_{\mu\nu} \frac{\delta S_H}{\delta g_{\mu\nu}} = 0$$

S_H è chiusa. Tutti i funzionali gauge invarianti lo sono

Gauge-fixing: $S_H \to S_H + (S, \Psi)$

$$\Psi = \int \bar{C}^\mu \left(G_\mu - \frac{1}{\lambda} B_\mu \right)$$ dove G_μ è la condizione di

gauge fixing $G_\mu = \partial^\nu g_{\mu\nu} - \frac{1}{2} \partial_\mu g_{\rho\sigma} \eta^{\rho\sigma}$

Ψ si chiama fermione di gauge

$$(S, \Psi) = - \int \frac{\delta_r S}{\delta K_\alpha} \frac{\delta \Psi}{\delta \Phi^\alpha} = - \int \frac{\delta_r S}{\delta K_\alpha^{\bar{C}}} \frac{\delta \Psi}{\delta \bar{C}^\alpha} - \int \cancel{\frac{\delta_r S}{\delta K_\alpha^B}} \frac{\delta \Psi}{\delta B^\alpha} +$$

$$- \frac{\delta_R S}{\delta K^{\mu\nu}} \frac{\delta \Psi}{\delta g_{\mu\nu}} \Big) = \int B^\mu \left(\mathcal{G}_\mu - \frac{1}{\lambda} B_\mu \right) +$$

$$+ \int \bar{c}^\alpha \frac{\delta \mathcal{G}_\alpha}{\delta g_{\mu\nu}} \left(g_{\mu\rho} \partial_\nu c^\rho + g_{\nu\rho} \partial_\mu c^\rho + c^\rho \partial_\rho g_{\mu\nu} \right)$$

B^μ appare algebricamente : $0 = \mathcal{G}_\mu - \frac{2}{\lambda} B_\mu \Rightarrow B_\mu = \frac{\lambda}{2} \mathcal{G}_\mu$

$$(S, \Psi) = \frac{\lambda}{4} \int \mathcal{G}^\mu \mathcal{G}_\mu - \int \bar{c}^\mu s_{diff}\, \mathcal{G}_\mu =$$

$$= \frac{\lambda}{4} \int \mathcal{G}^\mu \mathcal{G}_\mu + \int \bar{c}^\mu \partial^\nu \left(g_{\mu\rho} \partial_\nu c^\rho + g_{\nu\rho} \partial_\mu c^\rho + c^\rho \partial_\rho g_{\mu\nu} \right) +$$

$$- \frac{1}{2} \int \bar{c}^\alpha \partial_\alpha \left(g_{\mu\rho} \partial_\nu c^\rho + g_{\nu\rho} \partial_\mu c^\rho + c^\rho \partial_\rho g_{\mu\nu} \right) \eta^{\mu\nu}$$

$$= \frac{\lambda}{4} \int \mathcal{G}^\mu \mathcal{G}_\mu + \mathcal{L}_{ghost}$$

I risultati fisici non dipendono da Ψ

$\exists\ \Psi$ tale che si propagano solo le elicità fisiche: è la gauge di Coulomb

<u>QED</u>

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ + m)\psi$$

$$\delta A_\mu = \partial_\mu \Lambda \qquad \delta\psi = -ie\Lambda\psi \qquad \delta\bar{\psi} = ie\Lambda\bar{\psi}$$

$S(\Phi, K)$ tale che $(S,S) = 0$

$$\Phi^\alpha = (A_\mu, \psi, \bar{\psi}, c, \bar{c}, B)$$

$$K_\alpha = (K^\mu, K_\psi, \bar{K}_\psi, K_c \dots)$$

$$S = \int \mathcal{L} - \int \partial_\mu c K^\mu + ie\int c\psi K_\psi + ie\int \bar{\psi} c K_{\bar{\psi}} - \int B K_{\bar{c}}$$

$$\delta A_\mu \Big|_{\Lambda \to c} = (S, A_\mu) = -\frac{\delta_r S}{\delta K_\mu}$$

gauge-fixing

$$(S,B)=0 \qquad (S,\bar{c})=B$$

$$(S,A_\mu)=\partial_\mu c$$

$$S_{gf} = S + (S,\Psi)$$

$$(S,FG)=(S,F)G+(-1)^{\varepsilon_F}F(S,G)$$

$$\Psi = \int \bar{c}\left(\partial_\mu A^\mu + \frac{\lambda}{2}B\right)$$

$$(S,\psi) = \int B\Big(\overset{\vec{\nabla}\cdot\vec{A}}{\partial\cdot A} + \frac{\lambda}{2}B\Big) - \underbrace{\int \bar{c}\,\overset{\nabla^2}{\Box}\,c}_{S_{ghost}}$$

$$-\frac{1}{4}F_{\mu\nu}^2 + B\Big(\overset{\vec{\nabla}}{\partial}\cdot A + \frac{\lambda}{2}B\Big)$$

$$= (A_\mu, B)\begin{pmatrix} \cdots & & \cdots \\ & M & \\ \cdots & & \cdots \end{pmatrix}\begin{pmatrix} A_\nu \\ B \end{pmatrix}$$

$$\exists\, M^{-1}$$

$$\frac{\delta}{\delta B} = 0 \quad \text{dà} \quad \overset{\vec\nabla\cdot\vec A}{\partial\cdot A} + \lambda B = 0 \qquad B = -\frac{1}{\lambda}\,\overset{\vec\nabla\cdot\vec A}{\partial\cdot A}$$

$$-\frac{1}{4}\int F_{\mu\nu}^2 + \int B\Big(\overset{\vec\nabla\cdot\vec A}{\partial\cdot A} + \frac{\lambda}{2}B\Big) \longrightarrow -\frac{1}{4}\int F_{\mu\nu}^2 - \int\frac{1}{2\lambda}(\overset{\vec\nabla\cdot\vec A}{\partial\cdot A})^2 =$$

$$= -\frac{1}{2}\int(\partial_\mu A_\nu - \partial_\nu A_\mu)\partial_\mu A_\nu - \int\frac{1}{2\lambda}(\partial\cdot A)^2 =$$

$$= -\frac{1}{2}\int\Big[-A_\mu\Box A^\mu - (\partial\cdot A)^2 + \frac{1}{\lambda}(\partial\cdot A)^2\Big]$$

$\lambda = 1$: (gauge di Feynman)

$$\frac{1}{2}\int A_\mu\Box A^\mu - \int\bar c\,\Box\, c \qquad \leftarrow \text{azione quadratica}$$

μ p ν $= -\dfrac{i\eta^{\mu\nu}}{p^2+i\epsilon}$ 4

p $= \dfrac{i}{p^2}$ -2

tot $= 2$

Gauge di Coulomb $\qquad (S, \vec{A}) = \vec{\nabla} C$

$$\Psi = \int \bar{C} \left(\vec{\nabla} \cdot \vec{A} + \frac{\lambda}{2} B \right) \qquad (\lambda = 1)$$

$$(S, \Psi) = -\frac{1}{2} \int \left[- A_\mu \Box A^\mu - (\partial \cdot A)^2 + (\vec{\nabla} \cdot \vec{A})^2 \right] +$$

$$- \int \bar{C} \Delta C$$

$$\partial \cdot A = \partial_0 A_0 - \vec{\nabla} \cdot \vec{A}$$

$$= \frac{i}{\vec{p}^{\,2}}$$

Resta:

$$-\frac{1}{2} \left[- A_\mu \Box A_\mu - (\partial_0 A_0)^2 + 2 \partial_0 A_0 \vec{\nabla} \cdot \vec{A} \right] =$$

$$= -\frac{1}{2} \Big[- A_0 \cancel{\partial_0^2} A_0 + A_0 \Delta A_0 + A_i \Box A_i - \cancel{(\partial_0 A_0)^2} +$$

$$+ 2\partial_0 A_0 \vec{\nabla}\cdot\vec{A} \,]$$

$$\equiv M$$

$$= \frac{1}{2} \begin{pmatrix} A_0 , & A_i \end{pmatrix} \begin{pmatrix} \vec{k}^2 & k^0 k_j \\ k^0 k_i & \delta_{ij} k^2 \end{pmatrix} \begin{pmatrix} A_0 \\ A_j \end{pmatrix}$$

$$M^{-1} = \begin{pmatrix} a & d\, k_j \\ d\, k_i & b\,\delta_{ij} + c\, k_i k_j \end{pmatrix}$$

$$M M^{-1} = 1 = \begin{pmatrix} \vec{k}^2 a + d k^0 \vec{k}^2 & \vec{k}^2 d k_j + k^0 b k_j + c k^0 \vec{k}^2 k_j \\ a k^0 k_i \cancel{+ k^2 d k_i} & d k^0 \cancel{k_i} k_j + k^2 b \delta_{ij} + c k^2 \cancel{k_i} k_j \end{pmatrix}$$

$$b = \frac{1}{k^2} \quad \checkmark$$

$$d k^0 + c k^2 = 0$$

$$a k^0 + d k^2 = 0 \quad \checkmark$$

$$d \vec{k}^2 + b k^0 + c k^0 \vec{k}^2 = 0$$

$$a + dk^0 = \frac{1}{\vec{k}^2} \qquad a k^0 + d k^{0\,2} - d\vec{k}^2 = 0$$

$$k^0 (a + dk^0) = d\vec{k}^2$$

$$k^0 = d(\vec{k}^2)^2 \qquad d = \frac{k^0}{(\vec{k}^2)^2}$$

$$a = \frac{1}{\vec{k}^2} - \frac{(k^0)^2}{(\vec{k}^2)^2} = \frac{-k^2}{(\vec{k}^2)^2}$$

$$dk^0 + ck^2 = 0 \qquad c = -\frac{dk^0}{k^2} = -\frac{(k^0)^2}{k^2(\vec{k}^2)^2}$$

$$d\vec{k}^2 + bk^0 + ck^0\vec{k}^2 \overset{?}{=} 0$$

$$\frac{k^0}{\vec{k}^2} + \frac{k^0}{k^2} - \frac{(k^0)^3}{k^2(\vec{k}^2)} = \frac{k^0}{k^2(\vec{k}^2)}\left[k^2 + \vec{k}^2 - k^{0\,2}\right] = 0$$

$$M^{-1} = \begin{pmatrix} -\dfrac{k^2}{(\vec{k}^2)^2} & \dfrac{k_j k^0}{(\vec{k}^2)^2} \\ \dfrac{k_i k^0}{(\vec{k}^2)^2} & \dfrac{1}{k^2}\left(\delta_{ij} - \dfrac{(k^0)^2}{(\vec{k}^2)^2} k_i k_j\right) \end{pmatrix}$$

al polo : $k^2 = 0$

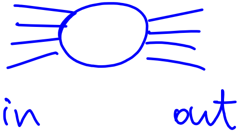
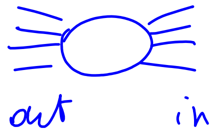
$$\begin{pmatrix} 0 & 0 \\ 0 & \dfrac{1}{k^2}\left(\delta_{ij} - \dfrac{k_i k_j}{\vec{k}^2}\right) \end{pmatrix}$$

$$k_i = (0, 0, 1)$$

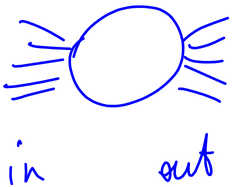
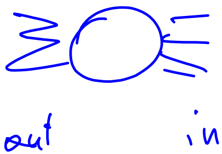
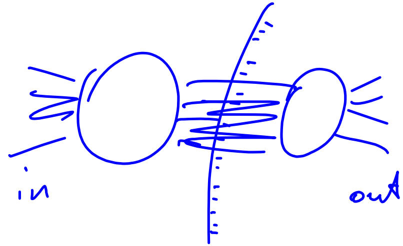
$$\delta_{ij} - \frac{k_i k_j}{\vec{k}^2} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} - \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad S S^{+} = 1$$

$S = 1 + iT \qquad S^+ = 1 - iT^+ \qquad SS^+ = 1$ dà $iT - iT^+ = -TT^+$

$iT \supset$  $\qquad -iT^+ =$  *

$SS^+ = 1 \qquad iT - iT^+ = -T \cdot T^+ \qquad \sum |n\rangle\langle n|$

 $+$  $^* = -$ 

In gravità si ottiene un risultato analogo nella gauge di Prenki ,

$\ell_\mu = \partial^i g_{i\mu} + a\, \delta^i_\mu \partial_i g_{\rho\sigma} \eta^{\rho\sigma} \qquad i = 1, 2, 3$

$(\Gamma, \Gamma) = 0$ Γ = funzionale generatore delle funzioni di Green irriducibili a una particella

$\Gamma(\Phi, K)$

QED $\quad \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \mathcal{L}_m \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$-\frac{1}{4}\int F_{\mu\nu}^2 = -\frac{1}{2}\int \partial_\mu A_\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) =$$

$$= \frac{1}{2}\int A_\mu(x)[\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu] A_\nu(x)\, d^4x \xrightarrow{\text{Fourier}}$$

$$= \frac{1}{2}\int \tilde{A}_\mu(k)\underbrace{\left[-\eta^{\mu\nu}k^2 + k^\mu k^\nu\right]} \tilde{A}_\nu(-k)\frac{d^4k}{(2\pi)^4}$$

$M_{\mu\nu}(k) \equiv -\eta^{\mu\nu}k^2 + k^\mu k^\nu$ non è invertibile

k^μ è un autovettore con autovalore zero:

$$M_{\mu\nu} k^\nu = -k_\mu k^2 + k_\mu k^2 = 0$$

Simmetria di gauge: $\delta A_\mu = \partial_\mu \Lambda \qquad \Lambda = \Lambda(x)$

$$\delta \mathcal{L} = \int A_\mu \underbrace{\left[\Box \eta^{\mu\nu} - \partial^\mu \partial^\nu\right]}\partial_\nu \Lambda = 0$$

Classicamente, basta imporre $\partial^\mu A_\mu = 0$

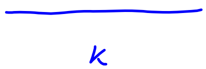
A livello della teoria dei campi quantistici è consigliabile preservare la località (= polinomialità nei campi e nelle derivate)

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{2}\varphi^2}_{} - \frac{\lambda}{4!}\varphi^4$$

Quantizzazione

$$(\Box + m^2)\, G(x) = -i\,\delta(x)$$

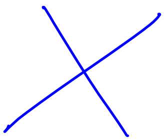
$$(k^2 - m^2)\,\delta(k^2 - m^2) = 0$$



$$= \frac{i}{k^2 - m^2 + i\epsilon}$$

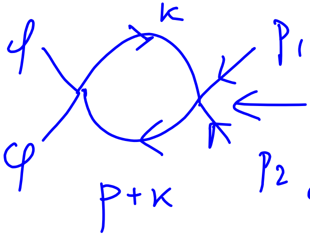
prescrizione di Feynman

propagatore



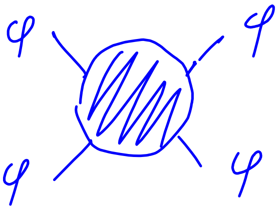
$$= -i\lambda$$

Costruire tutti i diagrammi di Feynman con queste regole (+ l'integrale sugli impulsi interni e un certo fattore combinatorico)

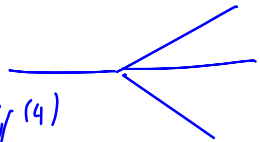


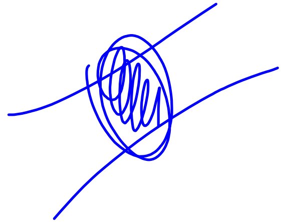
$$p = p_1 + p_2 \longrightarrow$$

$$\longrightarrow \int \frac{d^4k}{(2\pi)^4} (-i\lambda)^2 \frac{i}{k^2 - m^2 + i\epsilon} \cdot \frac{i}{(p+k)^2 - m^2 + i\epsilon}$$

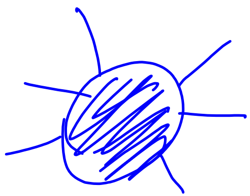


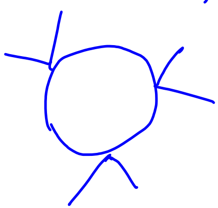
$$= \mathcal{M}^{(4)}$$

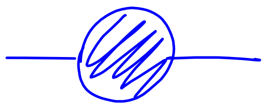


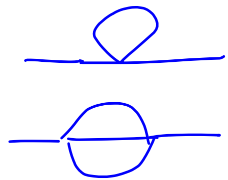


$$\Gamma \cdots \qquad + (-i)\frac{1}{4!}\,\varphi\,\overset{\varphi}{\underset{\varphi}{\mathcal{M}^{(4)}}}\,\varphi + \cdots$$









Teoria non locale

$$S = \frac{1}{2}\int (\partial_\mu \varphi) e^{-\Box/M^2} \partial^\mu \varphi \, d^4x \, +$$

$$+ \int \varphi(x)\varphi(y)\varphi(z)\varphi(w) \, V(x,y,z,w) \, dx\, dy\, dz\, dw$$

<u>Località</u> : quello che resta del principio di corrispondenza, assieme alla <u>rinormalizzabilità</u>

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2+i\epsilon} \frac{1}{(p+k)^2-m^2+i\epsilon}$$

è ben definito in tutti i domini finiti di k, ma diverge nell'ultravioletto (k grandi)

$$|k| \gg \Lambda \qquad \sim \int^{\Lambda} \frac{d^4k}{k^4} \sim \ln \Lambda$$

Località dei controtermini

$$\mathcal{M}^{(4)}(p) = \int^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(p+k)^2 - m^2 + i\epsilon}$$

$$\frac{\partial}{\partial p} \mathcal{M}^{(4)} < \infty$$

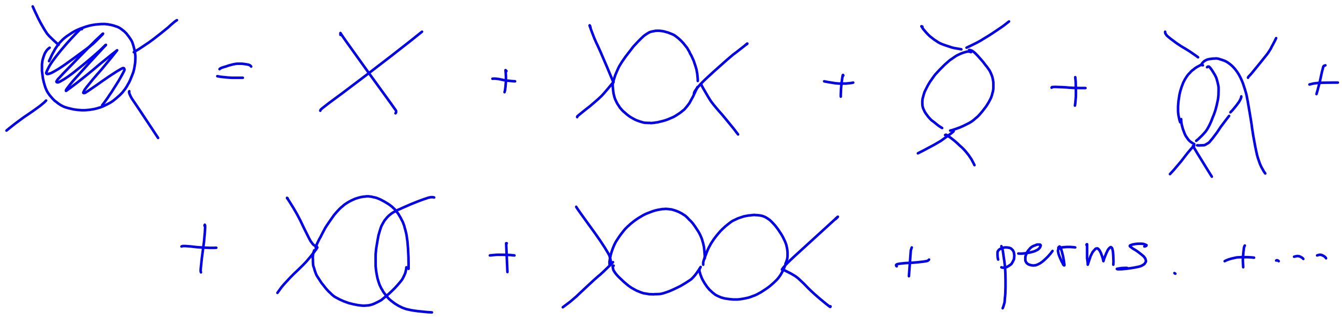
$\Longrightarrow \mathcal{M}^{(4)}_{div}$ non dipende da p

Le parti divergenti dei diagrammi sono sempre locali (in un senso algoritmico)

$\exists$ n finito

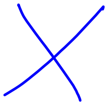
$$\frac{\partial^n}{\partial p_{\mu_1} \cdots \partial p_{\mu_n}} \mathcal{M}(p) < \infty$$

$$\frac{\partial^n}{\partial {m^2}^n} \mathcal{M}(p) < \infty$$



si può "rinormalizzare"

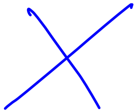
$(Z_\varphi^{1/2}\varphi)$

 $= -i\lambda$ non dip. da p

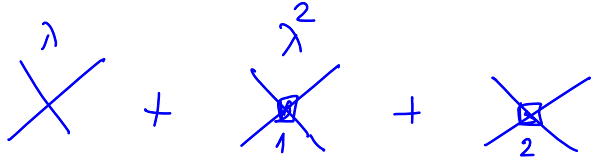
$$\mathcal{L}_R = Z_\varphi \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m^2}{2} Z_{m^2} Z_\varphi \varphi^2 - \frac{\lambda}{4!}\varphi^4 Z_\varphi^2 Z_\lambda =$$

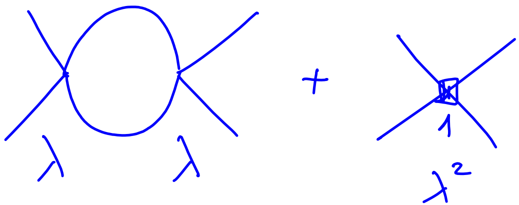
$$= \mathcal{L} + \text{controtermini (locali)}$$

$Z_{\varphi,m,\lambda} = 1 +$ correzioni

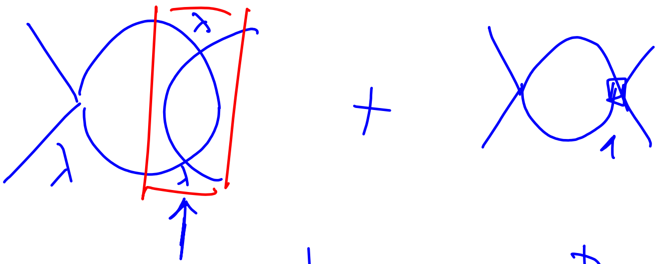
 $= -i Z_\varphi^2 Z_\lambda \lambda = -i\lambda + i\lambda^2 a + i\lambda^3 b \ldots =$

fisso a, b, ... per cancellare le parti divergenti

$=$  $+ \ldots$



fisso a per permettere $\lim_{\Lambda \to \infty}$ sulla somma

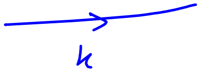


la somma ha parte divergente locale

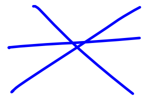
la sua parte div. è non locale $\left(\underline{\ln \Lambda}\ \underline{\ln p^2}\right)$

Problema: tutte le parti divergenti sono locali (algoritmicamente), quindi le posso sottrarre correggendo (rinormalizzando) i termini della Lagrangiana o aggiungendo altri termini locali. Non è detto che la lagrangiana contenga già tutti i tipi di termini locali generati come parti divergenti dei diagrammi costruiti coi suoi vertici

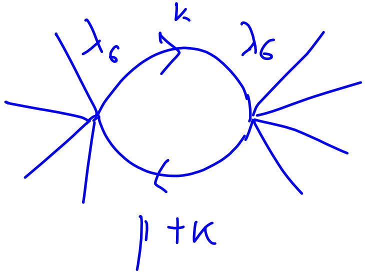
Esempio. $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{\lambda_6}{6!}\varphi^6 \quad \dots \ln\Lambda\, \varphi^8$



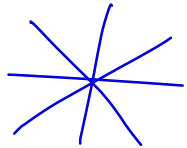
$= \frac{i}{k^2}$

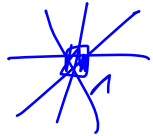


$= -i\lambda_6$



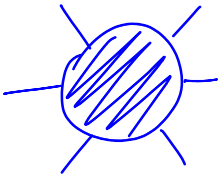
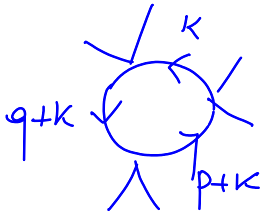
$\sim \ln\Lambda$





Se la procedura si chiude con una Lagrangiana contenente un numero finito di termini la teoria si dice rinormalizzabile, altrimenti si dice non rinormalizzabile

La teoria φ^4 è rinormalizzabile

 $< \infty$ 

$$\sim \int \frac{d^4k}{(k^2-m^2)((p+k)^2-m^2)((q+k)^2-m^2)}$$

Il modello standard è rinormalizzabile, la gravità (azione S_H) non lo è

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \underline{\frac{m^2}{2}\varphi^2} - \frac{\lambda}{4!}\varphi^4$$

Power counting (analisi dimensionale) $\hbar = c = 1$

$[p^\mu] = 1 = \underline{[m]}$ $\quad [x^\mu] = -1$ $\quad S = \int d^4x\, \mathcal{L}$

$[S] = 0$ $\quad [\mathcal{L}] = 4$ $\quad \underline{[\varphi] = 1} = [\partial_\mu]$ $\quad \underline{[\lambda] = 0}$

Se i campi hanno dimensione >0 e tutti i parametri hanno dimensione ≥ 0 allora la teoria è rinormalizzabile

$$\frac{J\varphi^6}{[\varphi]=1} \qquad \cancel{\frac{1}{p^\mu}} \quad \cancel{\frac{1}{p^2}} \quad \cancel{\frac{1}{\Box}}$$

$$[\varphi^6]=6 \qquad [J]=-2$$

non posso costruire tale J

$$\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{\lambda_6}{6!}\varphi^6 \qquad [\lambda_6]=-2$$

$$\ln\Lambda\ \lambda_6^2\varphi^8 \quad \text{ok} \qquad \ln\Lambda\ \lambda_6^3\varphi^{10} \ \ldots$$

$$-4+8=4 \qquad \lambda_6^2\varphi^2\Box^2\varphi^2 \ \ldots \qquad -6+10=4$$

La gravità (azione S_H) non è rinormalizzabile

$$S_H = -\frac{1}{2\kappa^2}\int\sqrt{-g}\,(R + 2\Lambda)$$

$$S_H \longrightarrow S_H + \frac{1}{2\lambda}\int g_\mu g^\mu + \int \mathcal{L}_{ghost}$$

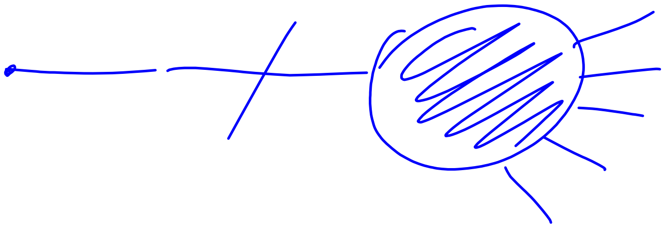
$$g_{\mu\nu} = \eta_{\mu\nu} + 2k\phi_{\mu\nu}$$

k è la "costante di gauge" della gravità

$$S_H + \frac{1}{2\lambda}\int g_\mu g^\mu = \cancel{\frac{1}{k^2}} + \cancel{\frac{\phi}{k}} + \underline{\phi^2} + k\phi^3 \cdots\cdots k^{n-2}\phi^n$$

ogni termine è la somma di un contributo con 2 derivate e uno senza derivate

$$g_\mu = \partial^\nu\phi_{\mu\nu} - \frac{1}{2}\partial_\mu\phi$$



Un diagramma fatto con un vertice con una sola gamba non è irriducibile a una particella

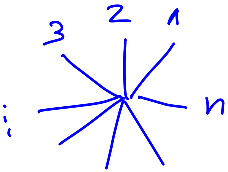
$$[g_{\mu\nu}] = 0 = [\eta_{\mu\nu}] =$$

$$= [\kappa \phi_{\mu\nu}] \Rightarrow [\kappa] = -1 < 0$$

$$[\phi_{\mu\nu}] = 1 \qquad [R] = +2 \qquad R \sim g^{-1} \partial^2 g$$

$$\sim \phi^2 : \quad \partial\phi\,\partial\phi + \lambda \phi^2 \qquad [\lambda] = 2$$

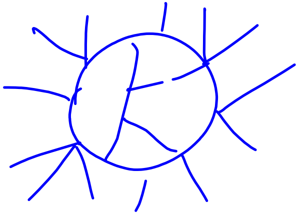
$$[\partial\phi\,\partial\phi] = 4 \Rightarrow [\phi_{\mu\nu}] = 1$$



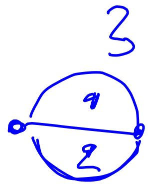
$\sim \kappa^{n-2}$

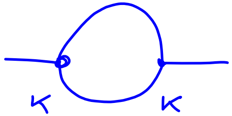
$L = 0$

$$\xrightarrow[p]{} \sim \frac{1}{p^2}$$

G'  $\sim (k^2)^{L-1}\, k^E$ $E = \#$ gambe esterne

$$\begin{cases} L = \#\text{ di loop} = I - V + 1 \\ I = \#\text{ gambe interne} \\ V = \#\text{ vertici} \end{cases}$$





$\sim k^2 \sim k^{E+2(L-1)}$ $L = 1$, $E = 2$

$$G: \prod_{i\ \text{vertici}} k^{n_i - 2} = k^{\sum_i n_i - 2V} = k^E\, k^{2(I-V)}$$

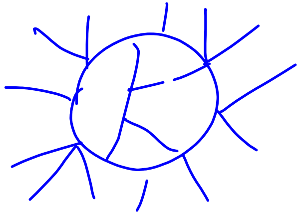
$\underline{I - V = L - 1}$

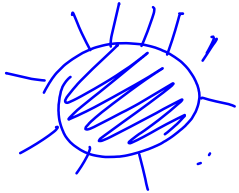
$$\sum_i n_i = E + 2I \qquad L = I - V + 1$$

Identità di Eulero: $2 = \#\text{facce} - \#\text{spigoli} + \#\text{vertici}$

$2 = L + 1 \qquad I \qquad + V$

Controtermini:

G'  $\sim (k^2)^{L-1}\, k^E$

G contribuisce a  E $\quad \phi^E$ in Γ

Γ:

$$(k^2)^{L-1}\, k^E \phi^E = \underline{\underline{(k^2)^L}} \; \underbrace{\frac{1}{k^2}\int (k\phi)^E}$$

<u>1 loop</u>

$$\int \sqrt{-g}\left[\alpha \underset{2}{R_{\mu\nu}} \underset{2}{R^{\mu\nu}} + \beta R^2 + \underset{2}{\Lambda}\, \underset{2}{R} + \underset{4}{\Lambda^2}\right]$$

$$[\alpha] = [\beta] = 0$$

2 loop : $\kappa^{2} \int \sqrt{-g} \left[Riem^3 + Ric^3 + R\Box R \ldots \right.$

$\kappa^{-2} \qquad\qquad \left. + R^2 + R + 1 \right]$

La rinormalizzazione richiede di aggiungere infiniti termini

Ogni divergenza di nuovo tipo rappresenta una nuova costante d'accoppiamento indipendente da misurare (a meno che non svanisca on shell, nel qual caso si può riassorbire con una ridefinizione dei campi)

$\ln \Lambda \quad - \quad (\ln \Lambda + c)$ $\qquad c =$ costante finita

$(\Lambda \to \infty)$

$$K = \frac{1}{M_{Pl}} \qquad K^2 \, \mathrm{Riem}^3 \sim \left(\frac{E^6}{M_{pl}^6}\right) M_{pl}^4$$

$$E \sim 10 \ \mathrm{TeV} \qquad M_{Pl} \sim 10^{19} \ \mathrm{GeV}$$

$$\frac{E}{M_{pl}} = \frac{10^4}{10^{19}} = 10^{-15}$$

Eq. del moto

$$\Lambda = 0 \qquad S_H = -\frac{1}{2K^2}\int \sqrt{-g}\, R \qquad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

$$1 \text{ loop}: \qquad \int \sqrt{-g}\left(\alpha R_{\mu\nu} R^{\mu\nu} + \beta R^2\right)$$

$$S_H \longrightarrow S_H - \int \sqrt{-g}\left(\alpha R_{\mu\nu} R^{\mu\nu} + \beta R^2\right) =$$

$$= S_H - \int \frac{\delta S_H}{\delta g_{\mu\nu}} \Delta g_{\mu\nu} = S_H\left(g_{\mu\nu} - \Delta g_{\mu\nu}\right) + \dots$$

$$\Delta g_{\mu\nu} = a R_{\mu\nu} + b g_{\mu\nu} R$$

L'azione S_H è finita a un loop, ma non lo è a due loop

$$\lambda_{nuova} \int \sqrt{-g}\, R_{\mu\nu\rho\sigma} R^{\rho\sigma\alpha\beta} R_{\alpha\beta}{}^{\mu\nu}$$ Goroff Sagnotti

S_H non è finita a un loop se accoppiata alla materia ('t Hooft - Veltman)

$$-\frac{1}{2\kappa^2}\int\sqrt{-g}\, R + \frac{1}{2}\int\sqrt{-g}\, \partial_\mu\varphi\, \partial_\nu\varphi\, g^{\mu\nu}$$

1 loop: $\int\sqrt{-g}\left[\alpha\, Ric^2 + \beta R^2 + \gamma R^{\mu\nu} \nabla_\mu\varphi \nabla_\nu\varphi + \right.$

$$+ \delta\, R\, \nabla_\mu \phi\, \nabla^\mu \phi + \xi\, (\Box \phi)^2 + \zeta\, \nabla_\mu \nabla_\nu \varphi\, \nabla^\mu \nabla^\nu \varphi$$

$$+ \dots \underbrace{(\nabla_\mu \varphi)(\nabla^\mu \varphi)(\nabla_\nu \varphi)(\nabla^\nu \varphi)} + \dots \Big]$$

Eq. del moto: $\quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2\, T_{\mu\nu}$

$$T \sim \nabla\varphi\, \nabla\varphi$$

Tutti i termini quadratici nelle curvature possono essere convertibili in termini cubici usando le eq. del moto

$R_{\mu\nu} R^{\mu\nu}$, R^2, $R \Box R$, $R_{iem} \Box R_{iem}$, ...

$\phi\, \Box^2 \phi$, $\quad \phi\, \Box^3 \phi$

$$R_{\mu\nu\rho\sigma} \nabla^{\cdot} \nabla^{\cdot} \cdots \nabla^{\cdot} R_{\alpha\beta\gamma\delta}$$

posso commutare le ∇ a piacimento a meno di termini cubici $[\nabla, \nabla] \sim R$

$$R_{\mu\nu\rho\sigma} \nabla^{\cdot} \cdots \nabla^{\cdot} \nabla^{\alpha} R_{\alpha\beta\gamma\delta} \to \text{Riem Ric}$$

$$\nabla_\mu R^\mu{}_{\nu\rho\sigma} = -\nabla_\sigma R_{\nu\rho} + \nabla_\rho R_{\nu\sigma}$$

Identità di Bianchi contratta

$$\text{Riem} \, \Box^n \, \text{Riem} = R_{\mu\nu\rho\sigma} \Box^n R^{\mu\nu\rho\sigma}$$

$n=0$ $\quad \sqrt{g}\,\text{Riem}^2 = \sqrt{g}\left(4\,\text{Ric}^2 - R^2\right) +$ derivata totale

$n>0$ $\quad R^{\mu\nu\rho\sigma} \Box^{n-1} \nabla^\alpha \nabla_\alpha R_{\mu\nu\rho\sigma}$

$$\nabla_\alpha R_{\mu\nu\rho\sigma} + \nabla_\nu R_{\alpha\mu\rho\sigma} + \nabla_\mu R_{\nu\alpha\rho\sigma} = 0$$

Identità di Bianchi non contratta

$$R^{\mu\nu\rho\sigma} \Box^{n-1} \nabla^\alpha \nabla_\alpha R_{\mu\nu\rho\sigma} = - R^{\mu\nu\rho\sigma} \Box^{n-1} \nabla^\alpha \left(\nabla_\nu R_{\alpha\mu\rho\sigma} + \nabla_\mu R_{\nu\alpha\rho\sigma} \right) \longrightarrow \propto Ric$$

L'azione classica della gravità quantistica

$$-\frac{1}{2k^2} \int \sqrt{-g} \left(2\Lambda + R + W^3 + \dots \right) + S_m$$

solo Weyl e almeno 3 W

S_m = azione della materia

Posso sempre eliminare Ricci e tenere solo Weyl

Eq. del moto:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = \kappa^2 \left(T^{(m)}_{\mu\nu} + T^{(g)}_{\mu\nu} \right)$$

$$T^{(g)}_{\mu\nu} \propto W^2$$

FLRW

$$T^{(m)}_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}$$

$$ds^2 = dt^2 - a^2(t) \left[\frac{dr^2}{1 - K r^2} + r^2 d\Omega^2 \right]$$

soddisfa

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = \kappa^2 \, T^{(m)}_{\mu\nu}$$

ed ha $W_{\mu\nu\rho\sigma} = 0$, quindi anche $T^{(g)}_{\mu\nu} = 0$,

quindi soddisfa anche

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = \kappa^2 \left(T^{(m)}_{\mu\nu} + T^{(g)}_{\mu\nu} \right)$$

— fine corso A (6 cfu) —

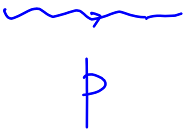
$$S_{HD} = -\frac{1}{2k^2} \int \sqrt{-g} \left[2\Lambda + \zeta R + \alpha R_{\mu\nu} R^{\mu\nu} + \beta R^2 \right]$$

è rinormalizzabile $\qquad \alpha$ e β non sono piccoli

$$g_{\mu\nu} = \eta_{\mu\nu} + 2k \phi_{\mu\nu} \qquad R_{\mu\nu} \sim \kappa \partial\partial\phi + O(\kappa^2 \phi^2)$$

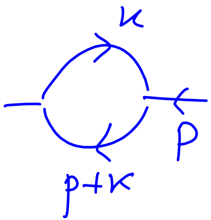
$$\alpha R_{\mu\nu} R^{\mu\nu} + \beta R^2 \sim \kappa^2 \phi \partial^4 \phi + O(\kappa^3 \phi^3)$$

Modificano il propagatore



$$\sim \frac{1}{(p^2)^2} \qquad \text{per } p^2 \gg 1$$

$$S(p,m) = \frac{1}{p^2(p^2-m^2)} = \frac{1}{m^2}\left(\frac{1}{p^2} - \frac{1}{p^2-m^2}\right)$$



$$= \int \frac{d^4k}{(2\pi)^4} S(k,m)\, S(p+k,m) < \infty$$

$$\sim \int \frac{d^4k}{k^4 \, k^4} \qquad \underline{k^2 \gg 1}$$

Power counting

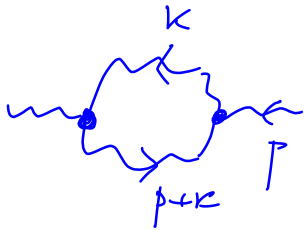
$$\underline{\underline{\frac{1}{2}(\partial_\mu \varphi)^2}} - \underline{\underline{\frac{m^2}{2}}}\varphi^2 - \frac{\lambda}{4!}\varphi^4$$

$[\varphi] = 1$

$$S_{HD} \sim \int \Big(\alpha\, \phi\, \partial^4 \phi + \beta\, \phi\, \partial^4 \phi + \underset{2}{J}\, \underset{0}{\phi}\, \underset{2}{\partial^2} \underset{0}{\phi} + \Lambda \phi^2 + \dots \Big)$$

$$[\alpha] = [\beta] = 0 \qquad [\partial] = 1 \qquad [\phi] = 0 \qquad [J] = 2 \qquad [\Lambda] = 4$$

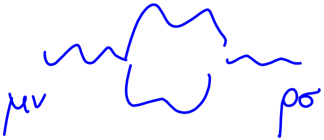
$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa\, \phi_{\mu\nu} \qquad [\kappa] = 0$$



$$I \equiv \int \frac{d^4k \;\; k^8}{(k^2)^2 \left((p+k)^2\right)^2}$$ diverge

$$\frac{\partial^5}{\partial p^5} I \sim \int \frac{d^4k \;\; k^8\, k\, k\, k\, k\, k}{(k^2)^2 \left((p+k)^2\right)^7} = \int d^4k \, \frac{k^{13}}{k^{18}} < \infty$$

I_{div} = polinomio di grado 4 in p



contribuisce alla $\langle \phi_{\mu\nu} \phi_{\rho\sigma} \rangle$

$$S_{HD} = -\frac{1}{2k^2}\int\sqrt{-g}\,\Big[2\Lambda + \xi R + \alpha R_{\mu\nu}R^{\mu\nu} + \beta R^2 + \gamma \underbrace{R\,\Box R} + \delta \underbrace{R_{\mu\nu}\,\Box R^{\mu\nu}} + R^3\Big]$$

è superrinormalizzabile (solo parametri di dimensione strettamente positiva, a parte i coefficienti γ, δ, dei termini quadratici dominanti nell'UV)

Power counting: $-\frac{1}{2k^2}\int \gamma R_{\mu\nu} \Box R^{\mu\nu}\sqrt{-g} =$

$$= \int \gamma \phi \partial^6 \phi \qquad [\gamma]=[\delta]=0 \qquad [\phi]=-1$$

$$g_{\mu\nu} = \eta_{\mu\nu} + 2k\,\phi_{\mu\nu} \qquad \underline{[k]=1} \qquad [\alpha]=[\beta]=2$$

$$[J]=4 \qquad [\Lambda]=6$$

Queste teorie hanno il problema dei ghost (se quantizzate con la prescrizione di Feynman)

Vediamo la controparte classica del problema dei ghost

Problemi dovuti alle derivate superiori

$\mathcal{L}(q, \dot{q})$ è rimpiazzata da una $\mathcal{L}(q, \dot{q}, \ddot{q}, \dots)$

Se aggiungo variabili ottengo $\mathcal{L}(q, \dot{q}, \xi, \dot{\xi}, \zeta, \dot{\zeta}, \dots)$,

ma l'energia non è limitata inferiormente

Radiazione di frenamento

Forza di Abraham_Lorentz

$$m\left(a - \tau \frac{da}{dt}\right) = F_{ext} \qquad \tau = \frac{2e^2}{3mc^3} > 0$$

Devo fissare $x(0)$, $\dot{x}(0)$, $\ddot{x}(0)$

Se fisso solo $x(0)$, $\dot{x}(0)$ ho soluzioni runaway

Eq. omogenea $\quad a = \tau \dot{a} \quad a(t) = a_0 e^{t/\tau}$

$$v(t) = a_0 \tau \left(e^{t/\tau} - 1\right) + v_0$$ Posso fissare $v_0 = 0$

$$x(t) = a_0 \tau^2 \left(e^{t/\tau} - 1\right) - a_0 \tau t + x_0$$ Posso fissare $x_0 = 0$

Esiste una maniera per liberarsi di queste soluzioni

$$m\left(1 - \tau \frac{d}{dt}\right) a = F_{ext}$$

$$ma = \underbrace{\frac{1}{1 - \tau \frac{d}{dt}}}_{\text{funzione di Green da definire.}} F_{ext} \equiv \langle F_{ext} \rangle$$

Se richiedo $\underline{\text{analiticità in}}$ $\underline{\tau}$ la funzione di Green è univocamente definita

$$ma = \frac{1}{\tau}\int_t^{+\infty} dt'\, e^{(t-t')/\tau}\, F_{ext}(t')$$ viola la micro-causalità

Soluz. generale (non analitica in τ) $\tau \sim 10^{-23}\,s$

$$ma = -\frac{1}{\tau}\int_{-\infty}^{t} dt'\, e^{(t-t')/\tau}\, F_{ext}(t') + m a_0 e^{t/\tau}$$

$$\Box(\alpha + \beta\Box)\phi = J$$

$$\Box\phi = \frac{1}{\alpha+\beta\Box} J \equiv \langle J \rangle$$

analiticità in β

viola la microcausalità

a $E \gg \sqrt{\frac{\alpha}{\beta}}$

Analiticità: $-u = (t-t')/\tau \qquad du = \frac{dt'}{\tau}$

$$ma = \int_0^{\infty} du\, e^{-u} F_{ext}(t+\tau u)$$

Si può fare anche nella teoria dei campi quantistici, in particolare nella gravità quantistica

$$\sqrt{\frac{\alpha}{\beta}} \sim 10^{-37} \text{ sec} \,, \quad 10^{-44} \text{ sec}$$

"Principio di corrispondenza" in QFT :

- Località
- Rinormalizzabilità
- Unitarietà (perturbativa)

Teorie di Proca, Pauli-Fierz e Rarita-Schwinger

spin 1 — $m \neq 0$

spin-2 — $m \neq 0$

Spin $\frac{3}{2}$ — $m = 0$, $m \neq 0$

Proca
$$S_P = \int \sqrt{-g} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\nu + \underbrace{\frac{\eta_P}{2} R^{\mu\nu} A_\mu A_\nu + \frac{\eta_P'}{2} R A_\mu A^\mu}_{\text{Accoppiamenti non minimali}} \right]$$

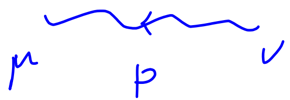
Nello spazio piatto:
$$\mathcal{L}_P = -\frac{1}{2} \partial^\mu A^\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{m^2}{2} A_\mu A^\mu$$

$$\Box A_\mu - \partial_\mu \partial \cdot A + m^2 A_\mu = 0$$

Prendo la divergenza: $m^2\,\partial\cdot A = 0$

Propagatore: $\left[(-p^2+m^2)\eta_{\mu\nu} + p_\mu p_\nu\right]^{-1}$

non rinormalizzabile

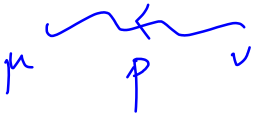


$$= -\frac{i}{p^2-m^2}\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right) \equiv P_{\mu\nu}(p)$$

$$-\frac{1}{p^2-m^2}\left(\eta_{\mu} - \frac{p_\mu p_\nu}{m^2}\right)\left[-(p^2-m^2)\eta_{\nu\rho} + p_\nu p_\rho\right] =$$

$$= \eta_{\mu\rho} - \cancel{\frac{p_\mu p_\rho}{m^2}} - \frac{1}{p^2-m^2}\cancel{\left(p_\mu - \frac{p^2 p_\mu}{m^2}\right)} p_\rho = \eta_{\nu\rho}$$

$p^\mu = (m,0,0,0)$ sistema di rif. a riposo



$$\underset{\text{polo}}{\simeq} \frac{i}{p^2-m^2}\left[\begin{pmatrix}-1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix} + \begin{pmatrix}1&&&\\&0&&\\&&0&\\&&&0\end{pmatrix}\right] =$$

$$= \frac{i}{p^2 - m^2} \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$P_{\mu\nu}(p) p^\nu = -\frac{i}{p^2 - m^2} \left(1 - \frac{p^2}{m^2} \right) p_\mu = \frac{i\, p_\mu}{m^2}$$: nessun polo

- Costruire il più generale $P_{\mu\nu}(p)$ / $P_{\mu\nu}(p) p^\nu$ non ha poli

$$P_{\mu\nu}(p) = A \eta_{\mu\nu} + B p_\mu p_\nu$$

$$P_{\mu\nu}(p) p^\nu = p_\mu (A + B p^2) \xrightarrow[\text{sul polo}]{} p_\mu (A + B m^2) = 0$$

$$B = -\frac{A}{m^2} \qquad P_{\mu\nu}(p) = A \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2} \right)$$

Azione di Pauli-Fierz $\quad \chi_{\mu\nu}$ tensore simmetrico $\rightarrow$

$\rightarrow$ 10 componenti

$\partial^\mu \chi_{\mu\nu}$: vettore (3) + scalare (1 : $\partial^\mu \partial^\nu \chi_{\mu\nu}$)

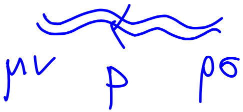
$\chi_{\mu\nu}\eta^{\mu\nu} = \chi$ traccia (1)

- Cerco $P_{\mu\nu\rho\sigma}(p)$ tale che $P_{\mu\nu\rho\sigma}(p)\,p^\sigma$ e $P_{\mu\nu\rho\sigma}(p)\,\eta^{\rho\sigma}$ non abbiano poli

symm

Esercizio

Il risultato è unico :



$$= \frac{i}{2}\,\frac{1}{p^2-m^2}\left(\pi_{\mu\rho}\pi_{\nu\sigma} + \pi_{\mu\sigma}\pi_{\nu\rho} - \frac{2}{3}\pi_{\mu\nu}\pi_{\rho\sigma}\right)$$

$$\pi_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2} \qquad \pi_{\mu\nu}\eta^{\mu\nu} = 4 - \frac{p^2}{m^2} \qquad \pi_{\mu\nu}\pi^{\nu}{}_{\rho} = \pi_{\mu\rho} + O(p^2-m^2)$$

$$P_{\mu}{}^{\mu}{}_{\rho\sigma} = \frac{i}{2}\frac{1}{p^2-m^2}\left(2\pi_{\rho\sigma} - \frac{2}{3}\left(4-\frac{p^2}{m^2}\right)\pi_{\rho\sigma}\right) =$$

$$= \frac{i}{2}\frac{1}{p^2-m^2}\pi_{\rho\sigma}\left(-\frac{2}{3}+\frac{2}{3}\frac{p^2}{m^2}\right) \quad \underline{\text{ok}}$$

Azione di Pauli-Fierz

come da azione di Hilbert parte quadratica attorno al piatto

$$S_{PF} = \frac{1}{2}\int\sqrt{-g}\Big[\big(\nabla_\rho \chi_{\mu\nu}\nabla^\rho\chi^{\mu\nu} - \nabla_\mu\chi\nabla^\mu\chi + 2\nabla_\mu\chi\nabla_\nu\chi^{\mu\nu} +$$

$$- 2\nabla_\mu\chi_{\nu\rho}\nabla^\rho\chi^{\nu\mu}\big) - m^2\left(\chi_{\mu\nu}\chi^{\mu\nu}-\chi^2\right)\Big]$$

$$S_{PF}^{\text{nonmin}} = \frac{1}{2}\int\sqrt{-g}\Big[a_1 R_{\mu\nu\rho\sigma}\chi^{\mu\rho}\chi^{\nu\sigma} + a_2 R_{\mu\nu}\chi^{\nu\rho}\chi^{\mu}_{\rho} +$$

$$+ a_3 R_{\mu\nu}\chi^{\mu\nu}\chi + a_4 R\chi_{\mu\nu}\chi^{\mu\nu} + a_5 R\chi^2\Big]$$

Nel piatto:

$$S_{PF} = \frac{1}{2}\int \Big[\partial_\rho \chi_{\mu\nu}\, \partial^\rho \chi^{\mu\nu} - \partial_\mu \chi\, \partial^\mu \chi + 2\, \partial_\mu \chi\, \partial_\nu \chi^{\mu\nu} +$$
$$- 2\, \partial_\mu \chi_{\nu\rho}\, \partial^\rho \chi^{\nu\mu} - m^2 \left(\chi_{\mu\nu} \chi^{\mu\nu} - \chi^2 \right) \Big]$$

$m = 0$ $\quad \delta \chi_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$ simmetria di gauge

Eq. del moto:

$$-\Box \chi_{\mu\nu} + \eta_{\mu\nu} \Box \chi - \eta_{\mu\nu} \partial_\alpha \partial_\beta \chi^{\alpha\beta} - \partial_\mu \partial_\nu \chi +$$
$$+ \partial_\mu \partial_\rho \chi^\rho_{\ \nu} + \partial_\nu \partial_\rho \chi^\rho_{\ \mu} - m^2 \chi_{\mu\nu} + m^2 \eta_{\mu\nu} \chi = 0$$

Divergenza: ∂^ν

$$\cancel{-\Box \partial^\nu \chi_{\mu\nu}} + \cancel{\partial_\mu \Box \chi} - \partial_\mu \cancel{\partial_\alpha \partial_\beta \chi^{\alpha\beta}} - \cancel{\partial_\mu \Box \chi} +$$
$$+ \cancel{\partial_\mu \partial_\alpha \partial_\beta \chi^{\alpha\beta}} + \cancel{\Box \partial_\rho \chi^\rho_{\ \mu}} - m^2 \partial^\nu \chi_{\mu\nu} + m^2 \partial_\mu \chi = 0$$

$$\partial^\nu \chi_{\mu\nu} = \partial_\mu \chi$$

Traccia $\eta^{\mu\nu}$:

$$-\Box\chi + 4\Box\chi - 4\,\partial_\alpha\partial_\beta\chi^{\alpha\beta} - \Box\chi + 2\,\partial_\alpha\partial_\beta\chi^{\alpha\beta} +$$

$$+ 3m^2\chi = 0$$

$$\cancel{2\Box\chi} - 2\,\partial_\alpha\cancel{\partial_\beta\chi^{\alpha\beta}} + 3m^2\chi = 0$$

$$\partial_\mu\chi^\mu_\nu = \partial_\nu\chi \qquad \Rightarrow \qquad \chi = 0$$

$$\partial_\mu\chi^\mu_\nu = 0$$

Azione di Rarita - Schwinger $\quad \psi_\mu$

$$\nabla_\mu\psi_\nu = \partial_\mu\psi_\nu + \frac{1}{8}[\gamma_a,\gamma_b]\,\omega_\mu^{ab}\,\psi_\nu - \Gamma_{\mu\nu}^{\rho}\,\psi_\rho$$

Nel piatto

$$\mathcal{L} = -\bar{\psi}_\mu \left(\varepsilon^{\mu\nu\rho\alpha} \gamma_5 \gamma_\nu \partial_\rho + m\,\sigma^{\mu\alpha}\right)\psi_\alpha =$$

$$= -\bar{\psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \psi_\rho - m\,\bar{\psi}_\mu \sigma^{\mu\nu} \psi_\nu$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \qquad \gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} =$$

$$= \frac{1}{3!}\left(\gamma^\mu\gamma^\nu\gamma^\rho + \dots \text{ (compl. antisimm.)}\right)$$

$$\mathcal{L} = -\bar{\psi}_\mu Q^{\mu\nu} \psi_\nu \qquad \vec{p} = -i\vec{\nabla} \quad i\partial_t = E \quad p^\mu = i(\partial_0, -\vec{\nabla})$$

$$Q^{\mu\nu} = (\eta^{\mu\nu} - \gamma^\mu\gamma^\nu)(i\partial\!\!\!/ - m) - i\gamma^\nu\partial^\mu + i\gamma^\mu\partial^\nu$$

$$p_\mu = i\partial_\mu \qquad Q^{\mu\nu} = (\eta^{\mu\nu} - \gamma^\mu\gamma^\nu)(p\!\!\!/ - m) + p^\nu\gamma^\mu - p^\mu\gamma^\nu$$

Occorre assicurarsi che propaghi solo spin $\frac{3}{2}$

$\partial^\mu \psi_\mu \,,\quad \gamma^\mu \psi_\mu$ sono spin $\frac{1}{2}$

Eq. del moto : $Q^{\mu\nu}\psi_\nu = 0 \Rightarrow p^\mu Q_{\mu\nu}\psi^\nu = \gamma^\mu Q_{\mu\nu}\psi^\nu = 0$

Da mostrare che ciò $\Rightarrow \partial\cdot\psi = \gamma\cdot\psi = 0$

$1\,,\ \gamma_5\,,\ \gamma_\mu\,,\ \sigma_{\mu\nu}$

$$p^\mu Q_{\mu\nu} = (p^\nu - \not{p}\gamma^\nu)(\not{p}-m) + p^\nu\not{p} - p^2\gamma^\nu =$$

$$= (p^\nu + \gamma^\nu\not{p} - 2p^\nu)(\not{p}-m) + \not{p}p^\nu - p^2\gamma^\nu =$$

$$= \cancel{-p^\nu\not{p}} + p^\nu m + \cancel{\gamma^\nu p^2} - \gamma^\nu\not{p}m + \cancel{\not{p}p^\nu} - \cancel{p^2\gamma^\nu} =$$

$$= m(p^\nu - \gamma^\nu\not{p}) \qquad \partial\cdot\psi = \gamma^\mu\not{\partial}\psi_\mu$$

$$\gamma^\mu Q_{\mu\nu} = -3\gamma^\nu(\not{p}-m)+4p^\nu-\not{p}\gamma^\nu =$$

$$= -3\gamma^\nu\not{p}+3m\gamma^\nu+4p^\nu+\gamma^\nu\not{p}-2p^\nu =$$

$$= -2\gamma^\nu\not{p}+3m\gamma^\nu+2p^\nu = 2(p^\nu-\gamma^\nu\not{p})+3m\gamma^\nu$$

Usando $\partial\cdot\psi = \gamma^\mu\not{\partial}\psi_\mu$, $\gamma^\mu Q_{\mu\nu}\psi^\nu = 0 \Rightarrow$

$$3m\,\gamma\cdot\psi = 0 \Rightarrow \gamma\cdot\psi = 0$$

$$\partial\cdot\psi = -\not{\partial}(\gamma\cdot\psi)+2\,\partial\cdot\psi \Rightarrow \partial\cdot\psi = 0$$

Propagatore : $P_{\mu\nu}Q^{\nu\rho} = i\,\delta_\mu^\rho$

$$P_{\mu\nu} = \frac{i}{p^2-m^2}\left[(\not{p}+m)\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{m^2}\right)+\frac{1}{3}\left(\gamma_\mu+\frac{p_\mu}{m}\right)(\not{p}-m)\left(\gamma_\nu+\frac{p_\nu}{m}\right)\right]$$

$P_{\mu\nu}\, p^\nu$ e $P_{\mu\nu}\, \gamma^\nu$ non hanno poli

$$P_{\mu\nu}\, p^\nu = \frac{i}{p^2-m^2}\,\frac{1}{3}\left(\gamma_\mu + \frac{p_\mu}{m}\right)\underbrace{(\not p - m)}\underbrace{\left(\not p + \frac{p^2}{m}\right)}_{\not p + m + \frac{p^2-m^2}{m}} \qquad \text{OK}$$

$$\frac{\not p}{m^2} = \frac{\not p - m}{m^2} + \frac{1}{m}$$

$$P_{\mu\nu}\, \gamma^\nu = \frac{i}{p^2-m^2}\left[(\not p + m)\left(\gamma_\mu - p_\mu \frac{\not p}{m^2}\right) + \frac{1}{3}\left(\gamma_\mu + \frac{p_\mu}{m}\right)\times\right.$$

$$\left.\times(\not p - m)\underbrace{\left(4 + \frac{\not p}{m}\right)}_{3 + \frac{\not p + m}{m}}\right] = \frac{i}{p^2-m^2}\left[(\not p + m)\left(\gamma_\mu - \frac{p_\mu}{m}\right) + \right.$$

$$\left. + \left(\gamma_\mu + \frac{p_\mu}{m}\right)(\not p - m) + O(p^2-m^2)\right] =$$

$$= \frac{i}{p^2 - m^2}\Big[\cancel{\not p \gamma_\mu} - \cancel{\not p \frac{p_\mu}{m}} \cancel{+m\gamma_\mu} - \cancel{p_\mu} - \cancel{\not p \gamma_\mu} + 2\cancel{p_\mu} +$$

$$\cancel{-m\gamma_\mu} + \cancel{\frac{p_\mu}{m}\not p} \cancel{-p_\mu} + O(p^2 - m^2)\Big] \qquad \underline{\text{ok}}$$

A $m=0$ ho un'ulteriore simmetria di gauge : $p^\mu Q_{\mu\nu} = 0$

$$\mathcal{L} = -\bar{\psi}_\mu \, \varepsilon^{\mu\nu\rho\sigma} \gamma_5 \gamma_\nu \partial_\rho \psi_\sigma \qquad \delta\psi_\mu = \partial_\mu \epsilon$$

$\rightarrow \partial\cdot\psi = 0$ è una condizione di gauge-fixing

Gauge residua : $\Box \epsilon = 0$

$\rightarrow \gamma\cdot\psi = 0$ è un'altra condizione di gauge fixing.
(algebrica)

Gauge residua : $\not\partial \epsilon = 0$

Effetto Unruh

Un rivelatore che si muove di moto accelerato rivela particelle nel vuoto con uno spettro di corpo nero

Rivelatore: particella puntiforme con livelli energetici discreti $E_0, E_1, \ldots$

Accoppiamo questo rivelatore a un campo scalare $\phi(x)$ quantizzato

τ = tempo proprio del rivelatore, $x^\mu(\tau)$ la sua traiettoria

L'interazione ("$J^\mu A_\mu$") la descriviamo come

$$c(\tau)\ \chi(\tau)\ \phi(x(\tau))$$

funzione interruttore — il rivelatore

$$\begin{cases} c(\tau) = 1 & \text{per } -\frac{\Omega}{2} \le \tau \le \frac{\Omega}{2} \\ c(\tau) = 0 & \text{per } |\tau| > \frac{\Omega}{2} \end{cases}$$

Manderemo poi Ω all'infinito

Stato iniziale : $|E_0\rangle |0\rangle_\varphi$

Stato finale possibile : $|E_i\rangle |\psi\rangle_\varphi$ qualunque

Ampiezza di transizione

$$A_i = i \int_{-\infty}^{+\infty} d\tau\, c(\tau) \langle E_i|\, {}_\varphi\langle\psi|\, \chi(\tau)\, \phi(x(\tau))\, |E_0\rangle |0\rangle_\varphi$$

$$= i\int_{-\infty}^{+\infty} d\tau \; c(\tau) \langle E_i | \mathcal{X}(\tau) | E_0 \rangle \, {}_\varphi\langle \psi | \phi(x(\tau)) | 0 \rangle_\varphi$$

$$\langle E_i | \mathcal{X}(\tau) | E_0 \rangle = e^{i(E_i - E_0)\tau} \langle E_i | \mathcal{X}(0) | E_0 \rangle$$

$$\mathcal{X}(\tau) = e^{iH\tau} \mathcal{X}(0) e^{-iH\tau}$$

Probabilità di transizione

$$P_i = \sum_\psi \int_{-\infty}^{+\infty} d\tau \int_{+\infty}^{+\infty} d\tau' \; c(\tau) c(\tau') e^{i(E_i - E_0)(\tau - \tau')} |\langle E_i | \mathcal{X}(0) | E_0 \rangle|^2 \times$$

$$\times \; {}_\varphi\langle 0 | \phi(x(\tau')) | \psi \rangle_\varphi \, {}_\varphi\langle \psi | \phi(x(\tau)) | 0 \rangle_\varphi$$

$$\mathbb{1} = \sum_\psi |\psi\rangle_\varphi \, {}_\varphi\langle \psi |$$

$$P_i = \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' \, c(\tau) c(\tau') \, |\langle E_i | \chi(0) | E_0 \rangle|^2 \cdot$$

$$\cdot \; e^{i(E_i - E_0)(\tau - \tau')} \, {}_\varphi\langle 0 | \phi(x(\tau')) \phi(x(\tau)) | 0 \rangle_\varphi$$

Campo scalare libero e massless

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = \frac{1}{4\pi^2} \frac{1}{(\vec{x} - \vec{y})^2 - (x^0 - y^0 - i\epsilon)^2}$$

Potenziali ritardati

Potenziali anticipato $(y = 0)$ $\quad \dfrac{1}{4\pi^2} \dfrac{1}{\vec{x}^2 - (x^0 + i\epsilon)^2} \qquad \dfrac{1}{(p^0 - i\epsilon)^2 - \vec{p}^2}$

Feynman: $\quad \dfrac{1}{4\pi^2} \dfrac{1}{x^2 - i\epsilon} \qquad \dfrac{1}{p^2 + i\epsilon}$

Moto rettilineo uniforme:

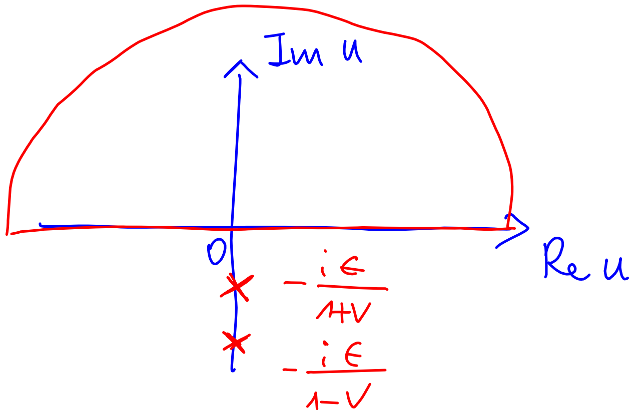
$$t(\tau) = t \qquad \vec{x}(\tau) = \vec{v}\,t + \vec{x}_0$$

$$\langle 0|\phi(x(\tau'))\,\phi(x(\tau))|0\rangle = \frac{1}{4\pi^2}\,\frac{1}{v^2(\tau'-\tau)^2-(\tau'-\tau-i\epsilon)^2}$$

$$u = \tau-\tau'$$

$$P_i = \frac{\Omega}{4\pi^2}\int_{-\infty}^{+\infty} du\,\frac{e^{iu(E_i-E_0)}}{v^2u^2-(u+i\epsilon)^2}\,|\langle E_i|\chi(0)|E_0\rangle|^2 = 0$$

per il teorema dei residui



$$E_i - E_0 > 0$$

$$\pm v u = u + i\epsilon$$

$$u\underbrace{(1\mp v)} = -i\epsilon$$

Moto naturalmente accelerato (iperbolico)

$$t = \frac{1}{g}\,\mathrm{senh}(g\tau) \qquad x = \frac{1}{g}\cosh(g\tau)$$

$$t^2 - x^2 = \frac{1}{g^2}(-1) = \mathrm{cost} \qquad \cancel{2}\,t\,dt = \cancel{2}\,x\,dx$$

$$\frac{dx}{dt} = \frac{t}{x} = \mathrm{tgh}(g\tau) \qquad \frac{dx^\mu}{dt} = (1, \mathrm{tgh}(g\tau), 0, 0)$$

$$\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = (\cosh(g\tau), \mathrm{senh}(g\tau), 0, 0) = u^\mu$$

$$\frac{dt}{d\tau} = \cosh(g\tau) \qquad u^\mu u_\mu = 1$$

$$a^\mu = \frac{du^\mu}{d\tau} = g(\mathrm{senh}(g\tau), \cosh(g\tau), 0, 0) \qquad \underline{a^\mu a_\mu = -g^2}$$

Dobbiamo calcolare $\qquad \bar{\epsilon} = g\epsilon$

$$\left(x(\tau') - x(\tau)\right)^2 - \left(\tau' - \tau - i\epsilon\right)^2 =$$

$$= \frac{1}{g^2}\left[\left(\cosh(g\tau') - \cosh(g\tau)\right)^2 - \left(\operatorname{senh}(g\tau') - \operatorname{senh}(g\tau) - i\bar{\epsilon}\right)^2\right] =$$

$$= \frac{2}{g^2}\left[1 - \cosh(g\tau')\cosh(g\tau) + \operatorname{senh}(g\tau')\operatorname{senh}(g\tau) + \right.$$
$$\left. + i\bar{\epsilon}\left(\operatorname{senh}(g\tau') - \operatorname{senh}(g\tau)\right)\right] =$$

$$= \frac{2}{g^2}\left[1 - \cosh\left(g(\tau' - \tau)\right) + i\bar{\epsilon}\left(\operatorname{senh}(g\tau') - \operatorname{senh}(g\tau)\right)\right] =$$

$$= -\frac{4}{g^2}\operatorname{senh}^2\left(g\frac{(\tau - \tau')}{2}\right) + \frac{2i\bar{\epsilon}}{g^2}\left(\operatorname{senh}(g\tau') - \operatorname{senh}(g\tau)\right) =$$

$$= -\frac{4}{g^2}\,\mathrm{senh}^2\left(\frac{g}{2}(\tau'-\tau-i\tilde{\epsilon})\right) = f(\tilde{\epsilon}) \qquad \text{per certa } \tilde{\epsilon}$$

$$= -\frac{4}{g^2}\,\mathrm{senh}^2\left(\frac{g}{2}(\tau'-\tau)\right) + \tilde{\epsilon}\,\underbrace{\left.\frac{df}{d\tilde{\epsilon}}\right|_{\tilde{\epsilon}=0}} + O(\tilde{\epsilon}^2)$$

$$\left.\frac{df}{d\tilde{\epsilon}}\right|_{\tilde{\epsilon}=0} = -\frac{4}{g^{\cancel{2}}}\,\cancel{2}\,\mathrm{senh}\left(\frac{g}{2}(\tau'-\tau)\right)\frac{\cancel{g}}{\cancel{2}}\cosh\left(\frac{g}{2}(\tau-\tau')\right)(-i) =$$

$$= \frac{2i}{g}\,\mathrm{senh}\left(g(\tau'-\tau)\right)$$

$$\left[\begin{aligned} -4\,\mathrm{senh}^2\alpha &= -\cancel{4}\left(\frac{e^{\alpha}-e^{-\alpha}}{\cancel{2}}\right)^2 = -\left(e^{2\alpha}+e^{-2\alpha}-2\right) = \\ &= 2-2\cosh(2\alpha) = 2\left(1-\cosh(2\alpha)\right) \end{aligned}\right]$$

$x = g\tau'$ $y = g\tau$ Dobbiamo mostrare che

$$\frac{\operatorname{senh} x - \operatorname{senh} y}{\operatorname{senh}(x-y)} > 0 \quad \forall x \ \forall y$$

$X = e^x$

$Y = e^y$

$$\frac{e^x - e^{-x} - e^y + e^{-y}}{e^{x-y} - e^{y-x}} = \frac{X - \frac{1}{X} - Y + \frac{1}{Y}}{\frac{X}{Y} - \frac{Y}{X}} =$$

$$= \frac{X^2Y - Y - XY^2 + X}{X^2 - Y^2} = \frac{\cancel{(X-Y)}(1+XY)}{\cancel{(X-Y)}(X+Y)} =$$

$$= \frac{1+XY}{X+Y} > 0 \quad \text{sempre}$$

$$P_i = \int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} d\tau' \, c(\tau) c(\tau') \, |\langle E_i | x(0) | E_0 \rangle|^2 \cdot$$

$$e^{i(E_i - E_0)(\tau - \tau')} \, {}_{\varphi}\langle 0 | \phi(x(\tau')) \phi(x(\tau)) | 0 \rangle_{\varphi} =$$

$u = \tau - \tau'$

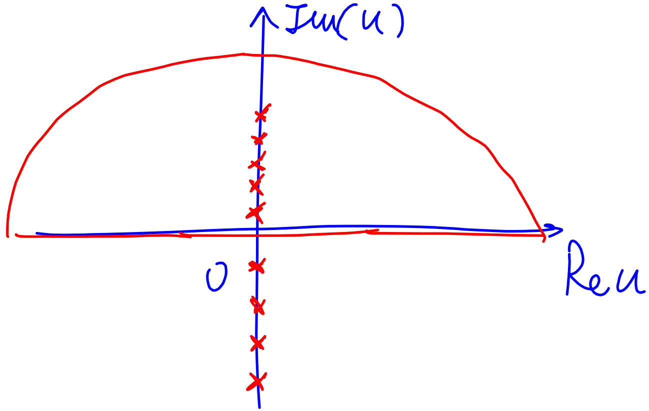
$$= -\frac{g^2 \Omega}{(4\pi)^2} |\langle E_i | x(0) | E_0 \rangle|^2 \int_{-\infty}^{+\infty} du \, \frac{e^{i(E_i - E_0)u}}{\operatorname{senh}^2\left(\frac{g}{2}(u + i\epsilon)\right)}$$

$$\frac{1}{\operatorname{senh}^2(\pi y)} = \frac{1}{\pi^2} \sum_{k=-\infty}^{+\infty} \frac{1}{(y + ik)^2} \qquad y = \frac{g}{2\pi}(u + i\epsilon)$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{n+1}} \qquad dy = \frac{g}{2\pi} du$$

$$\frac{\partial}{\partial y} = \frac{2\pi}{g} \frac{\partial}{\partial u} \qquad du = \frac{2\pi}{g} dy$$

Poli: $\frac{g}{2\pi}(u+i\epsilon) = -ik \qquad k \in \mathbb{Z}$



$$u = -ik\frac{2\pi}{g} - i\epsilon$$

$$\frac{P_i}{\Omega} = W_i = -\frac{g^2}{(4\pi)^2}\left|\langle E_i | \chi(0) | E_0\rangle\right|^2 \frac{1}{\pi^2} 2\pi i \left(\frac{2\pi}{g}\right)^2 \sum_{k=-1}^{-\infty} i(E_i - E_0) \cdot e^{i(E_i - E_0)\left(-\frac{2\pi}{g}ik - i\epsilon\right)} =$$

$$= \frac{E_i - E_0}{2\pi}\left|\langle E_i | \chi(0) | E_0\rangle\right|^2 \sum_{n=1}^{\infty} e^{-\frac{2\pi}{g}(E_i - E_0)n} =$$

$$= \frac{E_i - E_0}{2\pi} |\langle E_i | X(0) | E_0 \rangle|^2 \left[\frac{1}{1 - e^{-\frac{2\pi}{g}(E_i - E_0)}} - 1 \right] =$$

$$= \frac{E_i - E_0}{2\pi} \frac{|\langle E_i | X(0) | E_0 \rangle|^2}{e^{\frac{2\pi}{g}(E_i - E_0)} - 1} \qquad T = \frac{g}{2\pi}$$

$$\frac{1}{1 - e^{-a}} - 1 = \frac{e^{-a}}{1 - e^{-a}} = \frac{1}{e^{a} - 1}$$

Problema variazionale in gravità

$$\int R\sqrt{-g} = \int \mathcal{L}(g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \partial_\sigma \partial_\rho g_{\mu\nu})$$

$$\int dt\, \mathcal{L}(q, \dot{q}, \ddot{q})$$

$$\mathcal{L} = \frac{1}{2}\dot{q}^2 - V(q) \qquad S(q) = \int_{t_0}^{t_1} \mathcal{L}(q(t), \dot{q}(t))\,dt$$

$$\delta S = \int dt \left(\frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta \dot{q} \right) =$$

$$= \int dt \left(\frac{\partial \mathcal{L}}{\partial q}\delta q - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q \right) +$$

$$+ \int_{t_0}^{t_1} dt\, \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q \right) \quad " \quad \left. \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q \right|_{t_0}^{t_1}$$

Si impone $\delta q(t_1) = \delta q(t_0)$

$$\mathcal{L} = -\frac{1}{2} q \ddot{q} - V(q) \qquad S(q) = \int_{t_0}^{t_1} dt \; \mathcal{L}(q(t), \ddot{q}(t))$$

$$\delta S = \int dt \left(\frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{\partial \mathcal{L}}{\partial \ddot{q}} \delta \ddot{q} \right) =$$

$$= \int dt \left(\frac{\partial \mathcal{L}}{\partial q} \delta q - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \ddot{q}} \delta \dot{q} \right) + \left. \frac{\partial \mathcal{L}}{\partial \ddot{q}} \delta \dot{q} \right|_{t_0}^{t_1} =$$

$$= \int dt \left(\frac{\partial \mathcal{L}}{\partial q} \delta q + \frac{d^2}{dt^2} \frac{\partial \mathcal{L}}{\partial \ddot{q}} \delta q \right) - \left. \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \ddot{q}} \right) \delta q \right|_{t_0}^{t_1} +$$

$$+ \left. \frac{\partial \mathcal{L}}{\partial \ddot{q}} \delta \dot{q} \right|_{t_0}^{t_1}$$

Occorre imporre $\delta q(t_1) = \delta q(t_0) = \delta \dot{q}(t_1) = \delta \dot{q}(t_0) = 0$

$$0 = \frac{\partial \mathcal{L}}{\partial q} + \frac{d^2}{dt^2}\frac{\partial \mathcal{L}}{\partial \ddot{q}} = -\frac{\ddot{q}}{2} - \frac{\partial V}{\partial q} - \frac{\ddot{q}}{2}$$

$$\ddot{q} = -\frac{\partial V}{\partial q}$$

$$S(g, \Gamma(g)) = -\frac{1}{2k^2}\int_M \left[\partial_\lambda w^\lambda - \sqrt{-g}\, g^{\mu\nu}\left(\Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\lambda\alpha} - \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\nu\alpha}\right)\right]$$

$$w^\lambda = \sqrt{-g}\left(g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} - g^{\mu\lambda}\Gamma^{\nu}_{\mu\nu}\right)$$

$$\int_M \partial_\lambda w^\lambda = \int_{\partial M} w^\lambda \sigma_\lambda \qquad \sigma_\lambda = \sqrt{-g}\,\varepsilon_{\lambda\alpha\beta\gamma}\frac{dx^\alpha dx^\beta dx^\gamma}{3!}$$

Cerchiamo $\int_{\partial M} \Omega$ tale che $\delta \int_{\partial M} \Omega = \delta \int_{\partial M} w^\lambda \sigma_\lambda$

manifest. covariante

Sottovarietà

M = varietà di dimensione 4 , con metrica $g_{\mu\nu}$

Σ = sottovarietà di dimensione 3

$\exists$ n vettore normale a Σ ed è unico a meno della normalizzazione

Consideriamo una base V_1, V_2, V_3 di vettori tangenti a Σ in un dato punto p

Sia V_0 un vettore dello spazio tangente a M in p linearmente indipendente da V_1, V_2, V_3.

$\{V_0, V_1, V_2, V_3\} = \{V_\mu\}$ è una base di $T_p(M)$

Le componenti $v_\mu^{\ \nu}$ formano una matrice invertibile

Sia $G_{\mu\nu} = (v_\mu, v_\nu) = g(v_\mu, v_\nu) = g_{\alpha\beta} v_\mu^{\ \alpha} v_\nu^{\ \beta}$

Sia $G^{\mu\nu}$ l'inversa di $G_{\mu\nu}$

Il vettore normale è $\tilde{n} = G^{0\mu} v_\mu$ $\quad (\tilde{n}^\nu = G^{0\mu} v_\mu^{\ \nu})$

a meno della normalizzazione

$$(\tilde{n}, v_i) = g_{\alpha\beta} \tilde{n}^\alpha v_i^{\ \beta} = g_{\alpha\beta} G^{0\mu} v_\mu^{\ \alpha} v_i^{\ \beta} =$$

$$= G^{0\mu} G_{\mu i} = \delta_i^0 = 0$$

Sia $\tilde{n}'$ altro vettore normale a Σ in p.

$\exists\, b^\mu \,/\, \quad \tilde{n}' = b^\mu v_\mu$

Voglio $(\tilde{n}', v_i) = 0 \quad \forall i$

$$0 = g_{\alpha\beta}\, \tilde{n}'^{\alpha} v_i^{\beta} = g_{\alpha\beta}\, b^{\mu} v_{\mu}^{\alpha} v_i^{\beta} = b^{\mu} G_{\mu i}$$

$$\tilde{n}' = b^{\mu} v_{\mu} = b^{\mu} G_{\mu\alpha} G^{\alpha\nu} v_{\nu} =$$

$$= b^{\mu} G_{\mu 0} G^{0\nu} v_{\nu} = \underbrace{(b^{\mu} G_{\mu 0})}_{\text{costante di proporzionalità}} \tilde{n}$$

Il vettore normale è unico una volta normalizzato (se si può). Lo chiameremo n, n^{μ}

$$n^2 = (n, n) = g_{\alpha\beta}\, n^{\alpha} n^{\beta} = \begin{cases} 1 & \text{tipo tempo } (\Sigma \text{ tipo spazio}) \\ -1 & \text{tipo spazio } (\Sigma \text{ tipo tempo}) \\ 0 & \text{tipo luce} \end{cases}$$

Ci concentriamo sui casi $n^2 = \pm 1$

Posso considerare la base $w_{\mu} = (n, v_1, v_2, v_3)$

La metrica di Σ è $\quad h_{\mu\nu} = g_{\mu\nu} \mp n_\mu n_\nu \qquad (n^2 = \pm 1)$

$$n_\mu = g_{\mu\nu} n^\nu$$

$h_\mu^{\ \nu} = \delta_\mu^\nu \mp n_\mu n^\nu$ è un proiettore che proietta sullo spazio tangente a Σ

$$h_\mu^{\ \nu} h_\nu^{\ \rho} = \delta_\mu^\rho \mp 2 n_\mu n^\rho + n_\mu n^\nu n_\nu n^\rho = \delta_\mu^\rho \mp n_\mu n^\rho = h_\mu^{\ \rho}$$

$$h_\mu^{\ \nu} v_i^{\ \mu} = v_i^{\ \nu} \mp n_\mu n^\nu v_i^{\ \mu} = v_i^{\ \nu}$$

$$h_\mu^{\ \nu} n^\mu = n^\nu \mp n_\mu n^\nu n^\mu = n^\nu - n^\nu = 0$$

$$\tilde{h}_{\mu\nu} \equiv w_\mu^{\ \alpha} h_{\alpha\beta} w_\nu^{\ \beta} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & -\gamma & \\ 0 & & & \end{array} \right)$$

γ = metrica di Σ in P

Sia Σ definita da un'equazione

$$\chi(t,x,y,z) = 0 \qquad \chi(x) = 0$$

$n_\mu \propto \nabla_\mu \chi$ Sia γ : $x(\tau)$ una curva in Σ

$$\chi(x(\tau)) = 0 \quad \forall \tau \Rightarrow \frac{d}{d\tau}\chi(x(\tau)) = \frac{dx^\mu}{d\tau}\nabla_\mu \chi(x(\tau)) = 0$$

$$\Rightarrow n_\mu = \lambda \nabla_\mu \chi$$

Da adesso in poi assumiamo $n^2 = 1 = \lambda^2 \nabla_\mu \chi \nabla^\mu \chi$

Curvatura estrinseca

$$K_{\mu\nu} = h_\mu^{\ \rho} h_\nu^{\ \sigma} \nabla_\rho n_\sigma$$

Abbiamo anche $K_{\mu\nu} = h_\mu^{\ \rho} \nabla_\rho n_\nu$

Infatti $\quad K_{\mu\nu} = h_\mu{}^\rho \left(\delta_\nu^\sigma - \cancel{n_\nu n^\sigma} \right) \nabla_\rho n_\sigma = h_\mu^\rho \nabla_\rho n_\nu$

$$\nabla_\rho (n^2) = \nabla_\rho (1) = 0 = 2 n^\sigma \nabla_\rho n_\sigma$$

(assumendo compatibilità metrica)

Inoltre $\quad K_{\mu\nu} = K_{\nu\mu} \qquad\qquad n_\mu = \lambda \nabla_\mu \chi$

Infatti, $\quad K_{\mu\nu} - K_{\nu\mu} = h_\mu^\rho h_\nu^\sigma \left(\nabla_\rho n_\sigma - \nabla_\sigma n_\rho \right) =$

$$= h_\mu^\rho h_\nu^\sigma \left(\nabla_\rho \lambda \nabla_\sigma \chi + \cancel{\lambda \nabla_\rho \nabla_\sigma \chi} - \nabla_\sigma \lambda \nabla_\rho \chi - \cancel{\lambda \nabla_\sigma \nabla_\rho \chi} \right) =$$

$$= h_\mu^\rho h_\nu^\sigma \left(\frac{\nabla_\rho \lambda}{\lambda} n_\sigma - \frac{\nabla_\sigma \lambda}{\lambda} n_\rho \right) = 0$$

$K_{\mu\nu}$ è $\left(\frac{1}{2}\right)$ la derivata di Lie di $h_{\mu\nu}$ lungo n

$$K_{\mu\nu} \overset{?}{=} \frac{1}{2}\mathcal{L}_n h_{\mu\nu} = \frac{1}{2}\left[n^\rho \partial_\rho h_{\mu\nu} + h_{\mu\rho}\partial_\nu n^\rho + h_{\nu\rho}\partial_\mu n^\rho \right] =$$

$$= \frac{1}{2}\left[n^\rho \nabla_\rho h_{\mu\nu} + h_{\mu\rho}\nabla_\nu n^\rho + h_{\nu\rho}\nabla_\mu n^\rho \right] +$$

$$+ \frac{1}{2}\left[\cancel{n^\rho \Gamma^\sigma_{\rho\mu} h_{\sigma\nu}} + \cancel{n^\rho \Gamma^\sigma_{\rho\nu} h_{\sigma\mu}} - \cancel{h_{\mu\rho}\Gamma^\rho_{\nu\sigma} n^\sigma} - \cancel{h_{\nu\rho}\Gamma^\rho_{\mu\sigma} n^\sigma} \right] =$$

$$= \frac{1}{2}\left[-n^\rho \nabla_\rho (n_\mu n_\nu) + g_{\mu\rho}\nabla_\nu n^\rho - \cancel{n_\mu n_\rho \nabla_\nu n^\rho} + \right.$$

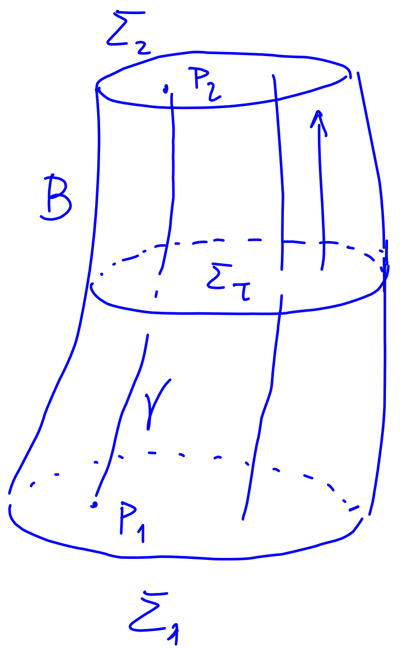
$$\left. + g_{\nu\rho}\nabla_\mu n^\rho - \cancel{n_\nu n_\rho \nabla_\mu n^\rho} \right] = \frac{1}{2}\left[-n^\rho \nabla_\rho n_\mu \, n_\nu + \right.$$

$$\left. - n^\rho \nabla_\rho n_\nu \, n_\mu + \nabla_\nu n_\mu + \nabla_\mu n_\nu \right]$$

$$= \frac{1}{2}\left[h_\mu^{\ \rho} \nabla_\rho n_\nu + h_\nu^{\ \rho}\nabla_\rho n_\mu \right] = \frac{1}{2}\left(K_{\mu\nu} + K_{\nu\mu} \right) =$$

$$= K_{\mu\nu}$$

Sia Σ_τ una famiglia di sottovarietà di tipo spazio parametrizzata da τ ("tempo")



$M = \bigcup_\tau \Sigma_\tau$ si dice foliazione della varietà

Σ_τ sia descritta da $\chi_\tau(x) = 0$

Per esempio $\chi_\tau(x) = \chi(x) - \tau$

$\Sigma_1 = \Sigma_{\tau_i}$ $\qquad \tau_i$ = tempo iniziale

$\Sigma_2 = \Sigma_{\tau_f}$ $\qquad \tau_f$ = tempo finale

Sia $\gamma: x(\tau)$ una curva tale che

$x(\tau_i) = P_1 \in \Sigma_1 \quad x(\tau_f) = P_2 \in \Sigma_2 \quad x(\tau) \in \Sigma_\tau \ \forall \tau$

Abbiamo $\chi(x(\tau)) - \tau = 0 \quad \forall \tau$

Se lo faccio $\forall P_1 \in \Sigma_1$ ottengo un flusso di diffeomorfismi che mappano Σ_τ in $\Sigma_{\tau'}$

Derivando $\chi(x(\tau)) - \tau = 0$ rispetto a τ ottengo

$$\nabla_\mu \chi \frac{dx^\mu}{d\tau} = 1 = n_\mu \frac{1}{\lambda} \frac{dx^\mu}{d\tau} \equiv n_\mu \delta^\mu$$

$\Rightarrow \delta^\mu$ campo vettoriale che mappa Σ_τ in $\Sigma_{\tau+\delta}$

δ^μ non è unico

Lo scompongo nella componente normale e nella componente tangente a Σ_τ

$$N^\mu \equiv \delta^\mu - n^\mu \delta^\nu n_\nu \qquad n_\mu N^\mu = n_\mu \delta^\mu - n_\mu n^\mu n_\nu \delta^\nu = 0$$

$$(n^2 = 1)$$

$$N \equiv n^\mu \delta_\mu \qquad \delta^\mu = N^\mu + n^\mu N$$

tangente, normale

Sia Σ_τ = tutto lo spazio a $t = \tau$

$$\chi(x) = t \qquad \chi(x(\tau)) - \tau = t - \tau$$

$\nabla_\mu \chi = (1, 0, 0, 0)$ Posso scegliere $\delta^\mu = (1, 0, 0, 0)$

$$n_\mu = \lambda \nabla_\mu \chi = (\lambda, 0, 0, 0) \qquad n_\mu = (N, \vec{0})$$

$$\underline{N^\mu = (1, \vec{0}) - n^\mu N} \qquad N = \delta^\mu n_\mu = n_0$$

$$N^\mu n_\mu = 0 = N N^0 \qquad N^0 = 0 = 1 - n^0 N \qquad n^0 = \frac{1}{N}$$

$$N^\mu = (0, N^i) \qquad N^i = -n^i N \qquad n^i = -\frac{N^i}{N}$$

$$n^\mu = \frac{1}{N}(1, -N^i) \qquad h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$$

$$h_{00} = g_{00} - N^2 \qquad h_{0i} = g_{0i} \qquad h_{ij} = g_{ij} \equiv -k_{ij}$$

k_{ij} = metrica su Σ

$$0 = n_i = g_{i\mu} n^\mu = \frac{1}{N}(g_{i0} - h_{ij} N^j)$$

$$g_{0i} = h_{ij} N^j = -k_{ij} N^j \equiv -N_i$$

$$n^0 = \frac{1}{N} = n_\mu g^{\mu 0} = N g^{00} \qquad g^{00} = \frac{1}{N^2}$$

$$1 = n_\mu n^\mu = n^\mu g_{\mu\nu} n^\nu = \frac{1}{N^2}(g_{00} - 2 g_{0i} N^i + g_{ij} N^i N^j)$$

$$N^2 = g_{00} + 2N_iN^i - N_iN^i \implies g_{00} = N^2 - N_iN^i$$

$$g_{\mu\nu} = \begin{pmatrix} N^2 - N_iN^i & -N_i \\ -N_i & -k_{ij} \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} \dfrac{1}{N^2} & -\dfrac{N^i}{N^2} \\ -\dfrac{N^j}{N^2} & \dfrac{N^iN^j}{N^2} - k^{ij} \end{pmatrix}$$

$$g_{\mu\nu}\, g^{\nu\rho} = \begin{pmatrix} 1 - \cancel{\dfrac{N_iN^i}{N^2}} + \cancel{\dfrac{N_iN^i}{N^2}} & \cancel{-N^i} + \cancel{\dfrac{N_kN^kN^i}{N^2}} - \cancel{\dfrac{N_kN^kN^i}{N^2}} \cancel{+N^i} \\ 0 & \cancel{\dfrac{N_jN^i}{N^2}} - \cancel{\dfrac{N_jN^i}{N^2}} + \delta^i_j \end{pmatrix}$$

$$= \delta_\nu^\rho$$

$$g = \det g_{\mu\nu} = \det \begin{pmatrix} N^2 - N_i N^i & -N_1 & -N_2 & -N_3 \\ -N_1 & -K_{11} & -K_{12} & -K_{13} \\ -N_2 & -K_{12} & -K_{22} & -K_{23} \\ -N_3 & -K_{13} & -K_{23} & -K_{33} \end{pmatrix} =$$

$$= -\det(K_{ij})\left(N^2 - N_\ell N^\ell\right) + N_1 \left[-N_1 K^{11} K - N_2 K K^{21} - N_3 K K^{31}\right] +$$

$$+ \dots = -K\left(N^2 - \cancel{N_i N^i}\right) - K \cancel{N_i K^{ij}} N_j = -K N^2$$

$$\sqrt{-g} = N\sqrt{K} \quad \text{assumendo } N > 0$$

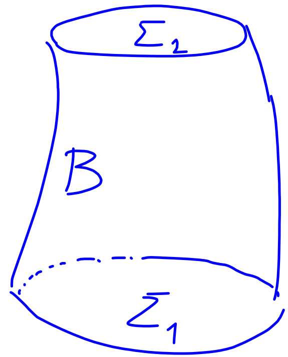
$$K_{\mu\nu} = h_\mu^{\ \rho} h_\nu^{\ \sigma} \nabla_\rho n_\sigma = h_\mu^{\ \rho} \nabla_\rho n_\nu \qquad \bar{K} = g^{\mu\nu} K_{\mu\nu}$$

Vogliamo dimostrare che

$$\delta \int_{\Sigma_2} w^\lambda \sigma_\lambda = 2\delta \int_{\Sigma_2} \sqrt{K}\, \bar{K}\, d^3x$$

$$K = g^{\mu\nu} h_\mu^{\rho} \nabla_\rho n_\nu = h^{\mu\nu} \nabla_\mu n_\nu = h_\mu^{\nu} \nabla_\nu n^\mu$$

$$\partial M = \Sigma_2 \cup \Sigma_1 \cup B$$



$$w^\lambda = g^{\mu\nu} \Gamma^\lambda_{\mu\nu} - g^{\mu\lambda} \Gamma^\nu_{\mu\nu}$$

$$\sigma_\lambda = \sqrt{-g}\, \varepsilon_{\lambda\alpha\beta\gamma} \frac{dx^\alpha dx^\beta dx^\gamma}{3!}$$

Problema variazionale: $\delta g_{\mu\nu} = 0$ su ∂M

$\delta N = 0$, $\delta N^i = 0$, $\delta K_{ij} = 0$, $\delta n_\mu = \delta n^\mu = 0$

$\delta h_{\mu\nu} = 0$ Non possiamo dire nulla su $\partial_\mu \delta g_{\rho\sigma}$ su ∂M, $\partial_\mu n^\nu$ ... ecc.

Su ∂M $\quad \partial_\mu \delta n^\nu = C^\nu n_\mu \quad C^\nu =$ certe funzioni

perché Σ_2 è dato da $\underbrace{\chi(x) = 0}$

$n_\mu \propto \partial_\mu \chi$ e su Σ_2 $\quad \delta n^\nu = 0$

$n_\mu \propto \partial_\mu \delta n^\nu \quad \forall \nu$

$$I \equiv \delta \int_{\Sigma_2} \sqrt{k}\, K \, d^3x = \int_{\Sigma_2} \delta \left(\sqrt{k}\, h^\nu_\mu \nabla_\nu n^\mu \right) d^3x =$$

$$= \int_{\Sigma_2} \sqrt{k}\, h^\nu_\mu \, \delta\left(\partial_\nu n^\mu + \Gamma^\mu_{\nu\rho} n^\rho \right) d^3x =$$

$$= \int_{\Sigma_2} \sqrt{k}\, h^\nu_\mu \left(\cancel{n_\nu C^\mu} + \delta \Gamma^\mu_{\nu\rho} n^\rho \right) d^3x =$$

$$= \int_{\Sigma_2} \sqrt{k}\, h_\mu^{\ \nu}\, \delta\Gamma^\mu_{\nu\rho}\, n^\rho\, d^3x = \int_{\Sigma_2} \sqrt{k} \left[\delta\Gamma^\mu_{\mu\rho}\, n^\rho - \underline{n_\mu n^\nu n^\rho \delta\Gamma^\mu_{\nu\rho}} \right] d^3x$$

$$h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$$

Possiamo anche scrivere

$$I = \delta \int_{\Sigma_2} \sqrt{k}\, h^{\mu\nu}\, \nabla_\mu n_\nu\, d^3x = \int_{\Sigma_2} \sqrt{k}\, h^{\mu\nu}\, \delta\left(\partial_\mu n_\nu - \Gamma^\rho_{\mu\nu} n_\rho\right) d^3x =$$

$$= \int_{\Sigma_2} \sqrt{k}\, h^{\mu\nu} \left[\cancel{n_\mu C_\nu} - \delta\Gamma^\rho_{\mu\nu}\, n_\rho \right] d^3x =$$

$$= \int_{\Sigma_2} \sqrt{k} \left[- g^{\mu\nu} \delta\Gamma^\rho_{\mu\nu}\, n_\rho + \underline{n^\mu n^\nu n_\rho\, \delta\Gamma^\rho_{\mu\nu}} \right] d^3x$$

$$I = \frac{1}{2} \int_{\Sigma_2} \sqrt{k} \left[\delta\Gamma^\mu_{\mu\rho}\, n^\rho - g^{\mu\nu}\, \underline{\delta\Gamma^\rho_{\mu\nu}}\, n_\rho \right] d^3x$$

$$\delta \int_{\Sigma_2} w^\lambda \sigma_\lambda = - \delta \int_{\Sigma_2} \left(g^{\mu\nu} \Gamma^0_{\mu\nu} - g^{\mu 0} \Gamma^\nu_{\mu\nu} \right) N \sqrt{k}\, d^3x =$$

$$\lambda = 0 \qquad n_\mu = (N, \vec{0}) \qquad \varepsilon_{0123} = -1$$

$$= - \delta \int_{\Sigma_2} n_\lambda \left(g^{\mu\nu} \Gamma^\lambda_{\mu\nu} - g^{\mu\lambda} \Gamma^\nu_{\mu\nu} \right) \sqrt{k}\, d^3x =$$

$$= \int_{\Sigma_2} \sqrt{k} \left[- g^{\mu\nu} \delta \Gamma^\lambda_{\mu\nu} n_\lambda + n^\mu \delta \Gamma^\nu_{\mu\nu} \right] d^3x = 2I$$

Pertanto, $\delta \int_{\Sigma_2} w^\lambda \sigma_\lambda = 2 \delta \int_{\Sigma_2} \sqrt{k}\, K\, d^3x$

$$S(g, \Gamma(g)) = - \frac{1}{2k^2} \int_M \partial_\lambda w^\lambda + S_{\Gamma\Gamma} = S_H = - \frac{1}{2k^2} \int \sqrt{-g} R$$

$$S_{\Gamma\Gamma} = S_H + \frac{1}{2k^2} \int_{\partial M} w^\lambda \sigma_\lambda$$

Azione traccia K (azione con termine di bordo)

$$S_K = S_H + \frac{1}{2k^2}\left[2\left(\int_{\Sigma_2} - \int_{\Sigma_1} \right) \sqrt{h}\, K \, d^3x + \right.$$

$$\left. + \int_B \sqrt{\gamma}\, H \, d^3x \right]$$

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$ metrica su B

$$n_\mu n^\mu = -1$$

$$\gamma = \det(\gamma_{\alpha\beta}) \qquad \gamma_{\mu\nu} = \begin{pmatrix} \times & \times & \times & \times \\ \times & & & \\ \times & & \gamma_{\alpha\beta} & \\ \times & & & \end{pmatrix}$$

$$H = \gamma_\mu{}^\nu \nabla_\nu n^\mu$$

$$\delta S_K = \delta S_{\Gamma\Gamma}$$ Problema variazionale ben definito

Cariche conservate

$$\partial_\mu F^{\mu\nu} = J^\nu \qquad Q(t) = \int d^3\vec{x}\, J^0(t,\vec{x}) =$$

$$= \int d^3\vec{x}\, \partial_\mu F^{\mu 0} =$$

$$= \int d^3\vec{x}\, \partial_i F^{i0} =$$

$$= \int_{S^2(\infty)} d\sigma\, F^{i0} n_i$$

Energia del campo gravitazionale

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \qquad S_{tot} = S_{HE} + S_m$$

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \Lambda g^{\mu\nu} = \kappa^2 T^{\mu\nu}$$

$D_\mu T^{\mu\nu} = 0$ sulle soluzioni delle eq. del moto

$T^{\mu\nu}$ è covariantemente conservato, ma NON conservato

Se fosse $\partial_\mu T^{\mu\nu} = 0 = \partial_0 T^{0\nu} + \partial_i T^{i\nu}$

$$P^\mu \equiv \int_{\mathbb{R}^3} d^3x \; T^{0\mu}(t, \vec{x}) = P^\mu(t)$$

$$\frac{dP^\mu}{dt} = \int_{\mathbb{R}^3} d^3x \, \partial_0 T^{0\mu}(t,\vec{x}) = -\int_{\mathbb{R}^3} d^3x \, \partial_i T^{i\mu}(t,\vec{x})$$

$$\underbrace{R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \Lambda g^{\mu\nu}}_{\equiv E_{\mu\nu}} = \kappa^2 T^{\mu\nu}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa \phi_{\mu\nu}$$

$$E_{\mu\nu}(g) = E_{\mu\nu}(\eta) + 2\kappa \int \frac{\delta E_{\mu\nu}}{\delta g_{\rho\sigma}}(\eta) \, \phi_{\rho\sigma} + \kappa^2 X_{\mu\nu}$$

$$X_{\mu\nu} = O(\phi^2)$$

$$E_{\mu\nu}(\eta) + 2\kappa \int \frac{\delta E_{\mu\nu}}{\delta g_{\rho\sigma}}(\eta) \, \phi_{\rho\sigma} \equiv \kappa^2 \tilde{T}_{\mu\nu} \qquad \tilde{T}_{\mu\nu} = T_{\mu\nu} - X_{\mu\nu}$$

$$\partial^\mu \tilde{T}_{\mu\nu} = 0$$

attorno a un background $\bar{g}_{\mu\nu}$ che soddisfa

$$\bar{R}^{\mu\nu} - \frac{1}{2}\bar{g}^{\mu\nu}\bar{R} - \Lambda \bar{g}^{\mu\nu} = 0$$

scrivo $g_{\mu\nu} = \bar{g}_{\mu\nu} + \varepsilon h_{\mu\nu}$ ε = parametro di espansione ($= 2\kappa$ attorno al piatto)

e definisco

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \Lambda g^{\mu\nu} \equiv \varepsilon K^{\mu\nu} + O(\varepsilon^2)$$

$$\nabla_\mu = \bar{\nabla}_\mu + O(\varepsilon) = \text{"}\partial_\mu + \Gamma_\mu\text{"} \qquad \Gamma_\mu = \bar{\Gamma}_\mu + O(\varepsilon)$$

$$0 = \nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R - \Lambda g^{\mu\nu} \right) = \left(\bar{\nabla}_\mu + O(\varepsilon) \right) \left(\varepsilon K^{\mu\nu} + O(\varepsilon^2) \right) = \varepsilon \bar{\nabla}_\mu K^{\mu\nu} + O(\varepsilon^2) \Longrightarrow \bar{\nabla}_\mu K^{\mu\nu} = 0$$

Poniamo $\varepsilon=1$ e definiamo (con $\bar{g}_{\mu\nu}=\eta_{\mu\nu}$)

$$R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R-\Lambda g^{\mu\nu}=K^{\mu\nu}-\kappa^2 t^{\mu\nu}\equiv\kappa^2 T^{\mu\nu}$$

$$\Rightarrow \quad K^{\mu\nu}=\kappa^2\tilde{T}^{\mu\nu} \qquad \tilde{T}^{\mu\nu}=T^{\mu\nu}+t^{\mu\nu}$$

$$\partial_\mu K^{\mu\nu}=0 \Rightarrow \partial_\mu\tilde{T}^{\mu\nu}=0$$

$$P^\mu=\int_{\mathbb{R}^3}d^3\vec{x}\,\tilde{T}^{\mu 0}(t,\vec{x}) \qquad \frac{dP^\mu}{dt}=0 \quad \forall\mu$$

Mettiamo $\Lambda=0$ per semplicità

$$R^\alpha{}_{\lambda\rho\sigma}=\partial_\rho\Gamma^\alpha_{\lambda\sigma}-\partial_\sigma\Gamma^\alpha_{\lambda\rho}+O(\Gamma^2)$$

$$R_\mu^{\ \nu} - \frac{1}{2}\delta_\mu^\nu R = -\frac{1}{4}\delta^{\nu\rho\sigma}_{\mu\alpha\beta} R^{\alpha\beta}_{\ \ \rho\sigma} = -\frac{1}{4}\begin{vmatrix} \delta_\mu^\nu & \delta_\mu^\rho & \delta_\mu^\sigma \\ \delta_\alpha^\nu & \delta_\alpha^\rho & \delta_\alpha^\sigma \\ \delta_\beta^\nu & \delta_\beta^\rho & \delta_\beta^\sigma \end{vmatrix} R^{\alpha\beta}_{\ \ \rho\sigma} =$$

$$= -\frac{1}{4}\left[2\delta_\mu^\nu R - 2\, R_\mu^{\ \nu}\ 2\right] \overset{\text{ok!}}{=}$$

$$= -\frac{1}{4} 2\,\delta^{\nu\rho\sigma}_{\mu\alpha\beta}\, \eta^{\beta\lambda}\, \partial_\rho \Gamma^\alpha_{\ \lambda\sigma} + O(h^2) =$$

$$= \frac{1}{2}\partial_\rho Q_\mu^{\ \rho\nu} + O(h^2) \qquad Q_\mu^{\ \rho\nu} = -\delta^{\nu\rho\sigma}_{\mu\alpha\beta}\,\eta^{\beta\lambda}\,\Gamma^\alpha_{\ \lambda\sigma}$$

$$K_\mu^{\ \nu} = \frac{1}{2}\partial_\rho Q_\mu^{\ \rho\nu} = \kappa^2 \tilde{T}_\mu^{\ \nu} \qquad Q_\mu^{\ \rho\nu} = -Q_\mu^{\ \nu\rho}$$

$$\Rightarrow \quad \partial_\nu K_\mu^{\ \nu} = 0 \quad \Rightarrow \quad \partial_\nu \tilde{T}_\mu^{\ \nu} = 0$$

$$P_\mu = \int_{\mathbb{R}^3} d^3\vec{x}\ \tilde{T}_\mu^{\ 0} = \frac{1}{2\kappa^2}\int_{\mathbb{R}^3} d^3x\, \partial_\rho Q_\mu^{\rho 0} =$$

$$= \frac{1}{2\kappa^2}\int_{\mathbb{R}^3} d\sigma_\nu\, \partial_\rho Q_\mu^{\rho\nu} = -\frac{1}{2\kappa^2}\int_{\mathbb{R}^3} \partial_\rho Q_\mu^{\rho\nu}\, \varepsilon_{\nu\alpha\beta\gamma}\frac{dx^\alpha dx^\beta dx^\gamma}{3!} =$$

$$= \frac{1}{\kappa^2}\int_{\mathbb{R}^3} dQ_\mu = \frac{1}{\kappa^2}\int_{S^2(\infty)} Q_\mu =$$

$$= \frac{1}{8\kappa^2}\int_{\mathbb{R}^3} \partial_\rho Q_\mu^{\xi\zeta}\, \underbrace{\overbrace{\varepsilon^{\rho\nu\lambda\tau}}\ \overbrace{\varepsilon_{\lambda\tau\xi\zeta}}}_{-2(\delta^\rho_\xi\delta^\nu_\zeta - \delta^\rho_\xi\delta^\nu_\zeta)}\ \overbrace{\varepsilon_{\nu\alpha\beta\gamma}}\frac{dx^\alpha dx^\beta dx^\gamma}{3!}$$

$$= \frac{1}{8\kappa^2} \int_{\mathbb{R}^3} \partial_\rho Q_\mu^{\xi\zeta} \, \varepsilon_{\lambda\tau\xi\zeta} \begin{vmatrix} \delta^\rho_\alpha & \delta^\lambda_\alpha & \delta^\tau_\alpha \\ \delta^\rho_\beta & \delta^\lambda_\beta & \delta^\tau_\beta \\ \delta^\rho_\gamma & \delta^\lambda_\gamma & \delta^\tau_\gamma \end{vmatrix} \frac{dx^\alpha dx^\beta dx^\gamma}{3!} =$$

$$= \delta^{\rho\lambda\tau}_{\alpha\beta\gamma}$$

$$= \frac{1}{8\kappa^2} \int_{\mathbb{R}^3} \sum \partial_\rho Q_\mu^{\xi\zeta} \, \varepsilon_{\lambda\tau\xi\zeta} \, dx^\rho dx^\lambda dx^\tau =$$

$$= \frac{1}{8\kappa^2} \int_{\mathbb{R}^3} d\left[Q_\mu^{\xi\zeta} \varepsilon_{\lambda\tau\xi\zeta} \, dx^\lambda dx^\tau \right] = \frac{1}{\kappa^2} \int_{\mathbb{R}^3} dQ_\mu$$

$$Q_\mu = \frac{1}{8} Q_\mu^{\xi\zeta} \varepsilon_{\lambda\tau\xi\zeta} \, dx^\lambda dx^\tau$$

Si dimostra che $P_0 \geq 0$ nella gauge sincrona

$g_{00} = 1 \qquad g_{0i} = 0 \qquad g_{\mu\nu} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & g_{ij} \end{array}\right)$

Condizioni di gauge-fixing sulle tetrade

$$e_\mu^{\ a} = \eta_{\mu b}\, e_\nu^{\ b}\, \eta^{\nu a}$$

gauge simmetrica (rompe sia Lorentz che i diffeomorfismi)

$$e_{\mu a} = \delta_\mu^{\ b}\, e_{\nu b}\, \delta_a^{\ \nu}$$

attorno al piatto:

$$g_{\mu\nu} = \eta_{\mu\nu} + 2K\phi_{\mu\nu} = e_{\mu a}\,\eta^{ab}\,e_{\nu b}$$

$$e_{\mu a} = \eta_{\mu a} + K\phi_{\mu a} + O(K^2) \qquad \phi_{\mu a} = \phi_{a\mu}$$

Si può preferire $e_{\mu a} = \eta_{\mu a} + \kappa \phi_{\mu a}$ $\quad \phi_{\mu a} = \phi_{a\mu}$

e allora $\quad g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa \phi_{\mu\nu} + \kappa^2 \phi_{\mu a} \phi^a_\nu$

La gauge simmetrica è algebrica anche nel settore ghost, perché è algebrica la simmetria di gauge

$$\delta_L e^a_\mu = \theta^a_b \, e^b_\mu \quad \cdots\cdots\rightarrow \quad \delta_L e^a_\mu = C^a{}_b \, e^b_\mu$$

$C^{ab} = -C^{ba}$ ghost di Faddeev-Popov della simmetria di Lorentz locale

$$\delta A_\mu = \partial_\mu \Lambda$$

$$\delta \omega^{ab} = -\nabla \theta^{ab} \qquad \delta \omega_\mu^{ab} = -\nabla_\mu \theta^{ab}$$

più simile a $\delta A_\mu = \partial_\mu \Lambda$

Un gauge-fixing derivativo per la simmetria di Lorentz locale è $[\ \partial \cdot A\ ]$

$$f^{ab} = \nabla^\mu \omega_\mu^{ab}$$

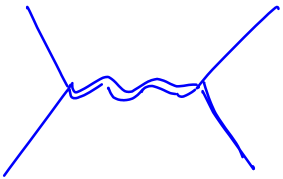
: rompe Lorentz ma NON i diffeomorfismi

Gauge sincrona per la tetrade:

$$e_{\mu a} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & e_{ij} \end{array}\right)$$

Cosa succede se si ipotizza che il gravitone abbia massa?

Discontinuità VDVZ (van Dam - Veltman - Zacharov)

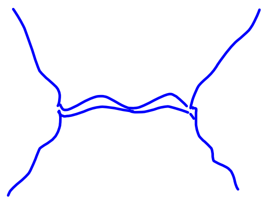


 = particella massiva

$$T^{\mu\nu} = m u^\mu u^\nu$$

u^μ = quadrivelocità

A riposo $T^{\mu\nu} = m \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right)$



 = fotone di energia E

$$T^{\mu\nu} = E u^\mu u^\nu \qquad u^\mu u_\mu = 0$$

$$T^\mu{}_\mu = 0$$

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \qquad S_m = S_{EM} = -\frac{1}{4}\int \sqrt{-g}\, F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma}$$

$$T_{\mu\nu} = -\frac{1}{4}\left(2\, F_{\mu\rho} F_\nu{}^\rho - \frac{1}{2} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right) 2 =$$

$$= -F_{\mu\rho} F_\nu{}^\rho + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

$g^{\mu\nu} T_{\mu\nu} = 0$: la radiazione ha tensore energia impulso a traccia nulla

S_{EM} è Weyl invariante: $g_{\mu\nu} \to g_{\mu\nu} e^{2\omega} \quad A_\mu \to A_\mu$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad e^{4\omega} e^{-2\omega} e^{-2\omega} = 1$$

Scalar fields

$$S_\varphi = \frac{1}{2}\int \sqrt{-g}\left[\nabla_\mu \varphi \, \nabla_\nu \varphi \, g^{\mu\nu} + \frac{\xi}{12} R \varphi^2 \right]$$

$e^{4\omega} \quad e^{-\omega} \quad e^{-\omega} \quad e^{-2\omega}$

$$g_{\mu\nu} \to g_{\mu\nu} e^{2\omega} \qquad \varphi \to \varphi e^{-\omega}$$

$$R \longrightarrow e^{-2\omega}\left(R - 6\nabla^2\omega - 6\nabla_\mu\omega\nabla^\mu\omega\right)$$ Esercizio

$$S_\varphi \to \frac{1}{2}\int\sqrt{-g}\left[e^{2\omega}\nabla_\mu(e^{-\omega}\varphi)\nabla^\mu(e^{-\omega}\varphi) + \frac{\xi}{12}\varphi^2 \cdot \left(R - 6\nabla^2\omega - 6\nabla_\mu\omega\nabla^\mu\omega\right)\right] =$$

$$= \frac{1}{2}\int\sqrt{-g}\left[\cancel{e^{2\omega}}\left(\cancel{e^{-\omega}}\nabla_\mu\varphi - \cancel{e^{-\omega}}\varphi\nabla_\mu\omega\right)\cdot\right.$$

$$\cdot\left(\cancel{e^{-\omega}}\nabla^\mu\varphi - \cancel{e^{-\omega}}\nabla^\mu\omega\,\varphi\right) + \frac{\xi}{12}\varphi^2 R +$$

$$-\frac{\xi}{2}\varphi^2\nabla^2\omega - \frac{\xi}{2}\varphi^2\nabla_\mu\omega\,\nabla^\mu\omega\Big] =$$

$$= \frac{1}{2}\int\sqrt{-g}\Big[\nabla_\mu\varphi\,\nabla^\mu\varphi - \underbrace{2\varphi\nabla_\mu\varphi}_{=\nabla_\mu\varphi^2}\nabla^\mu\omega + \cancel{\varphi^2\nabla_\mu\omega\,\nabla^\mu\omega} +$$

$$+\frac{\xi}{12}\varphi^2 R - \frac{\xi}{2}\varphi^2\nabla^2\omega - \cancel{\frac{\xi}{2}\varphi^2\nabla_\mu\omega\,\nabla^\mu\omega}\Big] =$$

$$= (\xi=2)\ \frac{1}{2}\int\sqrt{-g}\Big[\nabla_\mu\varphi\,\nabla_\nu\varphi\,g^{\mu\nu} + \frac{1}{6}\varphi^2 R\Big]$$

a meno di una derivata totale

$$T_{\mu\nu} = \partial_\mu\varphi\,\partial_\nu\varphi - \frac{1}{2}\eta_{\mu\nu}\partial_\alpha\varphi\,\partial^\alpha\varphi - \frac{1}{6}(\partial_\mu\partial_\nu - \Box\eta_{\mu\nu})(\varphi^2)$$

(nel piatto)

ESERCIZIO

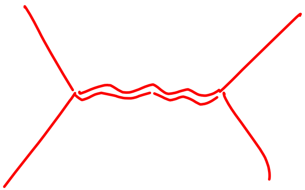
$$T_{\mu\nu}\,\eta^{\mu\nu} = -\partial_\mu\varphi\,\partial^\mu\varphi + \frac{1}{2}\Box(\varphi^2) =$$

$$= -\cancel{\partial_\mu\varphi\,\partial^\mu\varphi} + \frac{1}{2}\left(2\varphi\Box\varphi + \cancel{2\,\partial_\mu\varphi\,\partial^\mu\varphi}\right) =$$

$$= \varphi\Box\varphi = 0 \quad \text{on shell}$$

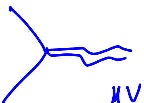
$$S_{Dirac} = \int e\,\bar\psi\, e^\mu_a\, i\nabla_\mu\,\gamma^a\,\psi \quad \text{è Weyl invariante}$$

con $\psi \to \psi\, e^{-\frac{3}{2}\omega}$ ESERCIZIO

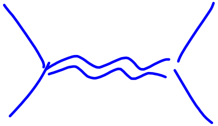


$$T^{\mu\nu} = m\,u^\mu u^\nu = \left(\begin{array}{c|c} m & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\,\frac{\delta S_m}{\delta g^{\mu\nu}} \qquad g^{\mu\nu} \simeq \eta^{\mu\nu} - 2\kappa\,\phi^{\mu\nu}$$

 $_{\mu\nu} = -i\,\kappa_m T_{\mu\nu}$

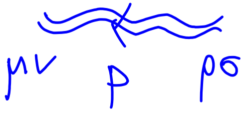
$$S_m \sim S_m(\eta) - \kappa_m \int T_{\mu\nu}\,\phi^{\mu\nu}$$

 $= (-ik)^2 \, T_{\mu\nu} P^{\mu\nu\rho\sigma} T'_{\rho\sigma} =$

$$\underline{p^\mu T_{\mu\nu} = 0}$$

$$= -\kappa_m^2 \, m_1 m_2 \, \frac{i}{2} \, \frac{1}{p^2 - m^2} \, \frac{4}{3}$$

Pauli-Fierz:

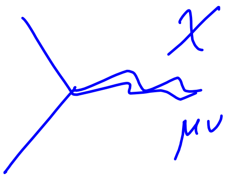


$$= \frac{i}{2} \, \frac{1}{p^2 - m^2} \left(\pi_{\mu\rho}\pi_{\nu\sigma} + \pi_{\mu\sigma}\pi_{\nu\rho} - \frac{2}{3}\pi_{\mu\nu}\pi_{\rho\sigma} \right) \equiv P^{\mu\nu\rho\sigma}$$

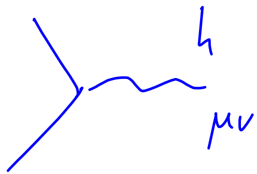
$$\pi_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}$$

$p^0 \sim 0$ (caso statico)

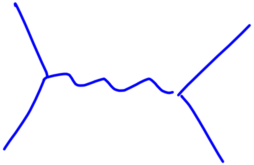
$$\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \frac{2}{3}\eta_{\mu\nu}\eta_{\rho\sigma} \right)_{0000} = \frac{4}{3}$$



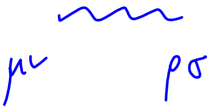
$= -i\kappa_m T_{\mu\nu}$



$= -i\kappa \, T_{\mu\nu}$



$$= -k^2 m_1 m_2 \frac{i}{2} \frac{1}{p^2} (1)$$



$$= \frac{i}{2p^2}\left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}\right) + \text{parti trasverse}$$

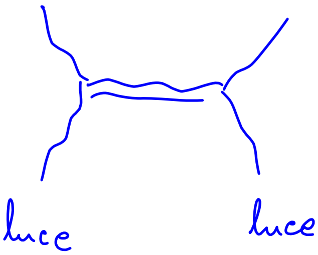
Legge di Newton $\quad -k^2 \frac{m_1 m_2}{p^{0\,2} - \vec{p}^{\,2}} = G \frac{m_1 m_2}{\vec{p}^{\,2}} \qquad G = k^2$

caso statico

$$\int \frac{d^3\vec{p}}{(2\pi)^3} \frac{e^{i\vec{p}\cdot\vec{x}}}{\vec{p}^{\,2}} \sim \frac{1}{|\vec{x}|}$$

$$-\frac{4}{3} k_m^2 \frac{m_1 m_2}{\cancel{p^{0\,2}} - \vec{p}^{\,2} - \cancel{m^2}} \qquad G = \frac{4}{3} k_m^2$$

Cosa cambia se studio la deflessione della luce da parte della gravità



Se scambio una particella di Pauli-Fierz

$$-\kappa_m^2 E_1 E_2 \frac{i}{2} \frac{1}{p^2-m^2} 2 =$$

$$= -\frac{3}{4} G E_1 E_2 \frac{i}{p^2-m^2}$$

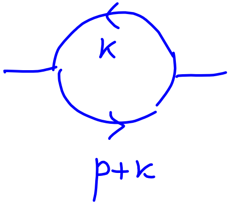
Se scambio h :

$$-\kappa^2 E_1 E_2 \frac{i}{2} \frac{1}{p^2} \cdot 2 =$$

$$= - G E_1 E_2 \frac{i}{p^2}$$

Discontinuità

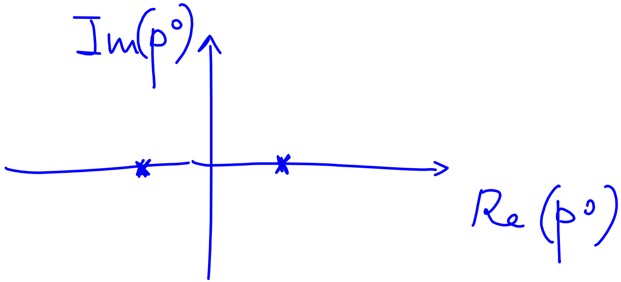
$$\frac{1}{p^2-m^2+i\epsilon} = S(p)$$



$$= \int \frac{d^4k}{(2\pi)^4} S(k)\, S(p+k)$$

$$\frac{1}{p^2-m^2} = \frac{1}{p^{0\,2}-\omega^2}$$

$$\omega(\vec{p}) = \sqrt{\vec{p}^{\,2}+m^2}$$

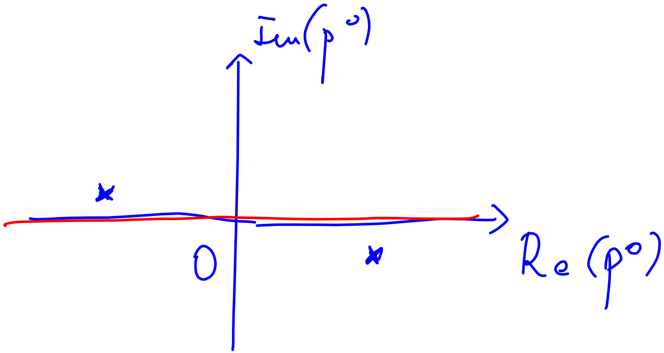


$p^0 \in \mathbb{C}$

$\vec{p} \in \mathbb{R}^3$

$$\frac{1}{p^2-m^2+i\epsilon} = \frac{1}{p^{0\,2}-\omega_\epsilon^2}$$

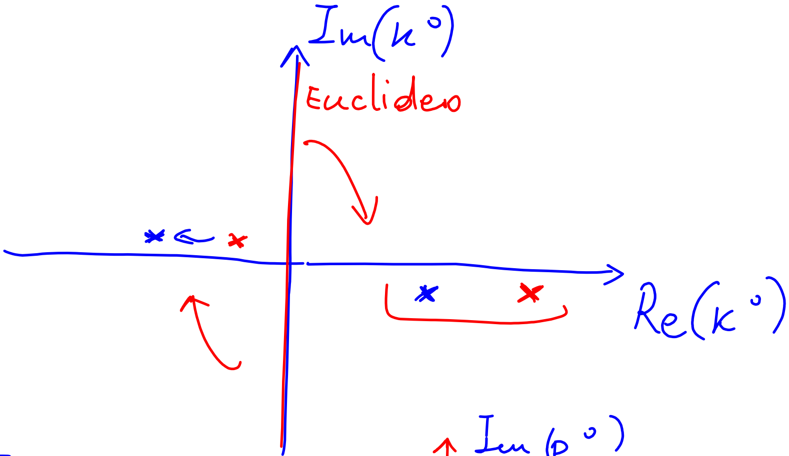
$$\omega_\epsilon = \sqrt{\vec{p}^{\,2}+m^2-i\epsilon}$$



I poli sono

$$p^0 = \pm\,\omega_\epsilon$$

$$\int \frac{d^4k}{(2\pi)^4} S(k) S(p+k) = \int_{-\infty}^{+\infty} \frac{dk^0}{2\pi} \int \frac{d^3\vec{k}}{(2\pi)^3} S(k) S(p+k) \equiv M(p)$$

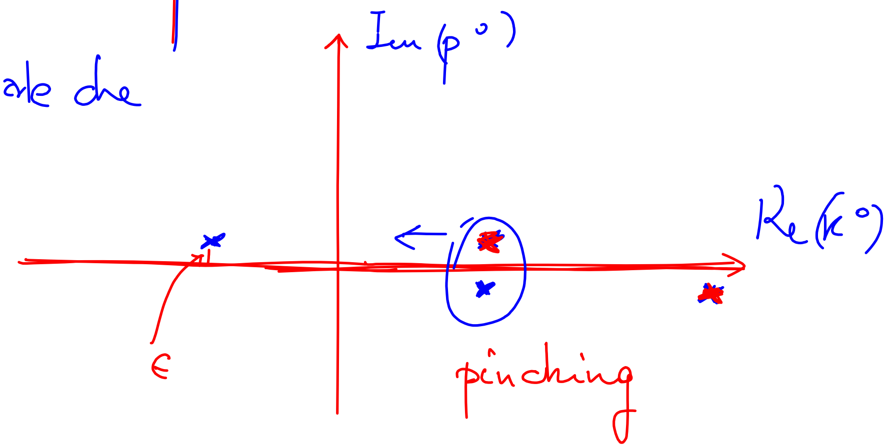


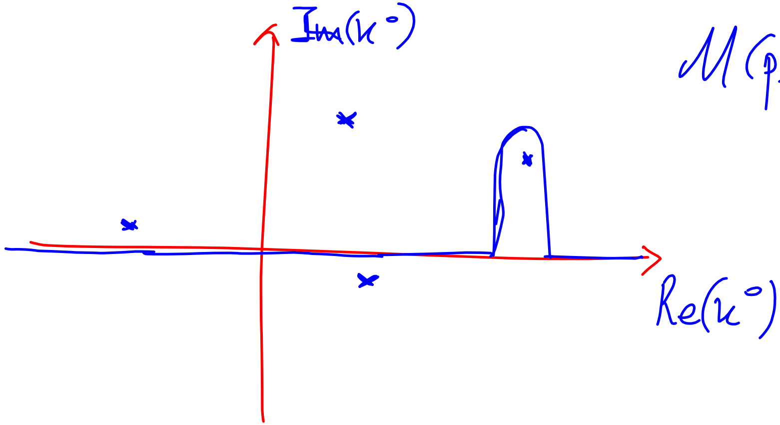
I poli di $S(p+k)$ sono

$$p^0 + k^0 = \pm\sqrt{(\vec{p}+\vec{k})^2 + m^2 - i\epsilon}$$

$$k^0 = -p^0 \pm \sqrt{(\vec{p}+\vec{k})^2 + m^2 - i\epsilon}$$

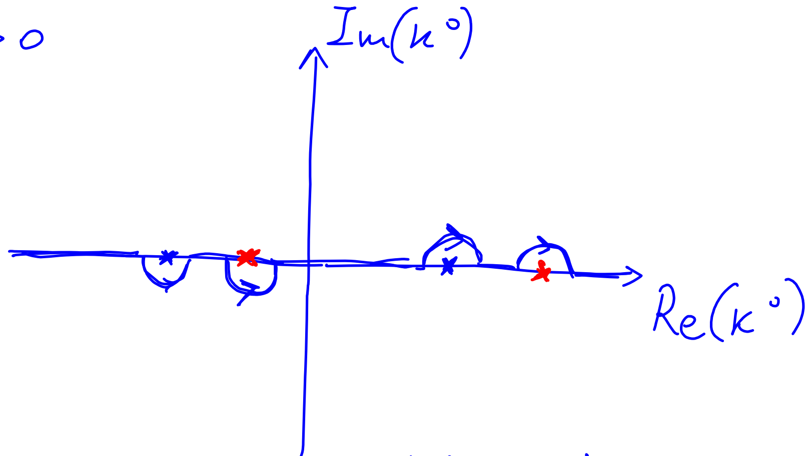
$\exists\, p^0$ tale che





$\mathcal{M}(p)$ è analitica finché posso deformare i domini di integrazione in modo che i poli dell'integrando non tocchino gli stessi domini di integrazione

$\epsilon \to 0$



Questa prescrizione (da sola) è incompleta

In corrispondenza del pinching non si può fare la deformazione. L'analiticità è violata da un punto di diramazione e relativo taglio

$$\int_{-\infty}^{+\infty} \frac{dk^0}{2\pi} \int \frac{d^3\vec{k}}{(2\pi)^3} S(k)\, S(p+k) \equiv \mathcal{M}(p)$$

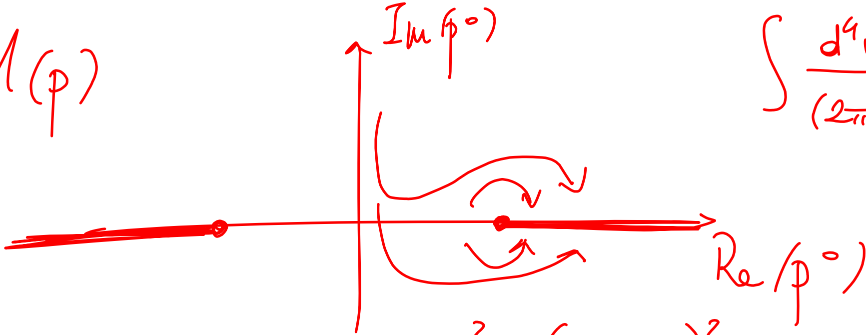
a $m=0$ $\quad \mathcal{M}(p) = \int^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 (p-k)^2} = \frac{1}{(4\pi)^2} \ln \frac{\Lambda^2}{-p^2}$

Sottrarre $\frac{1}{(4\pi)^2} \ln \frac{\Lambda^2}{\mu^2}$

$$\int^{\Lambda} \frac{d^4k}{k^4} = 2\pi^2 \ln \Lambda$$

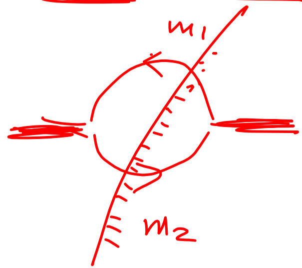
$$-\frac{1}{(4\pi)^2} \ln \frac{-p^2 - i\epsilon}{\Lambda^2}$$

$\mathcal{M}(p)$

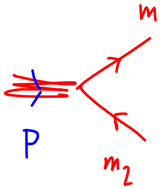


$p^2 = (m_1+m_2)^2$

$$\int \frac{d^4k}{(2\pi)^2}\, \underline{S(k,m_1)}\, \underline{S(p+k,m_2)}$$

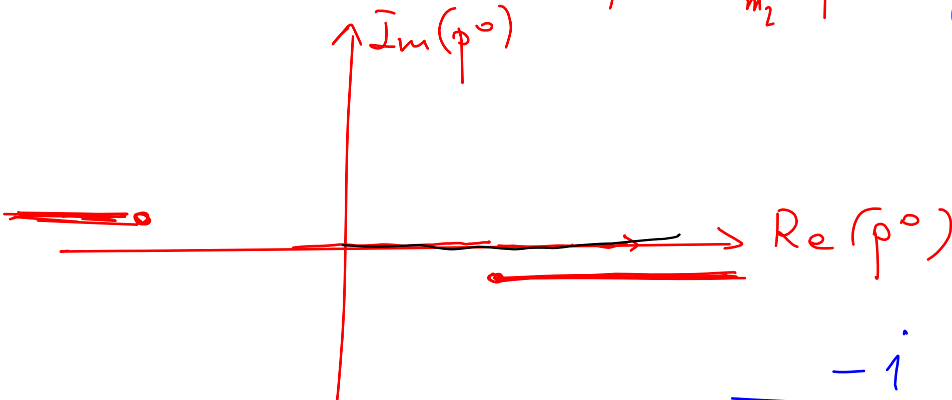


$$2\,\mathrm{Im}\,\mathcal{M}(p) = \int d\Phi \left| \quad \right|^2$$



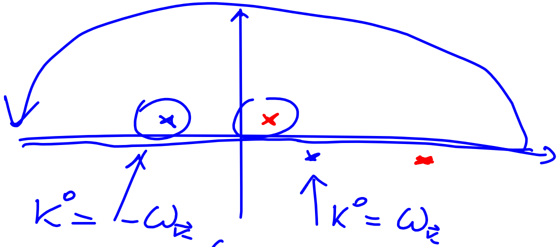
Teorema ottico

Φ = spazio delle fasi di m_1 e m_2



$$\frac{-i}{p^2 - m^2 + i\epsilon}$$

$$(2\pi i)\int \frac{d^3k}{(2\pi)^3}\, S(k)\,S(p+k)\Big|^{\text{Res}}_{k^0=-\omega_{\vec k}} + \left(\text{termine simile}\right)^{*}$$



$$S(k) = \frac{1}{k^2-m^2+i\epsilon} = \frac{1}{(k^0-\omega_{\vec k})(k^0+\omega_{\vec k})} = \frac{1}{2\omega_{\vec k}}\left(\frac{1}{k^0-\omega_{\vec k}} - \frac{1}{k^0+\omega_{\vec k}}\right)$$

$$\omega_{\vec k} = \sqrt{\vec k^2 + m^2 - i\epsilon}$$

$$\propto \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_{\vec k}}\,\frac{1}{2\omega_{\vec k+\vec p}}\left(\frac{1}{p^0-\omega_{\vec k}-\omega_{\vec k+\vec p}} - \frac{1}{p^0-\omega_{\vec k}+\omega_{\vec k+\vec p}}\right)$$

non dà pinching.
(si annulla col termine $*$)

Singolarità $\quad p^0 = \omega_{\vec k} + \omega_{\vec k+\vec p}$

Consideriamo $D=2$ $\vec{k} = (k_x)$

$$p^0 - \omega_{\vec{k}} = \omega_{\vec{p}+\vec{k}} \qquad p^0 - \sqrt{k^2+m^2} = \sqrt{(p+k)^2+m^2}$$

$$(p^0)^2 + \cancel{k^2} + \cancel{m^2} - 2p^0\sqrt{k^2+m^2} = p^2 + \cancel{k^2} + 2pk + \cancel{m^2}$$

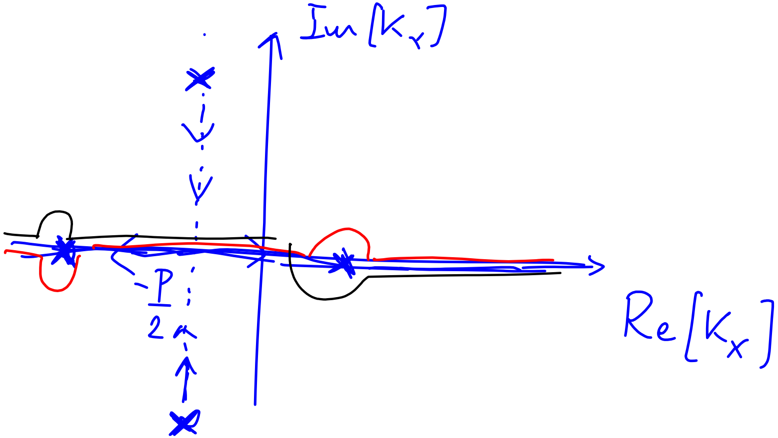
$$(p^0)^2 - p^2 - 2pk = 2p^0\sqrt{k^2+m^2} \qquad P^2 = (p^0)^2 - p^2$$

$$(P^2)^2 + \underbrace{4p^2k^2} - 4pkP^2 = \underbrace{4p^{0\,2}k^2} + 4p^{0\,2}m^2$$

$$4P^2k^2 + 4pkP^2 + 4p^{0\,2}m^2 - (P^2)^2 = 0$$

$$k_\pm = \frac{1}{4P^2}\left[-2pP^2 \pm \sqrt{4p^2(P^2)^2 - 16p^{0\,2}m^2P^2 + 4(P^2)^3}\right]$$

$$= -\frac{p}{2} \pm \frac{1}{2}\sqrt{p^2 + P^2 - \frac{4p^{0\,2}m^2}{P^2}} = -\frac{p}{2} \pm \frac{P_0}{2}\sqrt{1 - \frac{4m^2}{P^2}}$$



$$K_\pm = -\frac{P}{2} \pm \frac{P^0}{2}\sqrt{1-\frac{4m^2}{P^2}}$$

P = fisso

P^0 varia

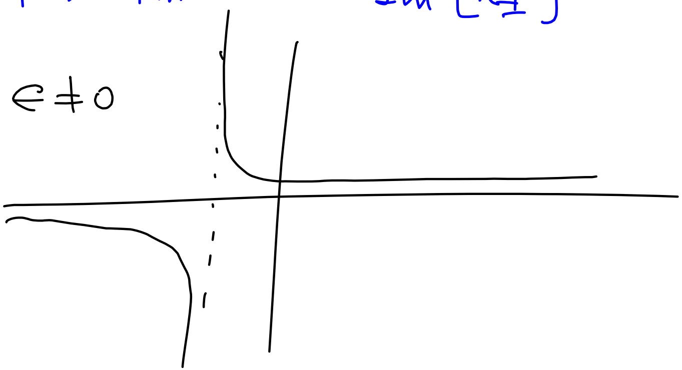
$P^2 < \underline{4m^2}$ $\quad Im[K_\pm] \gtrless 0$

sotto soglia

$P^2 = 4m^2$: pinching

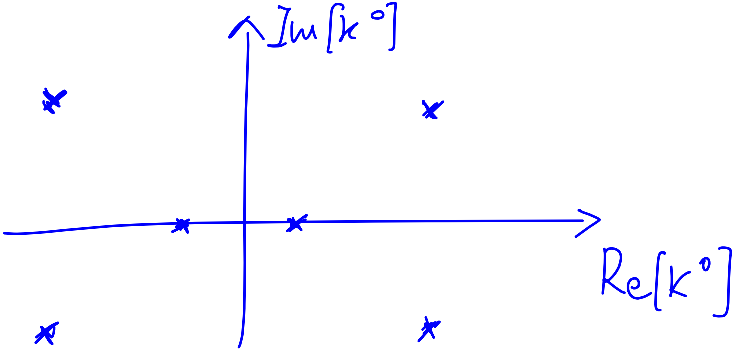
$P^2 > 4m^2$ $\quad Im[K_\pm] \approx 0$

$\epsilon \neq 0$



Modelli di Lee e Wick

$$S(k) = \frac{1}{(k^2 - m^2)\left[(k^2-\mu^2)^2 + M^4\right]}$$



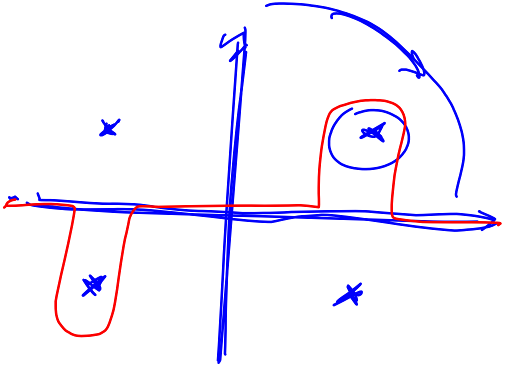
$$k^2 = \mu^2 \pm i M^2$$

$$k^{\circ 2} = \vec{k}^2 + \mu^2 \pm i M^2$$

$$k^\circ = \pm\sqrt{\vec{k}^2 + \mu^2 \pm i M^2}$$

Nella teoria $\int (R + R_{\mu\nu}^2 + R^2)\sqrt{-g}$ il propagatore va a zero come $\frac{1}{(p^2)^2}$ per $p^2 \to \infty$

Prendiamo $S(k) = \dfrac{1}{(k^2)^2 + M^4}$ $\qquad k^2 \to \infty \quad S(k) \sim \dfrac{1}{(k^2)^2}$



Se integro sul Minkowski <u>non</u> vale la località dei controtermini

$$\int \frac{d^4k}{(2\pi)^4} S(k)\, S(p+k)$$

Residuo in $k^0 = \pm\sqrt{\pm i M^2}$
$\vec{k} = 0$

$$k^0 = \pm \frac{1 \pm i}{2} \sqrt{2}\, M$$

Su <u>un</u> polo di $S(k)$ $\quad k^2 = \pm i M^2$

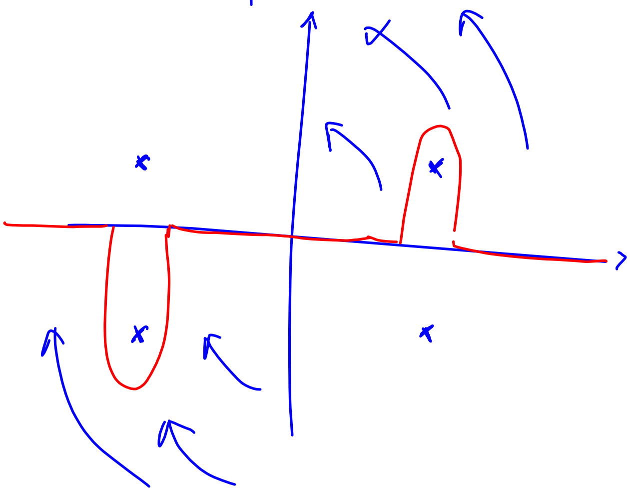
$$S(k) \simeq \frac{1}{|\vec{k}|} \qquad S(p+k) \simeq \frac{1}{(p^2 + \underline{k^2} + 2p\cdot k)^2 + M^4} =$$

$$= \frac{1}{(p^2 \pm i M^2 + 2p\cdot k)^2 + M^4} \simeq \frac{1}{(p\cdot k)^2}$$

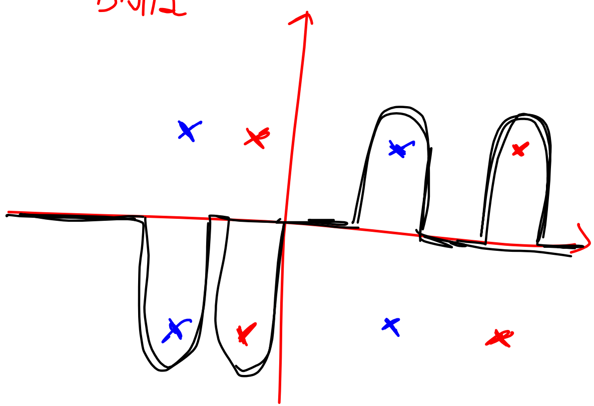
Le divergenze sono nonlocali : $\dfrac{\ln \Lambda_{UV}}{p^2}$

Una teoria dei campi va sempre definita nell' Euclideo e poi ruotata alla Wick.

Questo porta ai modelli di Lee e Wick

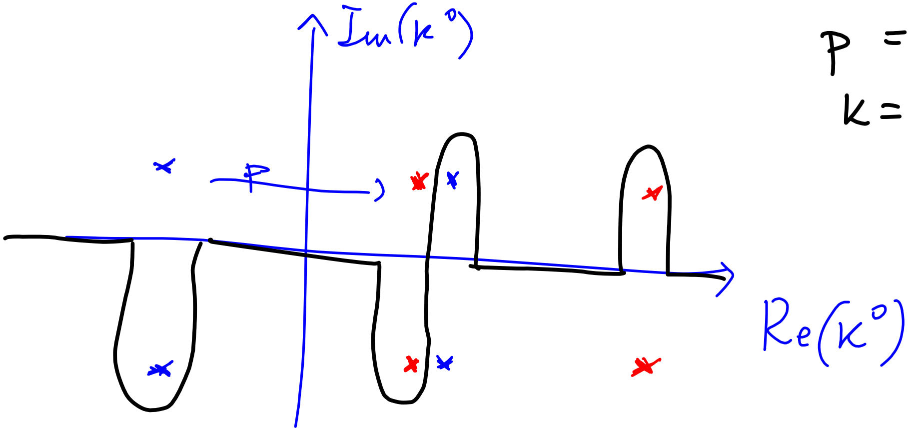


Bolla

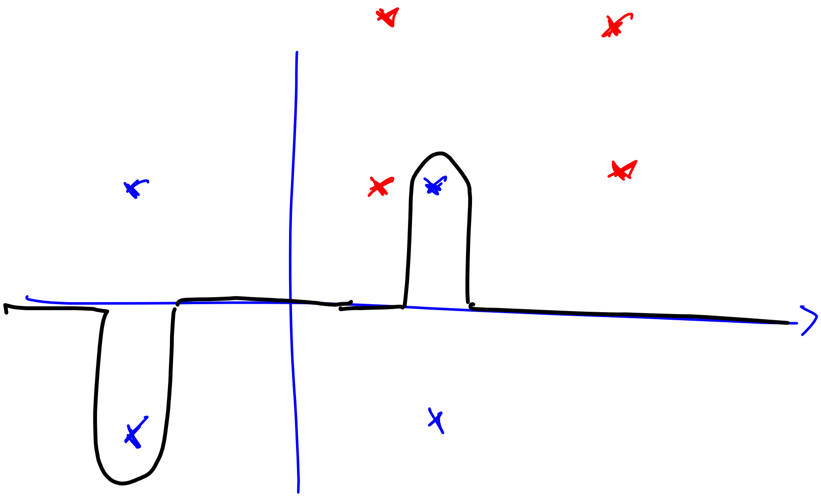


p = momento esterno

k = momento di loop



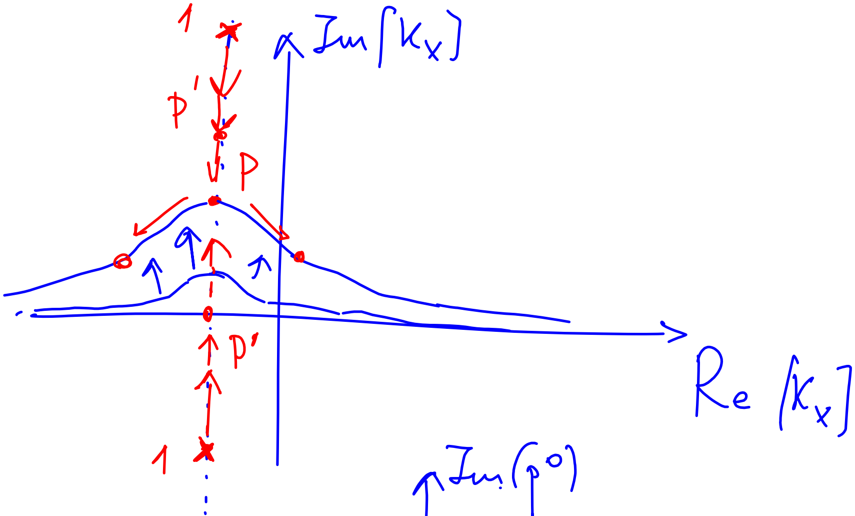
pinching 1

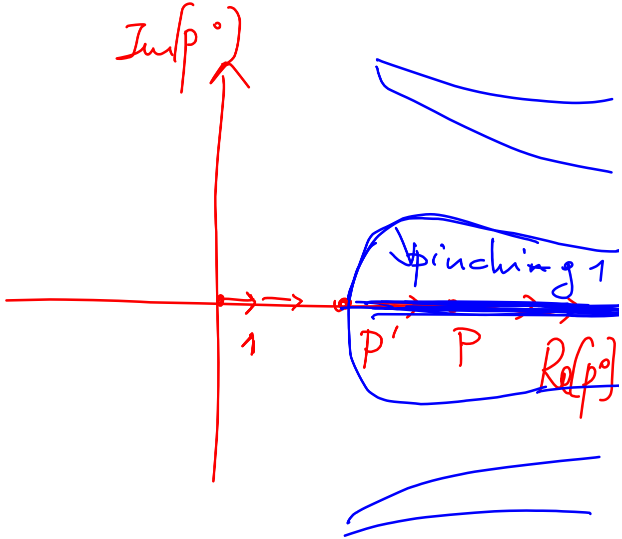


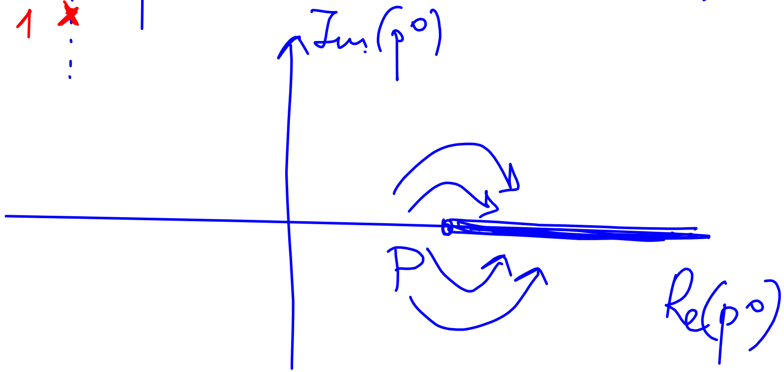
pinching 2

$D = 2$ $\int dk^0$ fatto col teorema dei residui

porta a $\int dk_x \, f(k_x)$ $\quad f(k_x)$ con poli $k_\pm$





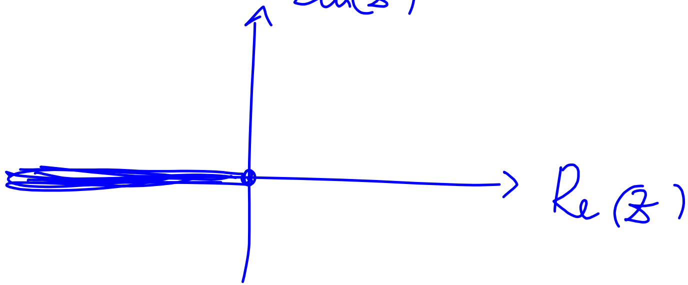


integrando su

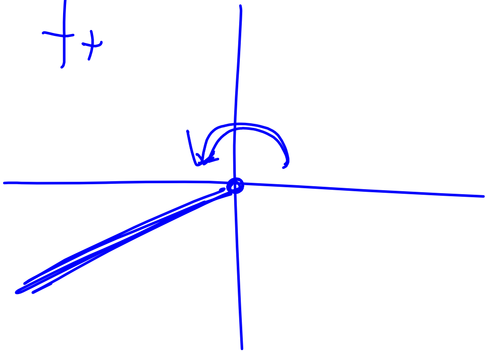
k_x reale

Continuazione media

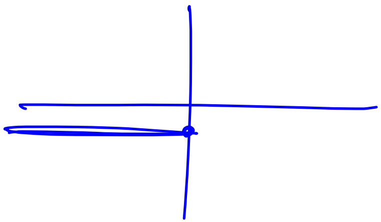
$\ln z$



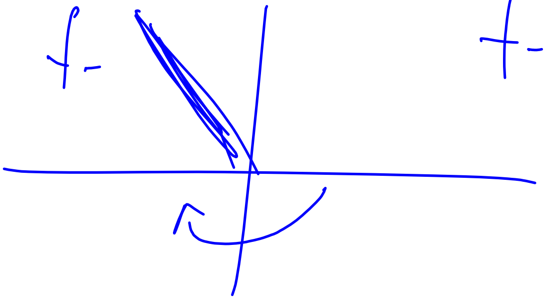
f_+



$$f_+ = \ln(z + i\epsilon)$$



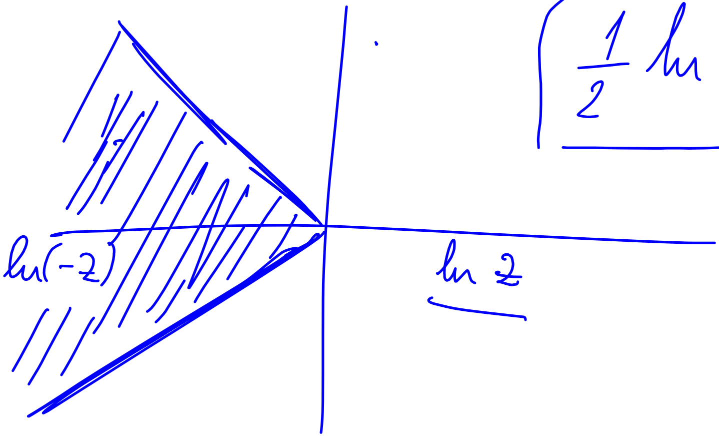
f_-



$$f_- = \ln(z - i\epsilon)$$

Media: $\frac{1}{2} \ln(z^2 + \epsilon^2)$

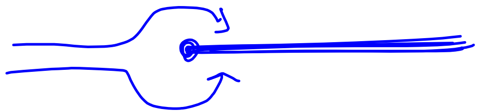
$$= \frac{1}{2} \ln z^2$$



$$\boxed{\frac{1}{2}\ln z^2}$$

Feynman : $\ln(-p^2 - i\epsilon)$

$z = -p^2 \quad \ln(z - i\epsilon)$



$$\operatorname{Im} \ln(-p^2 - i\epsilon) = i\pi\theta(p^2)$$

$$\frac{1}{2}\ln(-p^2)^2$$

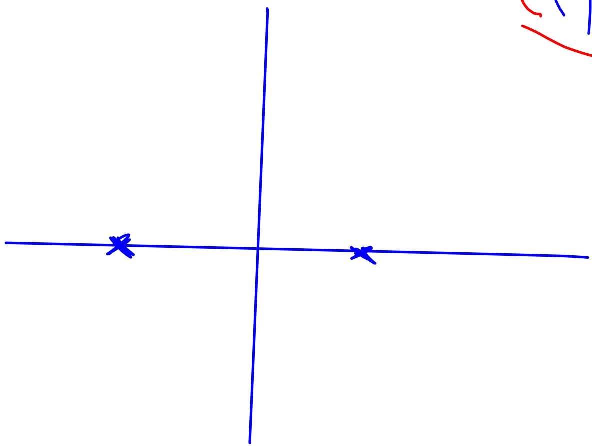
$$\frac{1}{p^2-m^2} \longrightarrow \frac{-1}{p^2-m^2+i\epsilon}$$

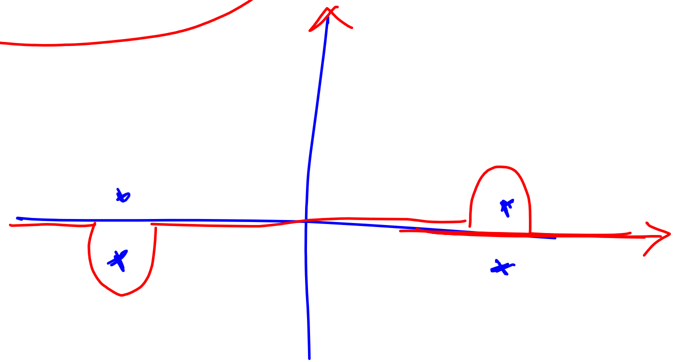
$$\cancel{P\frac{1}{x} = \frac{x}{x^2+\epsilon^2}}$$

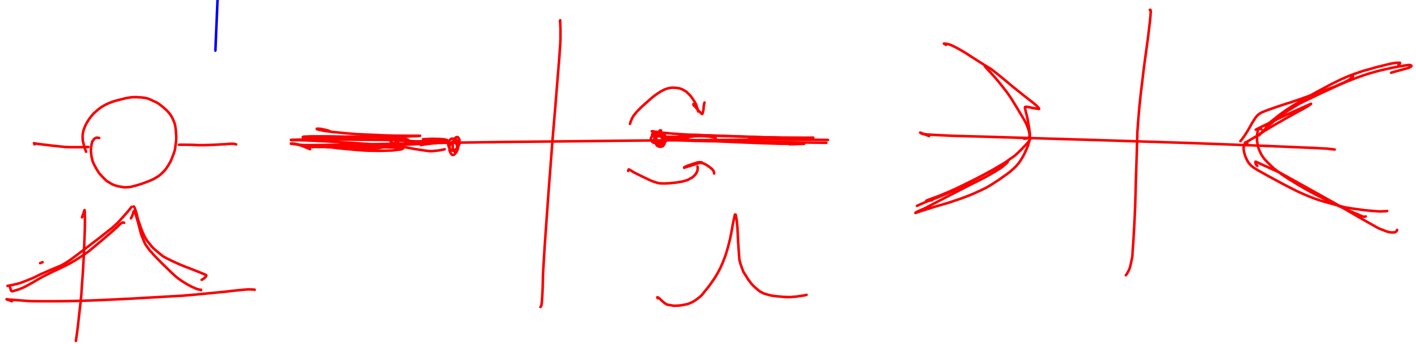
non è

$$\longrightarrow \frac{p^2-m^2}{(p^2-m^2)^2+\varepsilon^4}$$

$$\varepsilon \to 0$$

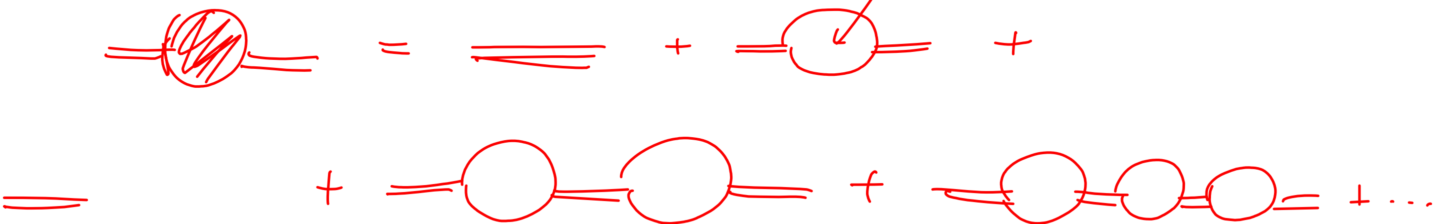




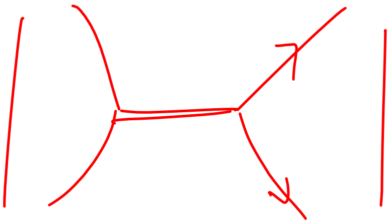
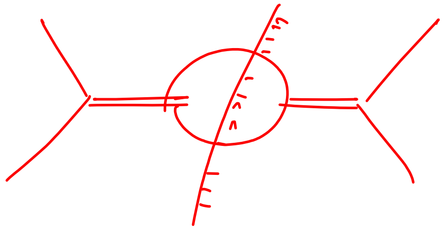


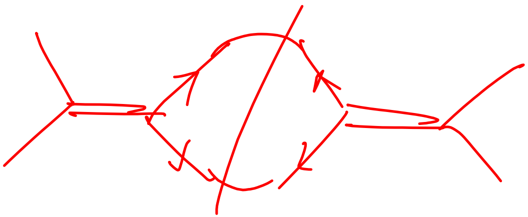
$$\frac{1}{p^2 - m^2 + }$$

particelle vere



particella finta

$\int d\Phi \,\Big|$  $\Big|^2 = 2\,\mathrm{Im}$ 



Normalmente

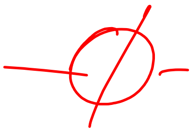
$$\frac{1}{p^2-m^2} \longrightarrow \frac{Z}{p^2-\bar{m}^2+i\bar{m}\Gamma}$$

Γ = larghezza $1/\Gamma = \tau$ = vita media

$\Gamma > 0$

Particella fake

$$\frac{-1}{p^2-m^2+i\bar{m}\Gamma}$$ a $p^2=m^2$ $$\frac{-1}{i\bar{m}\Gamma}$$



$\Gamma < 0$

$$\frac{\Gamma}{E-m+i\Gamma} \longrightarrow \operatorname{sgn}(t)\,\Theta(t\Gamma)\,e^{-imt-\Gamma t/2}$$

$$\int_{-\infty}^{+\infty} \frac{dE}{2\pi}\,\frac{e^{-iEt}}{E-m+i\Gamma}$$

$\Gamma > 0 \;:\; \Theta(t)$

$\Gamma < 0 \;:\; \Theta(-t)$

$$\Phi(t) = \int dt'\,\theta(t-t')\,J(t')$$

$$\int dt'\,\theta(t'-t)\,J(t')$$