3.1. Light cones in FRW

Exercise 1. Find the universe lines of massless particles for a flat Friedmann Robertson Walker universe using the cosmic time and the space coordinates $u = ax$.

The interval is of the form

$$ds^2 = dt^2 - a^2(t)dx^2$$

and the light cones are solution of

$$dt = \pm a dx$$

Now

$$du = adx + xda = adx + u\frac{da}{a}$$

which gives

$$dt = \pm du \mp u\frac{dt}{a}\frac{da}{dt}$$

and

$$\frac{du}{dt} - Hu = \pm 1$$

where $H = \dot{a}/a$. The complete solution can be found writing

$$\left(\frac{du}{dt} - Hu\right) e^{-\int_{t_0}^{t} H(t')dt'} = \pm e^{-\int_{t_0}^{t} H(t')dt'}$$

or

$$\frac{d}{dt}\left(ue^{-\int_{t_0}^{t} H(t')dt'}\right) = \pm e^{-\int_{t_0}^{t} H(t')dt'}$$

$$u(t) = u(t_0)e^{\int_{t_0}^{t} H(t')dt'} \pm e^{\int_{t_0}^{t} H(t')dt'} \int_{t_0}^{t} dt' e^{-\int_{t_0}^{t'} H(t'')dt''}$$

As a first example we consider the case of a constant Hubble parameter H. We obtain

$$\begin{aligned} u(t) &= u(t_0)e^{H(t-t_0)} \pm e^{H(t-t_0)} \int_{t_0}^{t} dt' e^{-Ht'} \\ &= u(0)e^{H(t-t_0)} \mp H^{-1}\left(1 - e^{H(t-t_0)}\right) \end{aligned}$$

In the case considered

$$a = e^{H(t-t_0)}$$

and the u coordinate of a comoving object is

$$u = e^{H(t-t_0)}x_0$$

Now let us consider the case

$$a = \left(\frac{t}{t_0}\right)^{\alpha}$$

which means

$$H = \frac{\alpha}{t}$$

and

$$e^{\int_{t_0}^{t} H(t')dt'} = e^{\int_{t_0}^{t} \frac{\alpha}{t'}dt'} = e^{\alpha \log \frac{t}{t_0}} = \left(\frac{t}{t_0}\right)^{\alpha}$$

$$\begin{aligned} u(t) &= u(t_0)\left(\frac{t}{t_0}\right)^{\alpha} \pm \left(\frac{t}{t_0}\right)^{\alpha} \int_{t_0}^{t} dt' \left(\frac{t'}{t_0}\right)^{-\alpha} \\ &= u(t_0)\left(\frac{t}{t_0}\right)^{\alpha} \pm t^{\alpha} \frac{1}{1-\alpha} \left(t^{1-\alpha} - t_0^{1-\alpha}\right) \\ &= u(t_0)\left(\frac{t}{t_0}\right)^{\alpha} \pm \frac{t_0}{1-\alpha} \left[\left(\frac{t}{t_0}\right) - \left(\frac{t_0}{t}\right)^{\alpha}\right] \end{aligned}$$

$$u(t) = \pm \frac{t_0}{1-\alpha} \left[\left(\frac{t}{t_0}\right) - \left(\frac{t_0}{t}\right)^{\alpha}\right]$$