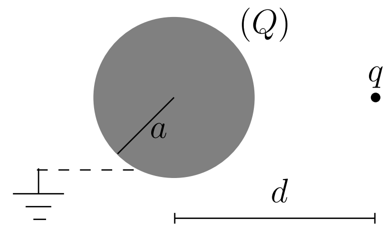
Point charge in front of a conducting sphere

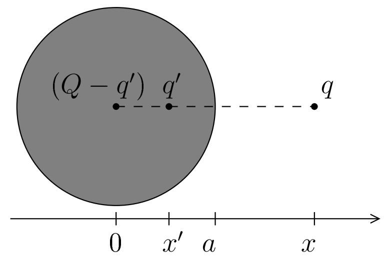
A point charge q is at distance d from the center of a conducting sphere of radius a. In both cases of a grounded sphere and of an isolated sphere having charge Q, find

a) the solution for the electric potential,
b) the force on the point charge,
c) the potential energy of the point charge.



Solution

a) The problem of finding the potential for a charge q in front of a grounded sphere and at a distance x from the center is solved by placing an image charge $q' = -(a/x)q$ at a distance $x' = a^2/x$ from the center of the sphere, along the same direction ($\hat{\mathbf{x}}$) from the center to q.



If the sphere is isolated and has a net charge Q, after placing q' we must distribute the additional charge $Q - q'$ in order that the potential remains uniform on the surface. This is achieved by distributing such charge uniformly on the surface, which corresponds to add a second image charge $Q - q'$ in the center of the sphere. The case $Q = 0$ corresponds to a charge neutral, isolated sphere.

b) The total force $\mathbf{f}$ on q is the sum of the force between q and each of the three image charges ($q_1 = q'$, $q_2 = -q'$, $q_3 = Q$), as if the image charges were real. Thus $\mathbf{f} = \sum_{i=1} 1^{i=3}\mathbf{f}_i$ where $\mathbf{f}_i = k_0\hat{\mathbf{x}}(qq_i)/(x-x_i)^2$ with $x_1 = x'$ and $x_2 = x_3 = 0$. Explicitly

$$f_1 = k_0\frac{qq'}{(x-x')^2} = -k_0\frac{q^2ax}{(x^2-a^2)^2}\,, \qquad f_2 = -k_0\frac{qq'}{x^2} = k_0q^2\frac{a}{x^3}\,, \qquad f_3 = k_0\frac{qQ}{x^2}\,. \tag{1}$$

c) To find the potential energy we displace the charge q along the x-axis from the initial position $x = d$ up to $x = +\infty$. The work done by each force term is given by

$$\begin{aligned} W_1 &= \int_d^\infty f_1 dx = k_0 \left.\frac{q^2a}{2(x^2-a^2)}\right|_d^\infty = -k_0\frac{q^2a}{2(d^2-a^2)}\,, \\ W_2 &= \int_d^\infty f_2 dx = -k_0 \left.\frac{q^2a}{2x^2}\right|_d^\infty = k_0\frac{q^2a}{2d^2}\,. \\ W_3 &= \int_d^\infty f_3 dx = k_0\frac{qQ}{d}\,. \end{aligned} \tag{2}$$

The potential energy $U = W_1$ for the grounded sphere and $U = W_1+W_2+W_3$ for the isolated sphere.

It may be interesting to compare the result for U with the potential energy U_{real} we would obtain if all the charges were real ($q_0 = q$):

$$\begin{aligned} U_{\text{real}} &= k_0\sum_{i<j}\frac{q_iq_j}{r_{ij}} \\ &= k_0\left(\frac{q_0q_1}{d-d'} + \frac{q_0q_2}{d} + \frac{q_0q_3}{d} + \frac{q_1q_2}{d'} + \frac{q_2q_3}{d'}\right) \\ &= k_0\left(-\frac{q^2a}{d^2-a^2} + \frac{q^2a}{d^2} + \frac{qQ}{d} - \frac{q^2}{d} - \frac{Qq}{a}\right)\,, \end{aligned} \tag{3}$$

where $d' = a^2/d$. From the expression of U_{real} as a sum of interaction energies between pair of charges, we obtain U by keeping *as a whole* the (0,3) term corresponding to the interaction of q with the actual charge on the sphere Q, then by adding *half* of the (0,1) and (0,2) terms corresponding to the interaction of q with the images $\pm q'$, and by neglecting the terms (1,2) and (1,4) corresponding to the mutual interaction of image charges.