Esercizio sulla conduzione del calore

$T_{1_0} < T_{2_0}$

Due corpi con stessa capacità C e $T_{1,0} < T_{2,0}$ a contatto tramite una sbarra con S, l, $\sigma \leftarrow$ conduc. termica

1) T_f ?

$$dQ_1 + dQ_2 = 0 \qquad (\text{sistema isolato})$$

$$dQ_1 = C\, dT_1$$
$$dQ_2 = C\, dT_2$$

$$\Rightarrow \quad C(T_f - T_{1_0}) + C(T_f - T_{2_0}) = 0$$

$$T_f = \frac{T_{1,0} + T_{2,0}}{2}$$

Nota: $Q < 0$ calore ceduto
$Q > 0$ calore assorbito
$\Rightarrow \xrightarrow{Q>0} \square \xrightarrow{Q<0}$

2) $\Delta T \equiv T_2 - T_1$. Trovare $\Delta T(t)$ in regime stazionario

$$I_1 \equiv \frac{dQ_1}{dt} = \sigma \frac{S}{l}(T_2 - T_1) = C \frac{dT_1}{dt}$$

$$\Rightarrow$$

$$I_2 \equiv \frac{dQ_2}{dt} = -\sigma \frac{S}{l}(T_2 - T_1) = C \frac{dT_2}{dt}$$

calore "esce"

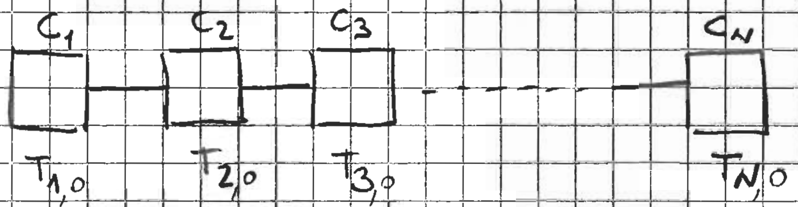
$$C \frac{d}{dt}(T_2 - T_1) = -2\sigma \frac{S}{l}(T_2 - T_1) \quad \Rightarrow$$

$$\frac{d}{dt}\Delta T = -2 \frac{\sigma}{C}\frac{S}{l}\Delta T \qquad \frac{d(\Delta T)}{\Delta T} = -2\frac{\sigma}{C}\frac{S}{l}\,dt$$

$$\Rightarrow \quad \Delta T(t) = \Delta T_0\, e^{-2\frac{\sigma}{C}\frac{S}{l}t}$$

con $\Delta T_0 = T_{2,0} - T_{1,0}$

3) Generalizziamo



Trovare T_f all'equilibrio termico:

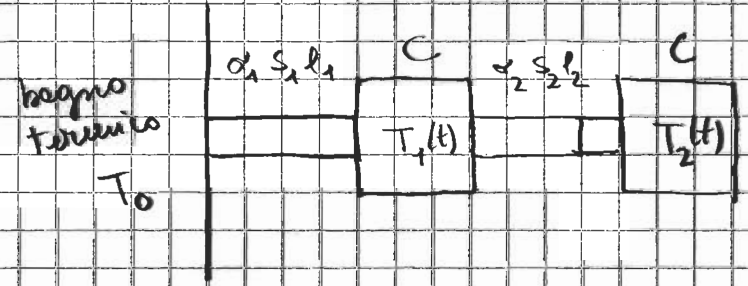
$$dQ_1 + dQ_2 + \dots + dQ_N = 0$$

$$C_1 (T_f - T_{1,0}) + C_2 (T_f - T_{2,0}) + \dots + C_N (T_f - T_{N,0}) = 0$$

$$\Rightarrow \quad (C_1 + C_2 + \dots + C_N)\, T_f = C_1 T_{1,0} + C_2 T_{2,0} + \dots + C_N T_{N,0}$$

$$\Rightarrow \quad T_f = \frac{C_1 T_{1,0} + C_2 T_{2,0} + \dots + C_N T_{N,0}}{C_1 + C_2 + \dots + C_N}$$

Adesso supponiamo $N = 3$, $C_1 \gg C_2 = C_3 \equiv C$



$$T_f = \frac{C_{bt} T_0 + C\, T_{1,0} + C\, T_{1,0}}{C_{bt} + C + C} \xrightarrow{C_{bt} \gg C} T_0 \quad (\text{ovvio})$$

$$\begin{cases} \dfrac{dQ_1}{dt} = \underbrace{\sigma_1 \dfrac{S_1}{\ell_1}}_{R_1} \left(T_0 - T_1(t)\right) - \underbrace{\dfrac{\sigma_2 S_2}{\ell_2}}_{R_2} \left(T_1(t) - T_2(t)\right) \\[2ex] \dfrac{dQ_2}{dt} = R_2 \left(T_1(t) - T_2(t)\right) \end{cases}$$

$$\Rightarrow \begin{cases} C \dfrac{dT_1}{dt} = R_1 (T_0 - T_1) - R_2 (T_1 - T_2) \\[2ex] C \dfrac{dT_2}{dt} = R_2 (T_1 - T_2) \end{cases}$$

$$\gamma_1 = \frac{R_1}{C} \qquad \gamma_2 = \frac{R_2}{C}$$

$$\begin{cases} \dfrac{dT_1}{dt} + (\gamma_1 + \gamma_2) T_1 - \gamma_2 T_2 = \gamma_1 T_0 \\[2mm] \dfrac{dT_2}{dt} \quad - \gamma_2 T_1 + \gamma_2 T_2 = 0 \end{cases}$$

Sol. particolare $\qquad T_1 = T_2 = T_0$

Sol. generale omogenea

$$\begin{matrix} T_1 = A_1 e^{\beta t} \\ T_2 = A_2 e^{\beta t} \end{matrix} \longrightarrow$$

$$\begin{cases} A_1 \beta e^{\beta t} + (\gamma_1 + \gamma_2) A_1 e^{\beta t} - \gamma_2 A_2 e^{\beta t} = 0 \\ A_2 \beta e^{\beta t} - \gamma_2 A_1 e^{\beta t} + \gamma_2 A_2 e^{\beta t} = 0 \end{cases}$$

$$\begin{cases} A_1 (\gamma_1 + \gamma_2 + \beta) - \gamma_2 A_2 = 0 \\ -\gamma_2 A_1 + (\gamma_2 + \beta) A_2 = 0 \end{cases} \Longleftrightarrow \begin{pmatrix} \gamma_1 + \gamma_2 + \beta & -\gamma_2 \\ -\gamma_2 & \gamma_2 + \beta \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

Soluzioni non banali se eq. linearmente dipendenti

$$\Rightarrow \quad \det \begin{vmatrix} \gamma_1 + \gamma_2 + \beta & -\gamma_2 \\ -\gamma_2 & \gamma_2 + \beta \end{vmatrix} = 0$$

$$\Rightarrow \quad \beta^2 + (\gamma_1 + 2\gamma_2)\beta + \gamma_1 \gamma_2 = 0$$

$$\beta = \frac{-(\gamma_1 + 2\gamma_2) \pm \sqrt{\gamma_1^2 + 4\gamma_2^2}}{2} = \beta_\pm$$

Soluzioni $\quad \begin{cases} A_1 = C_\pm (\gamma_2 + \beta_\pm) \\ A_2 = C_\pm \gamma_2 \end{cases}$

$C_\pm$ cost. arbitrarie che si trovano dalle condizioni iniziali