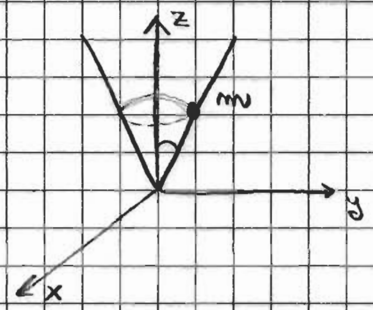
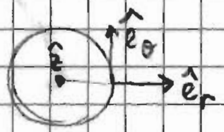
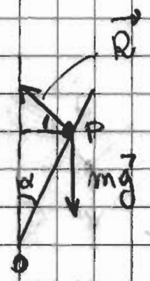
Esercizio della particella in un cono



$\alpha =$ semi apertura

Trovare le eqn del moto: usiamo coord. cilindriche





$$\begin{cases} m\ddot{z} = -mg + R\sin\alpha \\ m(\ddot{r} - r\dot{\theta}^2) = -R\cos\alpha \\ m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \end{cases}$$

Nota che $\frac{r}{z} = \text{tg}\,\alpha$ $\Rightarrow$

$$(1) \begin{cases} m\ddot{r} = -mg\,\text{tg}\,\alpha + R\sin\alpha\,\text{tg}\,\alpha \\ m(\ddot{r} - r\dot{\theta}^2) = -R\cos\alpha \\ m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \end{cases}$$

Altro modo di ricavare le eqn del moto:

$$E = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) + mgz$$

$$\vec{L}_O = m(\vec{r}_O \times \vec{v}_P) = m \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{z} \\ r & 0 & z \\ \dot{r} & r\dot{\theta} & \dot{z} \end{vmatrix}$$

$$= m\left[\hat{e}_r(-z r\dot{\theta}) - \hat{e}_\theta(r\dot{z} - z\dot{r}) + \hat{z}\, r^2\dot{\theta}\right]$$

$$\vec{M}_O = \vec{r}_O \times \vec{R} + \vec{r}_O \times m\vec{g} = -\sqrt{r^2+z^2}\,R\,\hat{e}_\theta + \sqrt{r^2+z^2}\,mg\sin\alpha\,\hat{e}_\theta$$

$$= \sqrt{r^2+z^2}\,(mg\sin\alpha - R)\,\hat{e}_\theta$$

$$\frac{d\vec{L}_0}{dt} = \vec{M}_0 \Rightarrow$$

$$\frac{d\hat{e}_r}{dt} = \dot\theta \hat{e}_\theta$$

$$m\Big[\left(-\dot z r\dot\theta - z\dot r\dot\theta - zr\ddot\theta\right)\hat{e}_r - zr\dot\theta\,\overbrace{\hat{e}_\theta\dot\theta}$$

$$-\left(\cancel{\dot r\dot z} + r\ddot z - \cancel{\dot z\dot r} - z\ddot r\right)\hat{e}_\theta - \left(r\dot z - z\dot r\right)\left(-\dot\theta\hat{e}_r\right)$$

$$+\hat{z}\left(2\dot r r\dot\theta + r^2\ddot\theta\right)\Big]$$

$$= \sqrt{r^2+z^2}\,(mg\sin\alpha - R)\,\hat{e}_\theta$$

$$\Rightarrow \begin{cases} 2\cancel{r}\dot r\dot\theta + r^{\cancel{2}}\ddot\theta = 0 \\ \cancel{r}\dot z\dot\theta + \cancel{z\dot r\dot\theta} + z\cancel{r}\ddot\theta + \cancel{r}\dot z\dot\theta - \cancel{\dot r\dot\theta z} = 0 \\ m\left(-zr\dot\theta^2 - r\ddot z + z\ddot r\right) = \sqrt{r^2+z^2}\,(mg\sin\alpha - R) \end{cases}$$

$$\Rightarrow \boxed{r\ddot\theta + 2\dot r\dot\theta = 0} \qquad (2)$$

$$z\ddot\theta + 2\dot z\dot\theta = 0 = \frac{r\ddot\theta + 2\dot r\dot\theta}{\operatorname{tg}\alpha} \Rightarrow \text{non dà altra informazione}$$

$$m\left(-\frac{r^{\cancel{2}}\dot\theta^2}{\operatorname{tg}\alpha} - \cancel{\frac{r\ddot r}{\operatorname{tg}\alpha}} + \cancel{\frac{r\ddot r}{\operatorname{tg}\alpha}}\right) = \cancel{r}\sqrt{1+\frac{1}{\operatorname{tg}^2\alpha}}\,(mg\sin\alpha - R)$$

$$-mr\dot\theta^2\frac{\cos\alpha}{\cancel{\sin\alpha}} = \sqrt{1+\frac{\cos^2\alpha}{\sin^2\alpha}}\,(mg\sin\alpha - R) = \frac{1}{\cancel{\sin\alpha}}(mg\sin\alpha - R)$$

$$\boxed{mr\dot\theta^2\cos\alpha = R - mg\sin\alpha} \qquad (3)$$

La III eq. dalla conservazione dell'energia:

$$\frac{dE}{dt} = 0 = \cancel{\frac{1}{2}}\cancel{m}\left(\cancel{2}\dot r\ddot r + \cancel{2}r\dot r\dot\theta^2 + \cancel{2}r^2\dot\theta\ddot\theta + \cancel{2}\dot z\ddot z\right) + \cancel{m}g\dot z$$

$$\boxed{\dot r\ddot r + r\dot r\dot\theta^2 + r^2\dot\theta\ddot\theta + \frac{\dot r\ddot r}{tg^2\alpha} + \frac{g\dot r}{tg\alpha} = 0} \qquad (4)$$

(2) – (4) sembrano diverse da (1) e però:

Da (2) so che $r\ddot\theta = -2\dot r\dot\theta \Rightarrow$ (4) diventa

$$\cancel{\dot r}\ddot r + r\cancel{\dot r}\dot\theta^2 + r\dot\theta(-2\cancel{\dot r}\dot\theta) + \frac{\ddot r\cancel{\dot r}}{tg^2\alpha} + \frac{g\cancel{\dot r}}{tg\alpha} = 0$$

$$\ddot r - r\dot\theta^2 + \frac{\ddot r}{tg^2\alpha} + \frac{g}{tg\alpha} = 0$$

$$\ddot r - r\dot\theta^2 + \ddot r\frac{\cos^2\alpha}{\sin^2\alpha} + g\frac{\cos\alpha}{\sin\alpha} = 0$$

$$\frac{\ddot r}{\sin^2\alpha} + g\frac{\cos\alpha}{\sin\alpha} - r\dot\theta^2 = 0$$

$$m\ddot r\frac{\cos\alpha}{\sin^2\alpha} + mg\frac{\cos^2\alpha}{\sin\alpha} = mr\dot\theta^2\cos\alpha \overset{(3)}{=} R - mg\sin\alpha$$

$$m\ddot r = R\frac{\sin^2\alpha}{\cos\alpha} - mg\frac{\sin^{\cancel{2}}\alpha}{\cancel{\sin\alpha}\cos\alpha} \Rightarrow$$ è uguale alla I eq. di (1)

Combinando questa eqn. con (3)

$$m\ddot r - mr\dot\theta^2 = R\frac{\sin^2\alpha}{\cos\alpha} - \cancel{mg\frac{\sin\alpha}{\cos\alpha}} - \frac{R}{\cos\alpha} + \cancel{mg\frac{\sin\alpha}{\cos\alpha}}$$

$$= R\frac{-(1-\sin^2\alpha)}{\cos\alpha} = -R\cos\alpha$$

$\Downarrow$

è la II eqn. di (1) !

Quanto deve essere $\dot{\theta}$ per avere un moto circolare?

Moto circ. $\Rightarrow$ $\dot{r} = \ddot{r} = 0$ $\Rightarrow$ (1) diventa

$$\begin{cases} -m r\dot{\theta}^2 = -R\cos\alpha \\ mg = R\sin\alpha \end{cases} \Rightarrow \quad \cancel{m} r\dot{\theta}^2 = \cancel{m} g\,\frac{\cos\alpha}{\sin\alpha}$$

$$\dot{\theta} = \sqrt{\frac{g}{r}\,\frac{\cos\alpha}{\sin\alpha}}$$

Trovare r_{min} e r_{max} (si potrà avere $r_{min} = 0$?)

$$E = \frac{1}{2} m\,\dot{r}^2\left(1 + \frac{1}{tg^2\alpha}\right) + \frac{1}{2} m r^2\dot{\theta}^2 + mg\,\frac{r}{tg\,\alpha}$$

$L_{o_z} = m\,r^2\,\dot{\theta}$ ed è costante $\Rightarrow$

$$E = \frac{1}{2} m\,\frac{\dot{r}^2}{\sin^2\alpha} + \frac{1}{2}\cancel{m}\,\cancel{r^2}\,\frac{L_{o_z}^2}{m^{\cancel{2}} r^{\cancel{4}\,2}} + mg\,r\,\frac{\cos\alpha}{\sin\alpha}$$

$$= \frac{1}{2} m\left(\frac{\dot{r}}{\sin\alpha}\right)^2 + \frac{L_{o_z}^2}{2mr^2} + mg\,\frac{r\cos\alpha}{\sin\alpha} = \text{cost.}$$

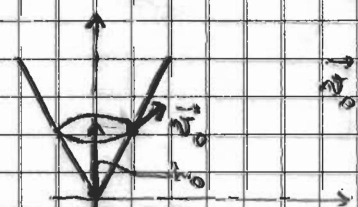
Se $r = 0$ $\Rightarrow$ $E = \infty$ $\Rightarrow$ non potrò mai avere $r = 0$

r_{max} e r_{min} $\Rightarrow$ $\dot{r} = 0$ $\Rightarrow$

$$E = \frac{L_{o_z}^2}{2mr^2} + mgr\,\frac{\cos\alpha}{\sin\alpha}$$

$$2m^2 g\,\frac{\cos\alpha}{\sin\alpha}\,r^3 - 2mr^2 E + L_{o_z}^2 = 0$$

Supponiamo che a $t = 0$



$\vec{v}_0 = v_0\,\hat{\theta}_0$

$r_0 = h_0\,tg\,\alpha$

$\Rightarrow$ $E = \frac{1}{2} m v_0^2 + mg h_0$

$L_{o_z} = m\,h_0\,tg\,\alpha\,v_0$

$$2\,m^2\,g\,r^3\,\frac{\cos\alpha}{\sin\alpha} - 2\,m\,r^2\left(\frac{1}{2}\,m v_0^2 + m g\,\frac{r_0}{\sin\alpha}\cos\alpha\right)$$

$$+\; m^2\,r_0^2\,\cancel{\frac{\cos^2\alpha}{\sin^2\alpha}}\,\cancel{\left(\frac{\sin\alpha}{\cos\alpha}\right)^2}\,v_0^2 = 0$$

$$2\,m^2 g\,\frac{\cos\alpha}{\sin\alpha}\,r^3 - m^2 v_0^2\left(r^2 - r_0^2\right) - 2\,m^2 g\,r^2\,r_0\,\frac{\cos\alpha}{\sin\alpha} = 0$$

$$2\,\cancel{m^2}\,g\,\frac{\cos\alpha}{\sin\alpha}\,r^2\,\cancel{(r - r_0)} - \cancel{m^2}\,v_0^2\,\cancel{(r - r_0)}\,(r + r_0) = 0$$

$$2g\,\frac{\cos\alpha}{\sin\alpha}\,r^2 - v_0^2\,r - v_0^2\,r_0 = 0$$

$$r = \frac{+v_0^2 \pm \sqrt{v_0^4 + 8\,g\,v_0^2\,\frac{\cos\alpha}{\sin\alpha}\,r_0}}{4g\,\frac{\cos\alpha}{\sin\alpha}}$$

$$\Rightarrow \begin{cases} r_{min} = \dfrac{v_0^2 \sin\alpha}{4\,g\cos\alpha}\left[1 + \sqrt{1 + \dfrac{8\,g\,r_0\cos\alpha}{v_0^2\sin\alpha}}\right] \\[2ex] r_{max} = r_0 \end{cases}$$