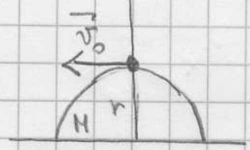
Esercizio della pallina su semisfera mobile

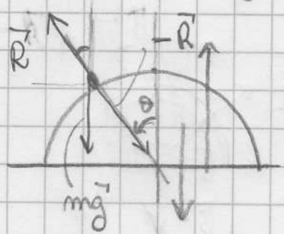
x casa
con $v_0 = 0$ e $M = \infty$

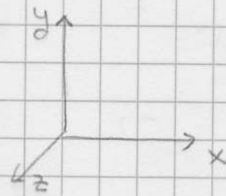


1) A che angolo si stacca?

Conservazione dell'energia si può applicare?

Forze: $m\vec{g}$, $M\vec{g}$, $\vec{R}_t$, $\vec{R}$, $-\vec{R}$





$m\vec{g}$ e $M\vec{g}$ conservative, le altre non fanno lavoro $\Rightarrow$

$$\frac{1}{2} m v_0^2 + mgr = \frac{1}{2} M V^2 + \frac{1}{2} m v^2 + mgr\cos\theta \qquad (1)$$

So che $\vec{v} = \vec{V} + \vec{\omega} \times \vec{r}$ $\qquad \vec{\omega} = +\dot{\theta}\,\hat{z}$

Inoltre lungo l'asse $\hat{x}$ ho conservazione della q.d.m. totale (forze esterne sono lungo y) $\Rightarrow$

$$\begin{cases} MV + m\dot{x} = -m v_0 \\ \dot{x} = V - r\dot{\theta}\cos\theta \\ \dot{y} = -r\dot{\theta}\sin\theta \end{cases} \Rightarrow \qquad MV + m(+V - \dot{\theta} r\cos\theta) = -m v_0$$

$$V = -\frac{m}{m+M}\left(v_0 - r\dot{\theta}\cos\theta\right)$$

$\Rightarrow$ (1) diventa:

$$\frac{1}{2} m v_0^2 + mgr = \frac{1}{2} M V^2 + \frac{1}{2} m \left(V^2 + \dot{\theta}^2 r^2 - 2V\dot{\theta} r\cos\theta\right) + mgr\cos\theta$$

$$\frac{1}{2} m v_0^2 + mgr = \frac{1}{2}\cancel{(m+M)}\, \frac{m^2}{(m+M)^{\cancel{2}}} \left(v_0 - r\dot\theta\cos\theta\right)^2$$

$$+ \frac{1}{2} m r^2 \dot\theta^2 - m r\dot\theta\cos\theta\, \frac{-m}{m+M}\left(v_0 - r\dot\theta\cos\theta\right) + mgr$$

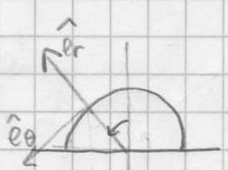
$$\frac{1}{2}\cancel{m} v_0^2 + \cancel{m} g r = \frac{1}{2}\frac{m^{\cancel{2}}}{M_T} v_0^2 + \cancel{\frac{1}{2}\frac{m^2}{M_T} r^2\dot\theta^2\cos^2\theta} - \cancel{\frac{m^2}{M_T} v_0 r\dot\theta\cos\theta}$$

$$+ \frac{1}{2}\cancel{m} r^2\dot\theta^2 + \cancel{\frac{m^2}{M_T} v_0 r\dot\theta\cos\theta} - \overset{1/2}{\cancel{1}}\frac{m^{\cancel{2}}}{M_T} r^2\dot\theta^2\cos^2\theta + \cancel{m} g r$$

$$\frac{1}{2} v_0^2 \left(\frac{M + \cancel{m} - \cancel{m}}{M_T}\right) + gr\left(1-\cos\theta\right) + \frac{1}{2} r^2\dot\theta^2\left[\frac{m}{M_T}\cos^2\theta - 1\right] =$$

$$\frac{1}{2}\frac{M}{M_T} v_0^2 + gr\left(1-\cos\theta\right) + \frac{1}{2} r^2\dot\theta^2\left[\frac{m}{M_T}\cos^2\theta - 1\right] = 0 \qquad (2)$$

Condizione perché si stacchi: $R = 0 \Rightarrow$



$$m a_r = R - mg\cos\theta \qquad \text{SR semisfera}$$

$$a_r = -\omega^2 r = -r\dot\theta^2$$

$$\Rightarrow R = 0 \Rightarrow \cancel{m} r\dot\theta^2 = \cancel{m} g\cos\theta \Rightarrow$$

$$\frac{1}{2}\frac{M}{M_T} v_0^2 + gr - \cancel{gr\cos\theta} + \frac{1}{2}\frac{m}{M_T} gr\cos^3\theta - \overset{3/2}{\cancel{\tfrac{1}{2}}}\, rg\cos\theta = 0$$

$$\boxed{\frac{m}{M_T} gr\cos^3\theta - 3gr\cos\theta + \frac{M}{M_T} v_0^2 + 2gr = 0} \qquad (3)$$

Eq. difficile. Più semplice per $M \gg m \Rightarrow$

$$-3\cancel{g}\cancel{r}\cos\theta + 2\cancel{g}\cancel{r}^{\,1} + \frac{v_0^2}{gr} = 0$$

$$3\cos\theta = 2 + \frac{v_0^2}{gr} \Rightarrow \cos\theta = \left(\frac{2gr + v_0^2}{3gr}\right)$$

2) Quanto deve essere v_0 perchè si stacchi subito?

Da (3) con $\theta = 0$

$$\underbrace{\frac{m}{M_T} gr - \cancel{3} gr}\; + \frac{M}{M_T} v_0^2 \cancel{+ 2gr} = 0$$

$$\left(\frac{\cancel{m} - \cancel{m} - M}{\cancel{M_T}}\right) gr + \frac{M}{\cancel{M_T}} v_0^2 = 0$$

$$v_0 = \sqrt{gr}$$

3) Trovare le eqn. del moto finchè m resta attaccato a M

a) I eqn. cardinali:

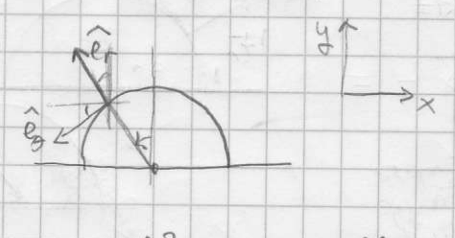
$$\begin{cases} m\ddot{x} = -R\sin\theta \\ m\ddot{y} = R\cos\theta - mg \\ M\ddot{x}_M = +R\sin\theta \\ M\ddot{y}_M = 0 = -Mg + R_t - R\cos\theta \end{cases}$$

$$m\ddot{x} + M\ddot{x}_M = 0 \quad (\text{f. esterne solo } /\!/ \hat{x})$$

$$\vec{a} = \vec{a}_{tr} + \vec{a}'$$

Nota che $\vec{v}' = r\dot{\theta}\,\hat{e}_\theta$

$$\vec{a}' = -r\dot{\theta}^2\hat{e}_r + r\ddot{\theta}\,\hat{e}_\theta$$



$$\Rightarrow \begin{cases} \ddot{x} = \ddot{x}_M + r\dot{\theta}^2\sin\theta - r\ddot{\theta}\cos\theta \\ \ddot{y} = -r\dot{\theta}^2\cos\theta - r\ddot{\theta}\sin\theta \end{cases}$$

$$\Rightarrow \begin{cases} m\,\ddot{x}_M + m r \dot{\theta}^2 \sin\theta - m r \ddot{\theta} \cos\theta = -R \sin\theta \\ - m r \dot{\theta}^2 \cos\theta - m r \ddot{\theta} \sin\theta = R \cos\theta - mg \\ M\,\ddot{x}_M = R \sin\theta \end{cases}$$

Risolvo

$$+\frac{m}{M} R \sin\theta + m r \dot{\theta}^2 \sin\theta - m r \ddot{\theta} \cos\theta = - R \sin\theta$$

$$\cos\theta \left(R \sin\theta \left(1 + \frac{m}{M} \right) \right) = \left[- m r \dot{\theta}^2 \sin\theta + m r \ddot{\theta} \cos\theta \right] \cos\theta$$

$$-\cancel{m} r \dot{\theta}^2 \sin\theta \cos\theta + \cancel{m} r \ddot{\theta} \cos^2\theta =$$

$$\sin\theta \left(1 + \frac{m}{M} \right) \left(\cancel{m} g - \cancel{m} r \dot{\theta}^2 \cos\theta - \cancel{m} r \ddot{\theta} \sin\theta \right)$$

$$r \ddot{\theta} \left(\underbrace{\sin^2\theta + \cos^2\theta}_{1} + \sin^2\theta \frac{m}{M} \right)$$

$$+ r \dot{\theta}^2 \sin\theta \cos\theta \left[\cancel{1} - \cancel{1} + \frac{m}{M} \right] - g \sin\theta \left(1 + \frac{m}{M} \right) = 0$$

$$r \ddot{\theta} \left(\underbrace{1 + \frac{m}{M}}_{\frac{m+M}{M}} - \frac{m}{M} \cos^2\theta \right) \frac{M}{m+M} + r \dot{\theta}^2 \sin\theta \cos\theta \frac{m}{\cancel{M}} \frac{\cancel{M}}{m+M}$$

$$- g \sin\theta = 0$$

$$r \ddot{\theta} \left(1 - \frac{m}{m+M} \cos^2\theta \right) + r \dot{\theta}^2 \sin\theta \cos\theta \frac{m}{m+M} - g \sin\theta = 0$$

$\dot{\theta}^2$ si ottiene dall'Eq. (2) (cons. energia)

b) dall'Eq. (2) : $E = \text{cost} \Rightarrow \frac{dE}{dt} = 0$

$$+ g \cancel{r} \sin\theta \cancel{\dot{\phi}} + \frac{\cancel{1}}{\cancel{2}} r^2 \cancel{2} \cancel{\dot{\phi}} \ddot{\theta} \left[\frac{m}{m+M} \cos^2\theta - 1 \right]$$

$$+ \frac{\cancel{1}}{\cancel{2}} r^2 \dot{\theta}^2 \left[- \frac{m}{m+M} \cancel{2} \sin\theta \cos\theta \cancel{\dot{\phi}} \right] = 0$$

$$r\ddot{\theta}\left[1-\frac{m}{m+M}\cos^2\theta\right]+r\dot{\theta}^2\sin\theta\cos\theta\,\frac{m}{m+M}-g\sin\theta=0$$

Assai più veloce che con a) !!

$$r\dot{\theta}^2=\frac{\cancel{\tfrac{1}{2}}\,\frac{M}{M_T}\,\frac{v_0^2}{r}+2g\cancel{r}\,(1-\cos\theta)}{\cancel{\tfrac{1}{2}}\cancel{r}\left(\frac{m}{M_T}\cos^2\theta-1\right)}$$

$$\Rightarrow\quad r\ddot{\theta}\left[1-\frac{m}{m+M}\cos^2\theta\right]+\frac{m}{m+M}\sin\theta\cos\theta\;\frac{\frac{M}{M+m}\frac{v_0^2}{r}+2g(1-\cos\theta)}{\left(\frac{m}{m+M}\cos^2\theta-1\right)}$$

$$-g\sin\theta=0$$