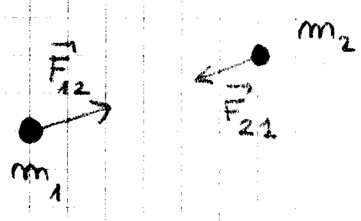
Esercizio sul moto in campo gravitazionale

Leggi di Keplero



$$\vec{F}_{12} = -\vec{F}_{21}$$

$$|\vec{F}_{12}| = |\vec{F}_{21}| = F(r) = \frac{G m_1 m_2}{r^2} \equiv \frac{k}{r^2}$$

$$k \doteq G m_1 m_2$$

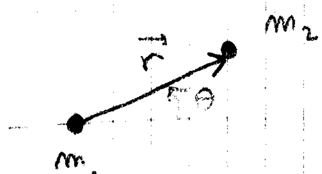
Eq. del moto:
$$\begin{cases} m_1 \ddot{\vec{r}}_1 = -\vec{F}_{21} \\ m_2 \ddot{\vec{r}}_2 = \vec{F}_{21} \end{cases}$$

Si possono riscrivere anche come:
$$\begin{cases} m_1 \ddot{\vec{r}}_1 + m_2 \ddot{\vec{r}}_2 = 0 \\ \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1 = \vec{F}_{21}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \end{cases}$$

$$\begin{cases} (m_1 + m_2)\, \ddot{\vec{r}}_{CM} = 0 \\ \frac{m_1 m_2}{m_1 + m_2} \ddot{\vec{r}} = \vec{F}_{21} \; ; \quad \vec{r} = \vec{r}_2 - \vec{r}_1 \end{cases}$$

$\Rightarrow$ CM. va di moto rettilineo uniforme $\Rightarrow$ mi metto nel SR CM. $\Rightarrow$

$$\boxed{\mu \ddot{\vec{r}} = -\frac{k}{r^2}\hat{r}}$$



equazione non banale da risolvere!

Procediamo per altra via: costanti del moto sono

1) energia E
2) mom. angolare $\vec{L}_{CM}$ (sono in SRCM)

1) $E = \frac{1}{2}\mu v^2 + U(r)$

$$U(r) - \underbrace{U(\infty)}_{=0} = -\int_{\infty}^{r} \left(-\frac{k}{x^2}\right) dx = -\frac{k}{r}$$

$$\Rightarrow \quad E = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right) - \frac{k}{r}$$

2) $\vec{L}_{CM} = \mu\, \vec{r} \times \vec{v}$

- $\vec{L} = m_1\, \vec{r}_1 \times \vec{v}_1 + m_2\, \vec{r}_2 \times \vec{v}_2$

$$\begin{cases} m_1 \vec{r}_1 + m_2 \vec{r}_2 = m_t \vec{r}_{CM} \\ m_1 \vec{v}_1 + m_2 \vec{v}_2 = m_t \vec{v}_{CM} \end{cases}$$

$$\begin{cases} \vec{r}_2 - \vec{r}_1 = \vec{r} \\ \vec{v}_2 - \vec{v}_1 = \vec{v} \end{cases} \Rightarrow \begin{cases} \vec{r}_1 = \vec{r}_{CM} - \frac{m_2}{m_t}\vec{r} \\ \vec{r}_2 = \vec{r}_{CM} + \frac{m_1}{m_t}\vec{r} \end{cases} \quad \begin{cases} \vec{v}_1 = \vec{v}_{CM} - \frac{m_2}{m_t}\vec{v} \\ \vec{v}_2 = \vec{v}_{CM} + \frac{m_1}{m_t}\vec{v} \end{cases}$$

$$\Rightarrow \vec{L} = m_1\left[\vec{r}_{CM} \times \vec{v}_{CM} - \cancel{\frac{m_2}{m_t}\vec{r}_{CM} \times \vec{v}} - \cancel{\frac{m_2}{m_t}\vec{r} \times \vec{v}_{CM}} + \frac{m_2^2}{m_t^2}\vec{r} \times \vec{v}\right]$$

$$+ m_2\left[\vec{r}_{CM} \times \vec{v}_{CM} + \cancel{\frac{m_1}{m_t}\vec{r}_{CM} \times \vec{v}} + \cancel{\frac{m_1}{m_t}\vec{r} \times \vec{v}_{CM}} + \frac{m_1^2}{m_t^2}\vec{r} \times \vec{v}\right]$$

$$= (m_1 + m_2)\,\vec{r}_{CM} \times \vec{v}_{CM} + \frac{m_1 m_2}{m_t^{\cancel{2}}}\cancel{(m_2 + m_1)}\,\vec{r} \times \vec{v}$$

$$= M_{tot}\,\vec{r}_{CM} \times \vec{v}_{CM} + \mu\,\vec{r} \times \vec{v} \quad \blacksquare$$

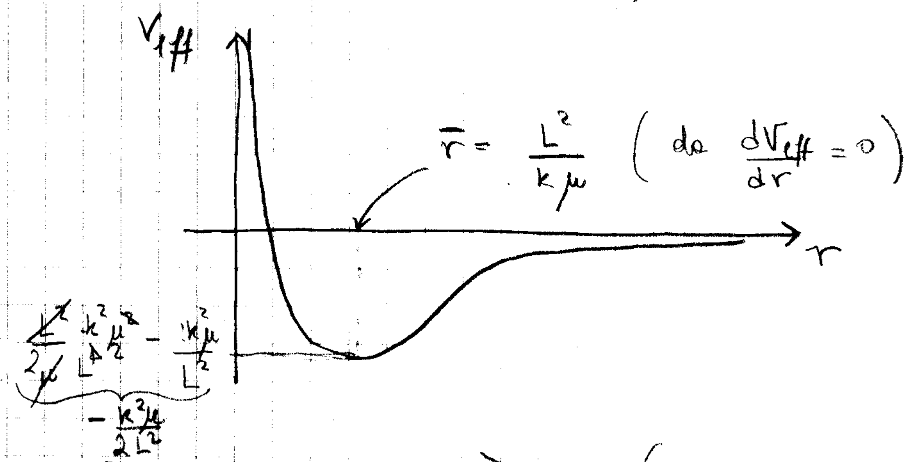
$$\Rightarrow \quad \vec{L}_{CM} = \mu r^2 \dot{\theta}\,\hat{z} \quad (\perp \text{ al foglio})$$

$\vec{L}_{CM}$ si conserva $\Rightarrow$ il <u>moto è piano</u> !!

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 - \frac{k}{r} = \frac{1}{2}\mu\dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{k}{r}$$

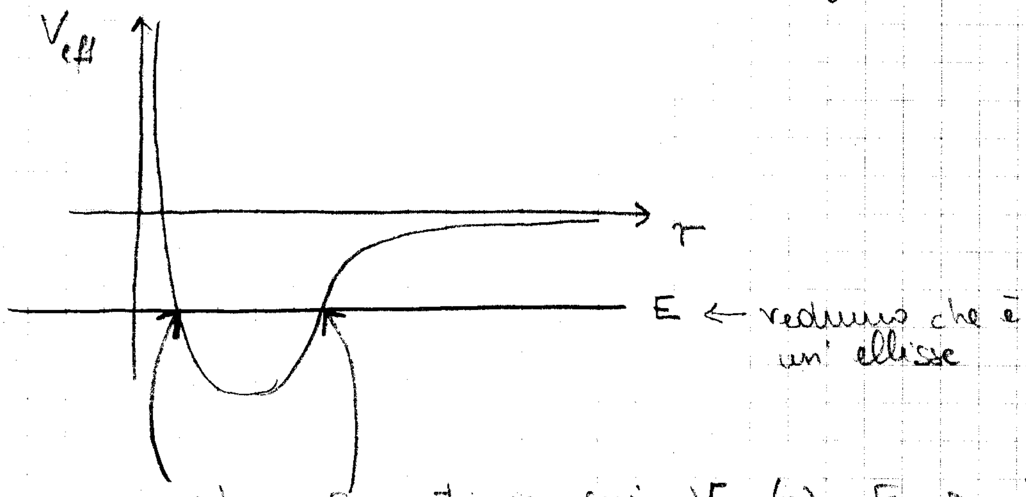
Che traiettoria fa?

Consideriamo $V_{eff}(r) \overset{\text{definisco}}{\equiv} \dfrac{L^2}{2\mu r^2} - \dfrac{k}{r}$



E può essere $\gtrless 0$ (per esempio a $t=0$ sono a $v_0 = 0$ e $r = r_0 \Rightarrow E = -\dfrac{k}{r_0}$)

$\boxed{E < 0}$



esistono 2 punti in cui $V_{eff}(r) = E \Rightarrow \dot{r} = 0$

$\Rightarrow$ sono r_{max} e r_{min}

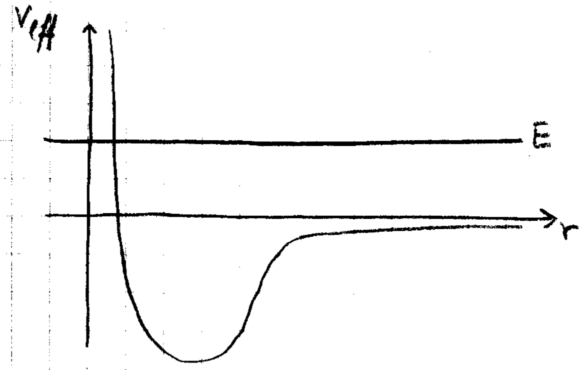
$\downarrow$ $E = -\dfrac{1}{2}\mu\dfrac{k^2}{L^2}$

$\Rightarrow$ traiettoria chiusa

In particolare per $E = V_{eff}(\bar{r})$ $\dot{r} = 0$ sempre

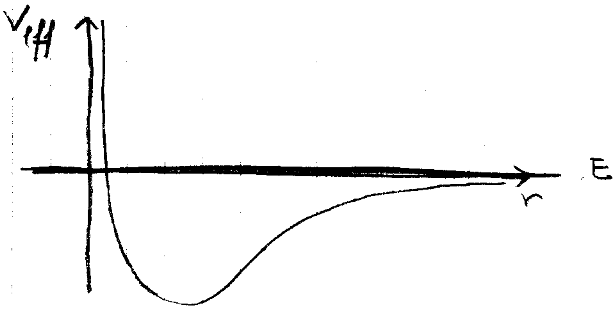
$\Rightarrow$ traiettoria circolare

$\boxed{E > 0}$



∃ una distanza minima, ma la traiettoria è aperta.
Vedremo che si tratta di un'iperbole

$\boxed{E = 0}$



La traiettoria è aperta e $\dot{r} = 0$ all'infinito
($E = 0$ e $V(r) = 0$ per $r = \infty$). Vedremo che si tratta di una parabola.

Risolviamo adesso per bene:

$$E = \frac{1}{2}\mu \dot{r}^2 + \frac{L^2}{2\mu r^2} - \frac{k}{r} \qquad L = \mu r^2 \dot{\theta}$$

$$= \frac{1}{2}\mu \left(\frac{dr}{d\theta}\right)^2 \dot{\theta}^2 + \frac{L^2}{2\mu r^2} - \frac{k}{r} = \frac{1}{2}\cancel{\mu} \left(\frac{dr}{d\theta}\right)^2 \frac{L^2}{\mu^{\cancel{2}} r^4} + \frac{L^2}{2\mu r^2} - \frac{k}{r}$$

$$\Rightarrow \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{2E\mu}{L^2} - \frac{1}{r^2} + \frac{2\mu k}{L^2}\frac{1}{r}$$

Introduco $u(\theta) := \dfrac{1}{r(\theta)} \Rightarrow \dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta} \Rightarrow$

$$\left(\frac{du}{d\theta}\right)^2 = \frac{2E\mu}{L^2} - u(\theta)^2 + \frac{2\mu k}{L^2} u(\theta)$$

$$A = \frac{2E\mu}{L^2} \quad ; \quad B = \frac{2\mu k}{L^2}$$

$$\left(\frac{du}{d\theta}\right)^2 = A + Bu - u^2$$

Soluzione: $\quad u(\theta) = \frac{B}{2} - \sqrt{A + \frac{B^2}{4}}\cos\theta$

■ Verifica: $\quad \frac{du}{d\theta} = \left(A + \frac{B^2}{4}\right)^{1/2}\sin\theta \Rightarrow$

$$\left(A + \frac{B^2}{4}\right)\sin^2\theta = A + \frac{B^2}{2} - \cancel{B\sqrt{A + \frac{B^2}{4}}\cos\theta}$$

$$- \frac{B^2}{4} - \left(A + \frac{B^2}{4}\right)\cos^2\theta + \cancel{B\sqrt{A + \frac{B^2}{4}}\cos\theta}$$

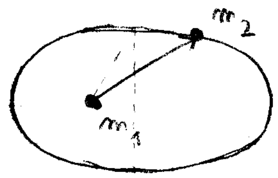
$$= \left(A + \frac{B^2}{4}\right)(1 - \cos^2\theta) \quad ■$$

$$\Rightarrow \begin{cases} r(\theta) = \dfrac{1}{\frac{B}{2} - \sqrt{A + \frac{B^2}{4}}\cos\theta} = \dfrac{l}{1 - e\cos\theta} \\[2ex] l = \dfrac{2}{B} = \dfrac{L^2}{\mu k} \\[2ex] e = \dfrac{2}{B}\sqrt{A + \dfrac{B^2}{4}} = \sqrt{1 + \dfrac{4A}{B^2}} = \sqrt{1 + \dfrac{\cancel{8}^{2} E \cancel{\mu}}{\cancel{L^2}} \dfrac{L^{\cancel{4}\,2}}{\cancel{4}\mu^{\cancel{2}} k^2}} \\[2ex] = \sqrt{1 + \dfrac{2EL^2}{\mu k^2}}\,. \end{cases}$$

$\boxed{e=0}$ $\quad 1+\frac{2EL^2}{\mu k^2}=0 \qquad E=-\frac{1}{2}\frac{\mu k^2}{L^2} \qquad r=\ell=\text{cost}$

$\Rightarrow$ circonferenza di raggio $\frac{L^2}{\mu k}$

$\boxed{0<e<1}$



ellisse. Ho

$r_{max}=\frac{\ell}{1-e}$

$r_{min}=\frac{\ell}{1+e}$

$\Rightarrow$ semiasse max $= \frac{r_{max}+r_{min}}{2} = \frac{\ell(1+\cancel{e})+\ell(1-\cancel{e})}{2(1-e^2)}$
$\qquad \| $
$\qquad a$

$= \frac{\ell}{1-e^2}$

(Relazione tra semiasse max e min dell'ellisse è $b=a\sqrt{1-e^2}$)

Altro modo di vederlo:

$r=\frac{\ell}{1+e\cos\theta} \qquad \begin{array}{l} x=r\cos\theta \\ y=r\sin\theta \end{array} \Rightarrow$

$\sqrt{x^2+y^2}-ex=\ell$

$x^2+y^2=\ell^2+e^2x^2+2\ell ex$

$x^2(1-e^2)+2\ell ex+\frac{\ell^2e^2}{(1-e^2)}-\frac{\ell^2e^2}{1-e^2}+y^2=\ell^2$

$\left[x(1-e^2)^{1/2}+\frac{\ell e}{(1-e^2)^{1/2}}\right]^2+y^2=\ell^2\left(1+\frac{e^2}{1-e^2}\right)$

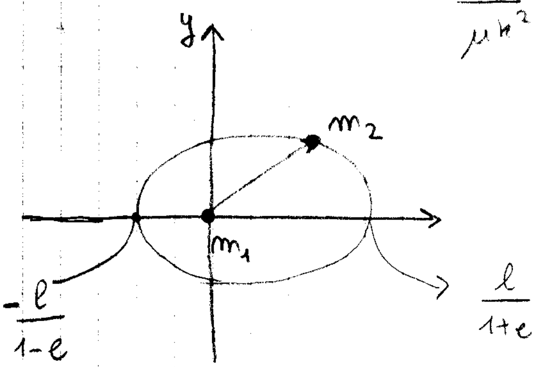
$(1-e^2)\left[x+\frac{\ell e}{1-e^2}\right]^2+y^2=\frac{\ell^2}{1-e^2}$

$X=x+\frac{\ell e}{1-e^2} \qquad \frac{X^2}{a^2}+\frac{y^2}{b^2}=1$ con

$$\begin{cases} a^2 = \dfrac{\ell^2}{(1-e^2)^2} \\ b^2 = \dfrac{\ell^2}{1-e^2} \end{cases} \Rightarrow \begin{cases} a = \dfrac{\ell}{1-e^2} \\ b = \dfrac{\ell}{\sqrt{1-e^2}} \end{cases} \qquad b = a\sqrt{1-e^2}$$

$$0 < e < 1 \quad \Leftrightarrow \quad 0 < 1 + \frac{2EL^2}{\mu k^2} < 1$$

$$\Rightarrow \quad \frac{EL^2}{\mu k^2} < 0 \qquad E < 0$$



$\boxed{e > 1}$

$$\begin{cases} r \to \infty \\ \cos\theta = \dfrac{1}{e} \end{cases} \Rightarrow \text{iperbole}$$

Ma vediamola meglio:

$$\sqrt{x^2+y^2} - ex - \ell$$

$$x^2(1-e^2) + 2\ell e x + \frac{\ell^2 e^2}{(1-e^2)} - \frac{\ell^2 e^2}{(1-e^2)} + y^2 = \ell^2$$

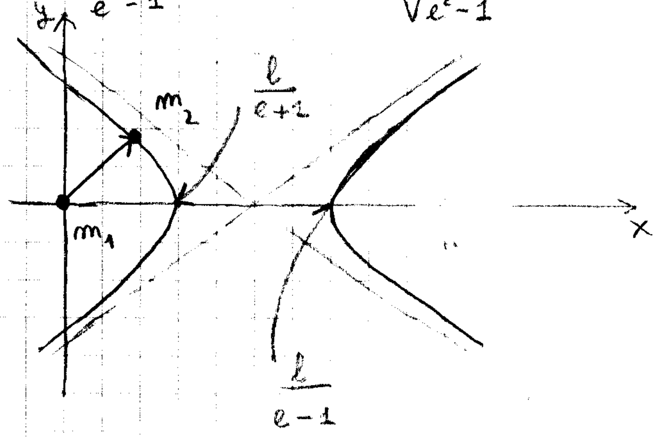
Noto che $1-e^2 < 0 \Rightarrow$

$$x^2(e^2-1) - 2\ell e x + \frac{\ell^2 e^2}{e^2-1} - \frac{\ell^2 e^2}{e^2-1} - y^2 = -\ell^2$$

$$(e^2-1)\left[X - \frac{\ell e}{e^2-1}\right]^2 - y^2 = -\ell^2\left(1 - \frac{\ell^2}{e^2-1}\right) = +\frac{\ell^2}{e^2-1}$$

$$X = x - \frac{\ell e}{e^2-1} \qquad \frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1 \quad \text{con}$$

$$a = \frac{l}{e^2-1}, \quad b = \frac{l}{\sqrt{e^2-1}} \longrightarrow \text{Iperbole}$$

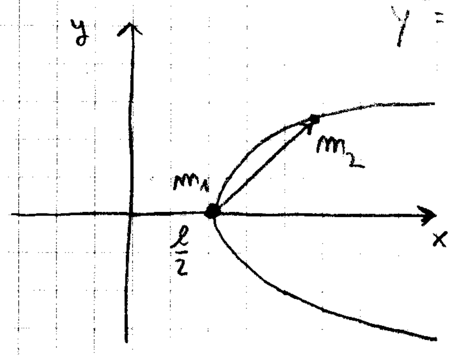


$$y = 0 \Rightarrow x^2 = a^2$$
$$X = \pm a$$
$$X - \frac{le}{e^2-1} = \pm \frac{l}{e^2-1}$$
$$(e^2-1)\,x = \pm l + le$$
$$x = \begin{cases} \frac{l}{e-1} \\ \frac{l}{e+1} \end{cases}$$

$$e > 1 \Rightarrow 1 + \frac{2EL^2}{\mu k^2} > 1 \qquad E > 0$$

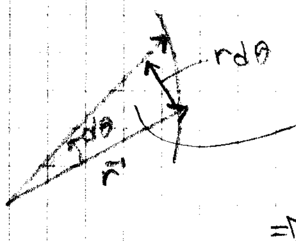
$\boxed{e = 1}$ $\quad r = \frac{l}{1-\cos\theta} \Rightarrow \theta = 0 \quad r \to \infty$ parabola

Di nuovo:
$$\sqrt{x^2+y^2} - x = l$$
$$\cancel{x^2} + y^2 = l^2 + \cancel{x^2} - 2lx$$
$$y^2 = l^2 - 2lx = l\,\underbrace{(l-2x)}_{X} \qquad y^2 = lX$$



$$e = 1 \Rightarrow 1 + \frac{2EL^2}{\mu k^2} = 1 \Rightarrow E = 0$$

Ultima cosa da notare: supponiamo $E<0$ ($\Rightarrow$ orbite chiuse)



$$dS = \frac{1}{2} r^2 d\theta \Rightarrow \dot{S} \equiv \text{velocità areale} = \frac{1}{2} r^2 \dot{\theta}$$

$$\Rightarrow \underset{\text{cost}}{\underset{\shortparallel}{L}} = \mu r^2 \dot{\theta} = 2\mu \dot{S} \Rightarrow \dot{S} \text{ è costante}$$

$$\dot{S} = \frac{S}{T} \overset{\text{ellisse}}{=} \frac{\pi a b}{T} \Rightarrow T = \frac{2\mu\pi a b}{L} = \frac{2\mu\pi a\, a\sqrt{1-e^2}}{\sqrt{k\mu l}} = \frac{2\mu\pi a^2 \cancel{\sqrt{1-e^2}}}{\sqrt{k\mu a (1-e^2)}}$$

$$T = 2\pi \sqrt{\frac{\mu}{k}}\, a^{3/2}$$

Suppongo $m_1 \gg m_2 \Rightarrow \mu = m_2 \leftarrow$ caso delle leggi di Keplero

1) Pianeti si muovono su un'orbita ellittica di cui il sole occupa uno dei fuochi [E cost. del moto < 0]

2) La velocità areale è costante [L è cost. del moto]

3) $T \propto a^{3/2}$ [Conseguenza delle precedenti]

Infine

$$a = \frac{\ell}{1-e^2} = \frac{\cancel{L}^2}{\mu \cancel{k}} \frac{1}{\cancel{1} - \cancel{1} - \frac{2E\cancel{L}^2}{\cancel{\mu} k^2}} = \frac{k}{2|E|}$$

$\Rightarrow$ a dipende solo da E !!