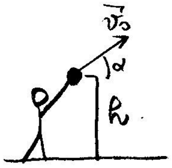
Esercizio sul lanciatore del peso

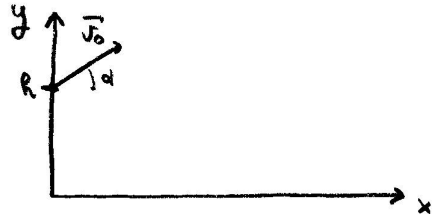


Lancio da h, con v_0.

Trovare α tale che il lancio sia massimo

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{g}\, t^2$$

Sagliamo gli assi:



$$\begin{cases} x = v_0 \cos\alpha\, t \\ y = h + v_0 \sin\alpha\, t - \frac{1}{2} g t^2 \end{cases}$$

Troviamo il tempo di caduta τ: τ t.c. $y(\tau) = 0$

$\Rightarrow$

$$h + v_0 \sin\alpha\, \tau - \frac{1}{2} g \tau^2 = 0$$

$$\tau = \frac{v_0 \sin\alpha}{g} + \sqrt{\frac{v_0^2 \sin^2\alpha}{g^2} + \frac{2h}{g}}$$

Sia $d = x(\tau)$, allora

$$d = \frac{v_0^2 \cos\alpha \sin\alpha}{g} + \cos\alpha \sqrt{\left(\frac{v_0^2}{g}\right)^2 \sin^2\alpha + \frac{2h v_0^2}{g}}$$

Voglio d_{max} $\Rightarrow$ devo trovare α t.c. $\frac{d}{d\alpha} d(\alpha) = 0$

$$\frac{d}{d\alpha} d(\alpha) = \frac{v_0^2}{g}\left(-\sin^2\alpha + \cos^2\alpha\right)$$

$$- \sin\alpha \sqrt{\left(\frac{v_0^2}{g}\right)^2 \sin^2\alpha + \frac{2h v_0^2}{g}}$$

$$+ \cos\alpha\, \cancel{\frac{1}{2}} \frac{1}{\sqrt{\left(\frac{v_0^2}{g}\right)^2 \sin^2\alpha + \frac{2h v_0^2}{g}}} \frac{v_0^4}{g^2} \cancel{2} \sin\alpha \cos\alpha = 0$$

Moltiplico per la radice al den.

$$\Rightarrow \quad \cancel{\frac{v_0^{2}}{g}}\,(\cos^2\alpha - \sin^2\alpha)\sqrt{\left(\frac{v_0^2}{g}\right)^2 \sin^2\alpha + \frac{2hv_0^2}{g}}$$

$$-\sin\alpha\,\frac{v_0^{\cancel{4}\,2}}{g^{\cancel{2}}}\sin^2\alpha - 2h\,\frac{\cancel{v_0^{2}}}{\cancel{g}}\sin\alpha$$

$$+\frac{v_0^{\cancel{4}\,2}}{g^{\cancel{2}}}\sin\alpha\,\cos^2\alpha = 0$$

$$\cos^2\alpha = 1-\sin^2\alpha \qquad \Rightarrow$$

$$(1-2\sin^2\alpha)\sqrt{\frac{v_0^4}{g^2}\sin^2\alpha + \frac{2hv_0^2}{g}} = \frac{v_0^2}{g}\sin^3\alpha + 2h\sin\alpha$$

$$-\frac{v_0^2}{g}\sin\alpha + \frac{v_0^2}{g}\sin^3\alpha$$

$$= 2\frac{v_0^2}{g}\sin^3\alpha + 2h\sin\alpha - \frac{v_0^2}{g}\sin\alpha$$

$$= \sin\alpha\left(2h - \frac{v_0^2}{g} + 2\frac{v_0^2}{g}\sin^2\alpha\right)$$

Quadriamo i 2 termini dell'uguaglianza $\Rightarrow$

$$(1+4\sin^4\alpha - 4\sin^2\alpha)\left(\frac{v_0^4}{g^2}\sin^2\alpha + \frac{2hv_0^2}{g}\right)$$

$$= \sin^2\alpha\left(4h^2 + \frac{v_0^4}{g^2} + 4\frac{v_0^4}{g^2}\sin^4\alpha - 4\frac{hv_0^2}{g} + 8\frac{hv_0^2}{g}\sin^2\alpha\right.$$

$$\left. - 4\frac{v_0^4}{g^2}\sin^2\alpha\right)$$

$$\cancel{\frac{v_0^4}{g^2}\sin^2\alpha} + \frac{2hv_0^2}{g} + \cancel{4\frac{v_0^4}{g^2}\sin^6\alpha} + 8\frac{hv_0^2}{g}\sin^4\alpha - \cancel{4\frac{v_0^4}{g^2}\sin^4\alpha} - 8\frac{v_0^2h}{g}\sin^2\alpha$$

$$= 4h^2\sin^2\alpha + \cancel{\frac{v_0^4}{g^2}\sin^2\alpha} + \cancel{4\frac{v_0^4}{g^2}\sin^6\alpha} - 4\frac{hv_0^2}{g}\sin^2\alpha + 8\frac{hv_0^2}{g}\sin^4\alpha - \cancel{4\frac{v_0^4}{g^2}\sin^4\alpha}$$

$$\cancel{2h}\,\frac{v_0^2}{g} - \overset{4}{\cancel{8}}\,\frac{v_0^2}{g}\cancel{h}\,\sin^2\alpha = \overset{2}{\cancel{4}}\,h^{2}\sin^2\alpha - \overset{2}{\cancel{4}}\,\cancel{h}\,\frac{v_0^2}{g}\sin^2\alpha$$

$$\frac{v_0^2}{g} = \sin^2\alpha\left(2h - 2\frac{v_0^2}{g} + 4\frac{v_0^2}{g}\right) = \sin^2\alpha\left(2h + 2\frac{v_0^2}{g}\right)$$

$$\Rightarrow \quad \sin^2\alpha = \frac{v_0^2}{2g}\,\frac{1}{h + \frac{v_0^2}{g}} = \frac{1}{2}\,\frac{1}{\left(\frac{gh}{v_0^2} + 1\right)}$$

<u>Controlli semplici:</u>

$$h = 0 \qquad \sin^2\alpha = \frac{1}{2} \quad \Rightarrow \quad \alpha = 45^\circ$$ (come doveva essere)

$$h \gg \frac{v_0^2}{g} \qquad \sin^2\alpha \approx \frac{1}{2\frac{gh}{v_0^2}} \ll 1 \quad \Rightarrow \quad \alpha \approx 0$$

Calcolare la gittata massima

$$d_{max} = d\left(\sin^2\alpha = \frac{1}{2}\,\frac{1}{\left(\frac{gh}{v_0^2} + 1\right)}\right)$$

$$\begin{cases} \sin\alpha = \dfrac{v_0}{\sqrt{2gh + 2v_0^2}} \\ \cos\alpha = \sqrt{1 - \dfrac{v_0^2}{2gh + 2v_0^2}} \end{cases}$$

$$d_{max} = \frac{v_0^2}{g}\,\frac{v_0}{\sqrt{2gh + 2v_0^2}}\sqrt{1 - \frac{v_0^2}{2gh + 2v_0^2}}$$

$$+ \sqrt{1 - \frac{v_0^2}{2gh + 2v_0^2}}\,\sqrt{\frac{v_0^4}{g^2}\,\frac{v_0^2}{2gh + 2v_0^2} + \frac{2hv_0^2}{g}}$$

$$d_{max} = \frac{v_0^3}{g\,(2gh+2v_0^2)} \sqrt{2gh+v_0^2}$$

$$+ \sqrt{\frac{2gh+v_0^2}{2gh+2v_0^2}} \; \frac{v_0}{\sqrt{g}} \sqrt{\frac{v_0^4}{g}\,\frac{1}{2hg+2v_0^2} + 2h}$$

$$= \frac{v_0^3}{g\,(2gh+2v_0^2)} \sqrt{2gh+v_0^2}$$

$$+ \sqrt{\frac{2gh+v_0^2}{2gh+2v_0^2}} \; \frac{v_0}{\sqrt{g}} \sqrt{\frac{v_0^4 + 2hg\,(2hg+2v_0^2)}{g\,(2hg+2v_0^2)}}$$

$$= \frac{v_0^3}{g} \frac{\sqrt{2gh+v_0^2}}{2gh+2v_0^2} + \frac{v_0}{\sqrt{g}} \sqrt{\frac{2gh+v_0^2}{2gh+2v_0^2}} \; \frac{2gh+v_0^2}{\sqrt{g\,(2hg+2v_0^2)}}$$

$$= \frac{v_0}{g} \frac{\sqrt{2gh+v_0^2}}{\cancel{2gh+2v_0^2}} \left[\cancel{v_0^2 + 2gh + v_0^2} \right]$$

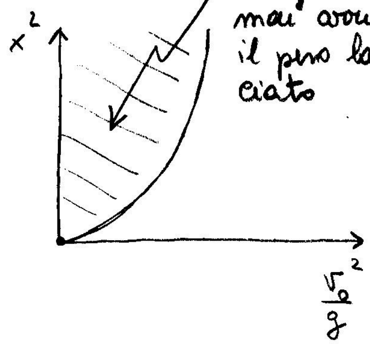
$$= \frac{v_0}{g} \sqrt{2gh+v_0^2}$$

Check
$$\begin{cases} h = 0 & d_{max} = \dfrac{v_0^2}{g} \quad \text{(come doveva essere)} \\[2ex] h \gg \dfrac{v_0^2}{g} & d_{max} \simeq \dfrac{v_0}{g}\sqrt{2gh} = \sqrt{\dfrac{2h}{g}}\, v_0 \end{cases}$$

sia $x \equiv d_{max} \Rightarrow$

$$x^2 = \frac{v_0^2}{g^2}(2gh+v_0^2) = \frac{v_0^4}{g^2} + 2h\frac{v_0^2}{g}$$

area in cui non potrà mai arrivare il peso lanciato



[record $\simeq$ 23 m
$v_0 \simeq$ 14 m/s