Studio del moto su spirale

Si ha $\quad r(\theta) = r_0 + \frac{b}{2\pi}\theta \qquad r_0$ e b dati

Questa è l'equazione di una spirale.

1) Si supponga che la spirale sia percorsa con $\dot{\theta} = \omega = \text{cost.}$ Studiare il moto.

$$\dot{\theta} = \omega \Rightarrow \theta(t) = \theta_0 + \omega t$$

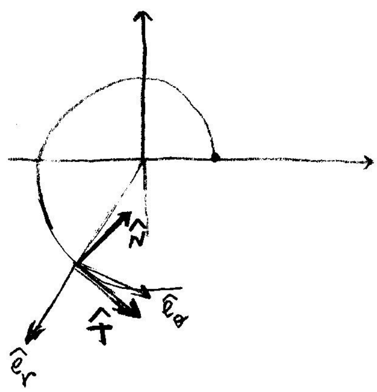
$$\Downarrow$$

$$r(t) = r_0 + \frac{b}{2\pi}\theta_0 + \frac{b\omega}{2\pi}t$$

Sia $r_0 = r(t=0) \Rightarrow \theta_0 = \theta(t=0) = 0$

$$\Rightarrow \quad r(t) = r_0 + \frac{b\omega}{2\pi}t \qquad \boxed{\frac{b\omega}{2\pi} \equiv v_0}$$

$$r(t) = r_0 + v_0 t$$



$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

$$\dot{r} = v_0$$

$$r\dot{\theta} = \omega(r_0 + v_0 t)$$

$$\Rightarrow \quad |\vec{v}| = \sqrt{v_0^2 + \omega^2(r_0+v_0t)^2}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta$$

$$= -(r_0+v_0t)\,\omega^2\,\hat{e}_r + 2\,v_0\omega\,\hat{e}_\theta$$

Trovare $\hat{T}$, $\hat{N}$ e $\rho \equiv$ raggio di curvatura

$$\hat{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{v_0}{v}\,\hat{e}_r + \frac{\omega(r_0+v_0t)}{v}\,\hat{e}_\theta$$

$$\hat{N}\cdot\hat{T} = 0 \Rightarrow \qquad \frac{v_0}{v}N_r + \frac{\omega}{v}(r_0+v_0t)N_\theta = 0$$

$$\vec{N} = -\frac{\omega}{v}(r_0 + v_0 t)\,\hat{e}_r + \frac{v_0}{v}\,\hat{e}_\theta$$

$$|\vec{N}| = \frac{\omega^2 (r_0 + v_0 t)^2 + v_0^2}{v^2} = 1 \quad \Rightarrow \quad \hat{N} = -\frac{\omega}{v}(r_0 + v_0 t)\,\hat{e}_r + \frac{v_0}{v}\,\hat{e}_\theta$$

$$\vec{a} = \dot{v}\,\hat{T} + \frac{v^2}{\rho}\hat{N} = -(r_0 + v_0 t)\,\omega^2\,\hat{e}_r + 2 v_0 \omega\,\hat{e}_\theta$$

$$\dot{v} = \frac{1}{\cancel{2}}\,\frac{\omega^2 \cancel{2}(r_0 + v_0 t)\,v_0}{v} = \omega^2 \frac{v_0}{v}(r_0 + v_0 t)$$

$$\Rightarrow \quad \frac{v_0}{v}\,\omega^2 \frac{v_0}{v}(r_0 + v_0 t)\,\hat{e}_r + \frac{\omega}{v}(r_0 + v_0 t)\,\omega^2 \frac{v_0}{v}(r_0 + v_0 t)\,\hat{e}_\theta$$

$$+ \frac{v^2}{\rho}\left(-\frac{\omega}{v}\right)(r_0 + v_0 t)\,\hat{e}_r + \frac{v^2}{\rho}\frac{v_0}{v}\,\hat{e}_\theta = -(r_0 + v_0 t)\,\omega^2\,\hat{e}_r + 2 v_0 \omega\,\hat{e}_\theta$$

$$\Rightarrow \begin{cases} \left(\dfrac{v_0 \cancel{\omega}}{v}\right)^2 \omega\,\cancel{(r_0 + v_0 t)} - \dfrac{\cancel{\omega} v}{\rho}\,\cancel{(r_0 + v_0 t)} = -\cancel{(r_0 + v_0 t)}\,\omega^{\cancel{2}} \\[2ex] \dfrac{\omega^3 \cancel{v_0}}{v^2}(r_0 + v_0 t)^2 + \dfrac{v \cancel{v_0}}{\rho} = 2 \cancel{v_0}\,\omega \end{cases}$$

$$\Rightarrow \begin{cases} \rho = \dfrac{v}{\left(\dfrac{v_0}{v}\right)^2 \omega + \omega} = \dfrac{v^3}{\omega (v_0^2 + v^2)} = \dfrac{\left[\sqrt{v_0^2 + \omega^2 (r_0 + v_0 t)^2}\,\right]^3}{\omega \left(2 v_0^2 + \omega^2 (r_0 + v_0 t)^2\right)} \\[3ex] \rho = \dfrac{v}{\omega\left(-\dfrac{\omega^2}{v^2}(r_0 + v_0 t)^2 + 2\right)} = \dfrac{v^3}{\omega\left(\cancel{-v^2} + v_0^2 + \cancel{2} v^2\right)} = + \dfrac{v^3}{\omega (v_0^2 + v^2)} \end{cases}$$

I due ρ sono uguali come <u>deve essere</u>.

2) Si supponga che la spirale sia percorsa con $r\dot{\theta} = w = \text{cost}$.

$$r\dot{\theta} = w = \left(r_0 + \frac{b}{2\pi}\theta\right)\frac{d\theta}{dt}$$

$$\Rightarrow \int_{\theta_0}^{\theta}\left(r_0 + \frac{b}{2\pi}\theta\right)d\theta = \int_0^t w\,dt$$

$$r_0\theta + \frac{b}{4\pi}\theta^2 + K_0 = wt$$

$K_0 =$ cost che dipende dalle condizioni iniziali: per $t=0$ voglio $\theta = 0$ $\Rightarrow K_0 = 0$

$$\frac{b}{4\pi}\theta^2 + r_0\theta - wt = 0$$

$$b\theta^2 + 4\pi r_0\theta - 4\pi wt = 0$$

soluzione fisica

$$\theta = \frac{-2\pi r_0 \pm \sqrt{4\pi^2 r_0^2 + 4\pi b w t}}{b}$$

$$\Rightarrow \theta = \frac{2\pi r_0}{b}\left[\sqrt{1 + \frac{bwt}{\pi r_0^2}} - 1\right]$$

$$\frac{bw}{\pi r_0^2} = \frac{\cancel{[L]}\,\cancel{[L]}\,[T]^{-1}}{\cancel{[L]}^2} \approx [T]^{-1}$$

$$\boxed{\frac{bw}{\pi r_0^2} = \omega_0} \quad \Rightarrow \theta = \frac{2\pi r_0}{b}\left(\sqrt{1+\omega_0 t} - 1\right)$$

$$\Rightarrow r(t) = \cancel{r_0} + \frac{\cancel{b}}{\cancel{2\pi}}\,\frac{\cancel{2\pi} r_0}{\cancel{b}}\left(\sqrt{1+\omega_0 t} \cancel{- 1}\right)$$

$$= r_0\sqrt{1+\omega_0 t}$$

$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

$$= r_0\,\frac{1}{2}\,\frac{\omega_0}{\sqrt{1+\omega_0 t}}\,\hat{e}_r + w\,\hat{e}_\theta$$

$$= \frac{\omega_0 r_0}{2\sqrt{1+\omega_0 t}}\,\hat{e}_r + w\,\hat{e}_\theta \quad \Rightarrow$$

$$|\vec{v}| = \sqrt{\frac{\omega_0^2 r_0^2}{4(1+\omega_0 t)} + w^2} \,.$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\,\hat{e}_\theta$$

$$\ddot{r} = \frac{\omega_0 r_0}{2}\left(-\frac{1}{2}\right)\frac{\omega_0}{(1+\omega_0 t)^{3/2}} = -\frac{\omega_0^2 r_0}{4(1+\omega_0 t)^{3/2}}$$

$$r\dot{\theta}^2 = (r\dot{\theta})\dot{\theta} = w\,\frac{\cancel{2}\pi r_0}{b}\,\frac{1}{\cancel{2}}\,\frac{\omega_0}{\sqrt{1+\omega_0 t}} = w\,\frac{\cancel{\pi r_0}}{\cancel{b}}\,\frac{\cancel{b} w}{\cancel{\pi} r_0^{\cancel{2}}}\,\frac{1}{\sqrt{1+\omega_0 t}}$$

$$= \frac{w^2}{r_0}\frac{1}{\sqrt{1+\omega_0 t}}$$

$$r^2\dot{\theta} = r(r\dot{\theta}) = w\, r_0\sqrt{1+\omega_0 t} \Rightarrow$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{w}{r}\,\frac{r_0\omega_0}{2\sqrt{1+\omega_0 t}} = \frac{w\,\omega_0}{2(1+\omega_0 t)}$$

$$\Rightarrow \vec{a} = \left(-\frac{\omega_0^2 r_0}{4(1+\omega_0 t)^{3/2}} - \frac{w^2}{r_0}\frac{1}{\sqrt{1+\omega_0 t}}\right)\hat{e}_r + \frac{w\,\omega_0}{2(1+\omega_0 t)}\,\hat{e}_\theta$$

$$\hat{T} = \frac{\omega_0 r_0}{2\sqrt{1+\omega_0 t}\, v}\,\hat{e}_r + \frac{w}{v}\,\hat{e}_\theta$$

$$\hat{N} = -\frac{w}{v}\,\hat{e}_r + \frac{\omega_0 r_0}{2v\sqrt{1+\omega_0 t}}\,\hat{e}_\theta$$

$$\dot{v} = \frac{1}{2}\,\frac{1}{v}\,\frac{\omega_0^2 r_0^2}{4}(-)\frac{\omega_0}{(1+\omega_0 t)^2} = -\frac{\omega_0^3 r_0^2}{8v(1+\omega_0 t)^2}$$

$$\vec{a} = \dot{v}\,\hat{T} + \frac{v^2}{\rho}\hat{N} = -\frac{\omega_0^3 r_0^2}{8v(1+\omega_0 t)^2}\,\frac{\omega_0 r_0}{2v\sqrt{1+\omega_0 t}}\,\hat{e}_r - \frac{\omega_0^3 r_0^2}{8v(1+\omega_0 t)^2}\,\frac{w}{v}\,\hat{e}_\theta$$

$$+ \frac{v^2}{\rho}\left(-\frac{w}{v}\,\hat{e}_r\right) + \frac{v^2}{\rho}\,\frac{\omega_0 r_0}{2\cancel{v}\sqrt{1+\omega_0 t}}\,\hat{e}_\theta =$$

$\Rightarrow$ dal ~~termine~~ α a $\hat{e}_\theta$ (più semplice)

$$-\frac{\omega_0^{\cancel{3}\,2} r_0^2 w}{\cancel{8}_4\, v^2 (1+\omega_0 t)^{\cancel{2}}} + \frac{\cancel{\omega_0} r_0 v}{\cancel{2}\rho\sqrt{1+\omega_0 t}} = \frac{w\cancel{\omega_0}}{\cancel{2}\cancel{(1+\omega_0 t)}}$$

$$\frac{r_0 v\sqrt{1+\omega_0 t}}{\rho} = w\left[1+\frac{\omega_0^2 r_0^2}{4v^2(1+\omega_0 t)}\right]$$

$$\rho = \frac{r_0 v\sqrt{1+\omega_0 t}}{w\,\dfrac{4v^2(1+\omega_0 t)+\omega_0^2 r_0^2}{4v^2(1+\omega_0 t)}}$$

$$= \frac{4 r_0 v^3 (1+\omega_0 t)^{3/2}}{w\left[\omega_0^2 r_0^2 + 4v^2(1+\omega_0 t)\right]}$$

Facciamo alcuni controlli:

$$[\rho] = \frac{[L]\,\cancel{[L]^3}\,\cancel{[T]^{-3}}}{\cancel{[L][T]^{-1}}\,\cancel{[L]^2[T]^{-2}}} = [L] \quad \checkmark$$

Se $b \to 0$ $\quad r(t) = r_0 = \text{cost}.\quad \dot{\theta} = \dfrac{w}{r_0} = \text{cost} \Rightarrow$ ho moto circ. uniforme.

$b \to 0 \Rightarrow \omega_0 = 0$, $v = w$ $\quad\Rightarrow\quad \rho = \dfrac{\cancel{4} r_0 \cancel{w^3}}{\cancel{w}\,\cancel{4}\,\cancel{w^2}} = r_0 \quad \checkmark$

$$\hat{T} = \hat{e}_\theta\ ;\quad \hat{N} = -\hat{e}_r \quad \checkmark$$