Lyapunov Exponents and Plasma Dynamics

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Plasma Physics, recent results and future perspectives
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Topics

- Finite Time Lyapunov Exponent for dynamic systems
  - Extension of Lyapunov Exponent for infinite time
  - Field line "tracer" for a vector field, e.g. fluid velocity
  - Rate of divergence/convergence of nearby field lines is related to underlying structures of the 3D vector field, as it evolves in time
- First application to plasmas: MHD sawtooth crash
  - **B**, **V** vector fields
  - Nonlinear MHD numerical simulation
- New view of MHD turbulence, incompressibility
- Future developments

Lyapunov Exponent

- Lyapunov exponents are related to phase space behavior over infinite time; the maximal one is related to overall system stability
- Measure the separation of trajectories

  $\delta \mathbf{Z} = \delta \mathbf{Z}_0 \exp(\gamma t)$

  $\gamma = \lim_{t \to \infty} \lim_{\delta \mathbf{Z}_0 \to 0} (1/t) \ln (|\delta \mathbf{Z}(t)|/|\delta \mathbf{Z}_0|)$

- Spectrum of values, one per dimension
- Maximal value related to system stability

Modern Finite Time Lyapunov Exponent

- Finite Time Lyapunov Exponents measure the local rate of divergence (or convergence) of the field lines of a vector field over a finite Δt interval; real space

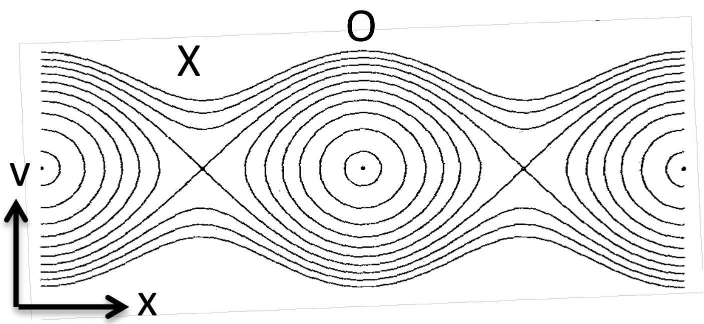
  $$\delta L = \delta L_o \exp(\gamma t_{eff}), \text{ over } \quad t_{eff} = t - t_0 = L \,/\, |V|$$

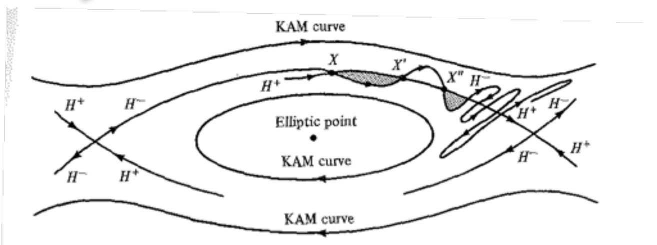
- Lagrangian Coherent Structures (LCS) are attracting or repelling velocity structures defined by the flow manifolds; persistent; coherent evolution over time. Usually hard to see.
- Mathematically rigorous relation to FTLE ⟷ LCS proven
  - G. Haller, S. Shadden, et al., 2001-2007+; generalized to *n* dimensions
- Many practical applications; rapidly evolving
- Many other FTLE-like quantities exist, but no rigorous relation to LCS yet proven, even for finite distance (FSLE).

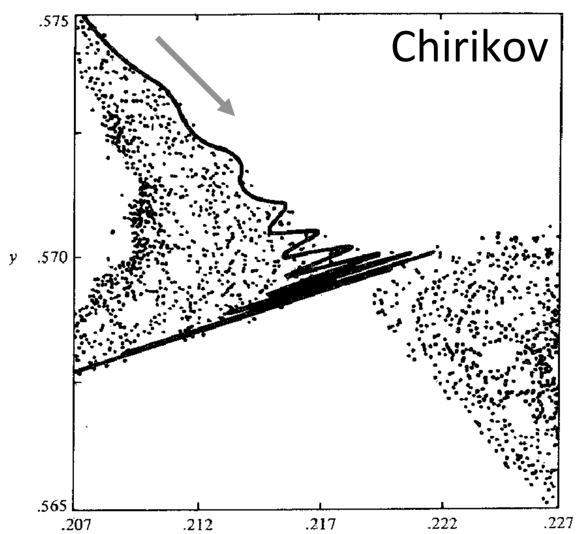
Structure inside chaos is common in fluids

- A vector field, perturbed
- Fluid velocity field **V** (or any reasonable vector field) has invariant flow manifolds (oriented surfaces)
- Disturbance leads to an asymptotic splitting of these invariant manifolds into stable and unstable manifolds
  - Depends on the direction in which the vector field lines are followed
- Unstable manifold develops oscillations around the stable one. Flow can follow either and cross original surfaces.
- Simple example − Hamiltonian system

(Homoclinic) **tangle in a Hamiltonian system**







- 1D pendulum in (x,v) phase space
- Hamiltonian system, periodic in x:

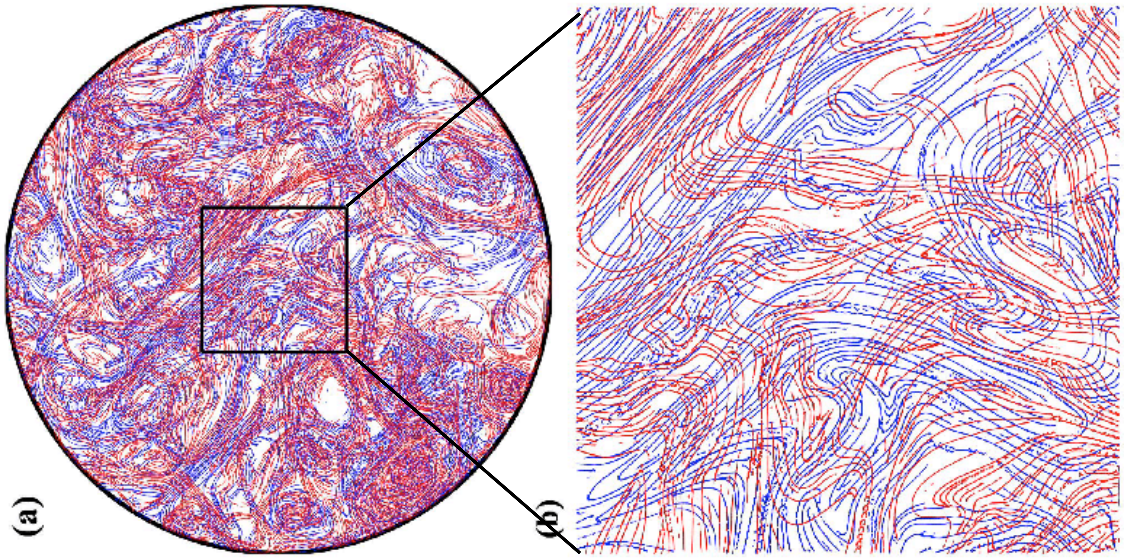
$$d^2x/dt^2 + \sin 2\pi x = 0, v = dx/dt$$

- Transverse perturbation anywhere on separatrix, e.g., by a forcing term at different frequency, causes the separatrix surface (a manifold) to split into 2 different asymptotic limits, with complicated behavior near an X-point.
- Trajectories formed by the extended loops on the two sides of the X-point intersect many times and become chaotic. Similar trajectory splitting occurs at each new X-point.
- Plasma – **B** field is Hamiltonian at each time *t*

Lichtenberg and Lieberman,
Regular and Chaotic Dynamics (1992)

Fluid experiments confirm LCS properties

- Fluid experiments relate the LCS manifolds to divergence/ convergence of the flow - ``ridges'' and ``valleys''



Flow divergence from tracer particles is used to reconstruct LCS manifolds
Red – positive divergence -> ridge, Blue – negative -> valley

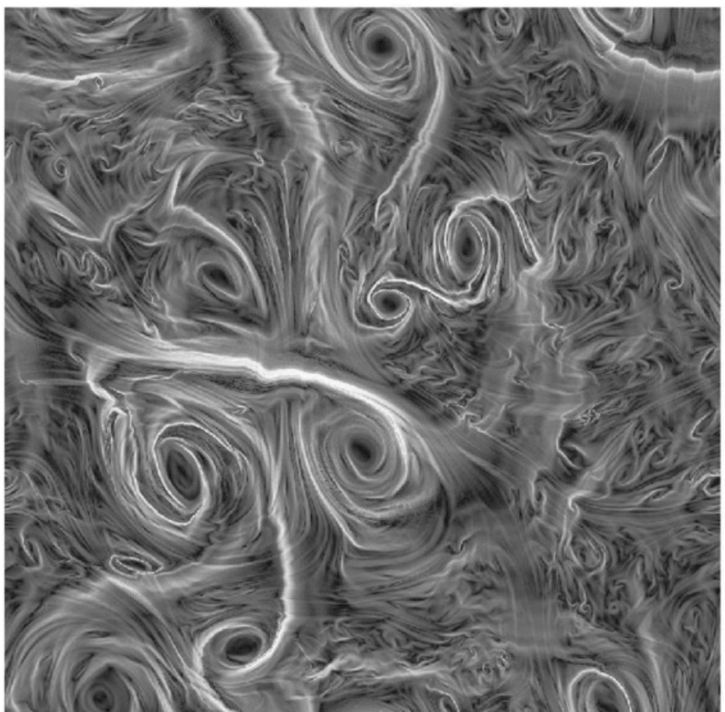
Turbulent fluid (top view of cylinder)
(M. Mathur, et al, PRL 2007)

Finite Time Lyapunov Exponent in a fluid

Local divergence or convergence of nearby field lines ("strain") over a chosen time interval, calculated as an exponential rate

$$\delta L = \delta L_o \exp(\gamma t_{eff}), \text{ over } \quad t_{eff} = t - t_0 = L \,/\, |V|$$

- Scalar function of space $\gamma_{teff}(\mathbf{x}, t_0)$
- Picks out boundaries in flow, including vortices in turbulent fluids



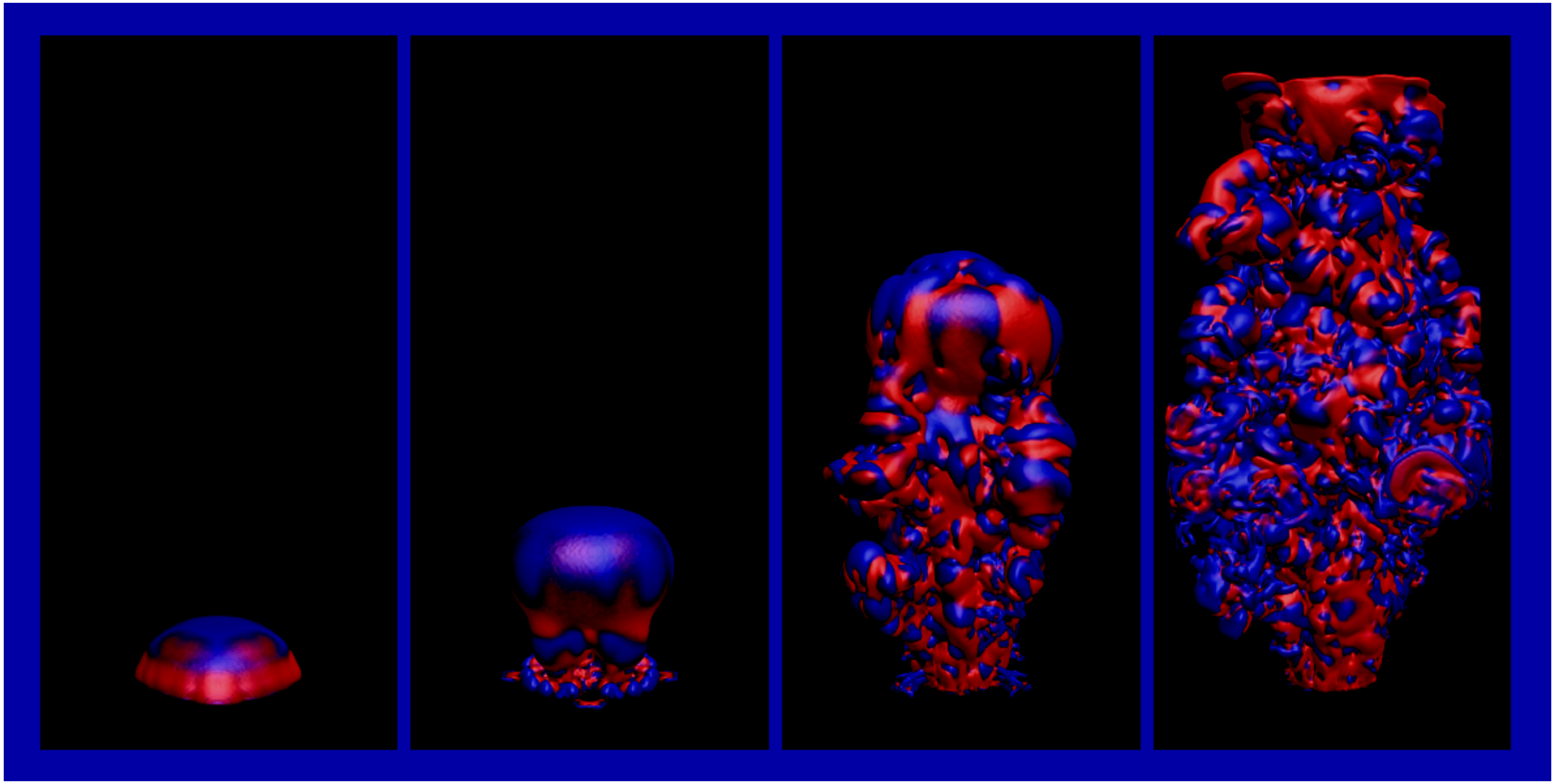
Decaying 2D fluid turbulence, with multiple vortices and fine scale structure
(G. Lapeyre, Chaos, 2002)

Light = large FTLE = fast divergence of flow lines -> boundaries of different flows

Dark = small = little divergence -> centers of vortices

Traced for longer -> more fine scale structure visible

Fluid jet with turbulence: FTLE picture



FTLE rendering for a jet of fluid moving upward into a stationary fluid. Forward time (diverging flow) regions are red, and backward time (converging) in blue (C. Garth et al., *SciDAC Review* **15** December (2009))

First application of FTLEs to plasma

- A plasma has many possible fields
- Simplest: MHD **B** and **V** fields at a single time, from simulation
- Work with visualization experts: H. Krishnan (LBNL), H. Childs and grad student Ryan Bliele (Univ Oregon) implemented a high accuracy FTLE calculation in the VisIt open source visualization package.
  - Developed an ocean circulation problem (Özgökmen, et al., Ocean Model. (2012) with H. Krishnan)
  - Parallelized version (Electronic Imaging conference 2017; Bleile, Sugiyama, et al)
- Application – sawtooth crash in tokamak with M3D code
  - Simple structure
  - Shortest tracing distance of 1 helical circuit around (q=1)

Resistive MHD

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p}{nm} + \frac{\mathbf{J} \times \mathbf{B}}{nm} + \mu_\perp \nabla^2 \mathbf{v}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}$$

$$\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p = -\Gamma p \nabla \cdot \mathbf{v} + \nabla \cdot n \left[\boldsymbol{\kappa}_\perp \cdot \nabla_\perp \frac{p}{n} + \boldsymbol{\kappa}_\parallel \cdot \nabla_\parallel \frac{p}{n} \right]$$

$$\frac{\partial n}{\partial t} + \mathbf{v} \cdot \nabla n = -n \nabla \cdot \mathbf{v} \qquad p = nT, \qquad \Gamma = 5/3$$

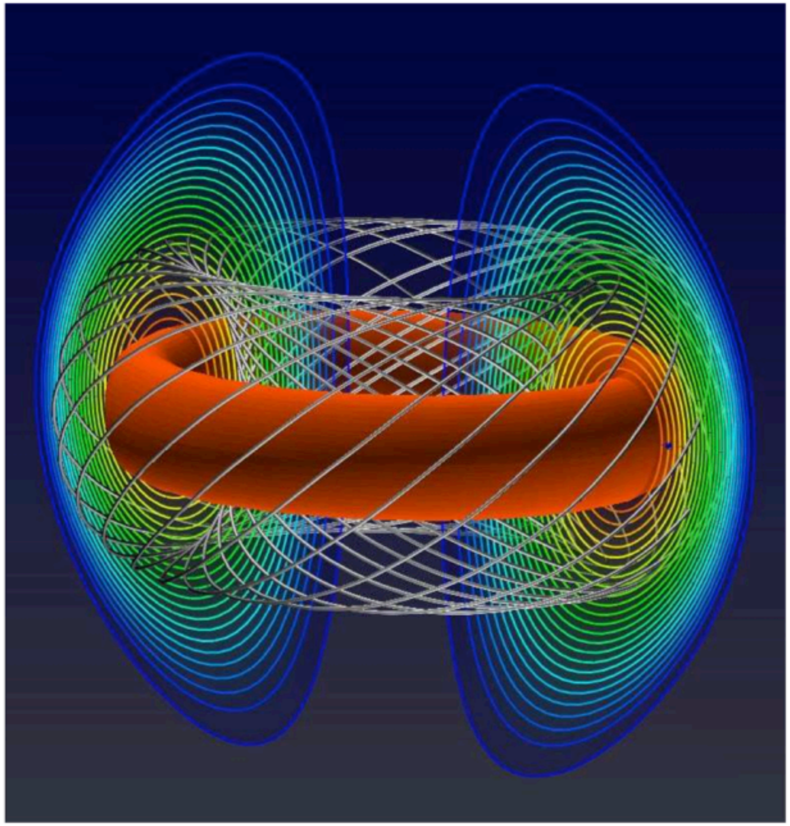
$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \qquad \mathbf{J} = \nabla \times \mathbf{B} \qquad \nabla \cdot \mathbf{B} = 0$$

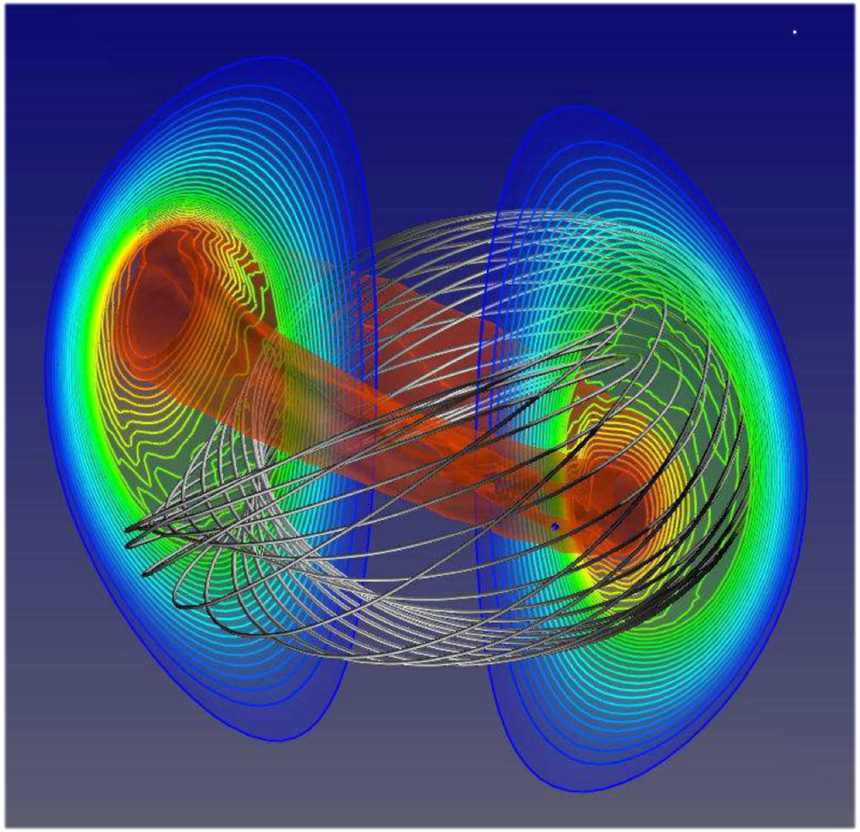
Anisotropic thermal conductivities $\boldsymbol{\kappa}_\parallel \gg \boldsymbol{\kappa}_\perp$ by a factor $10^5\times$ or more

Very small resistivity η (10^{-7} to 10^{-9} (ITER))

Sawtooth Instability

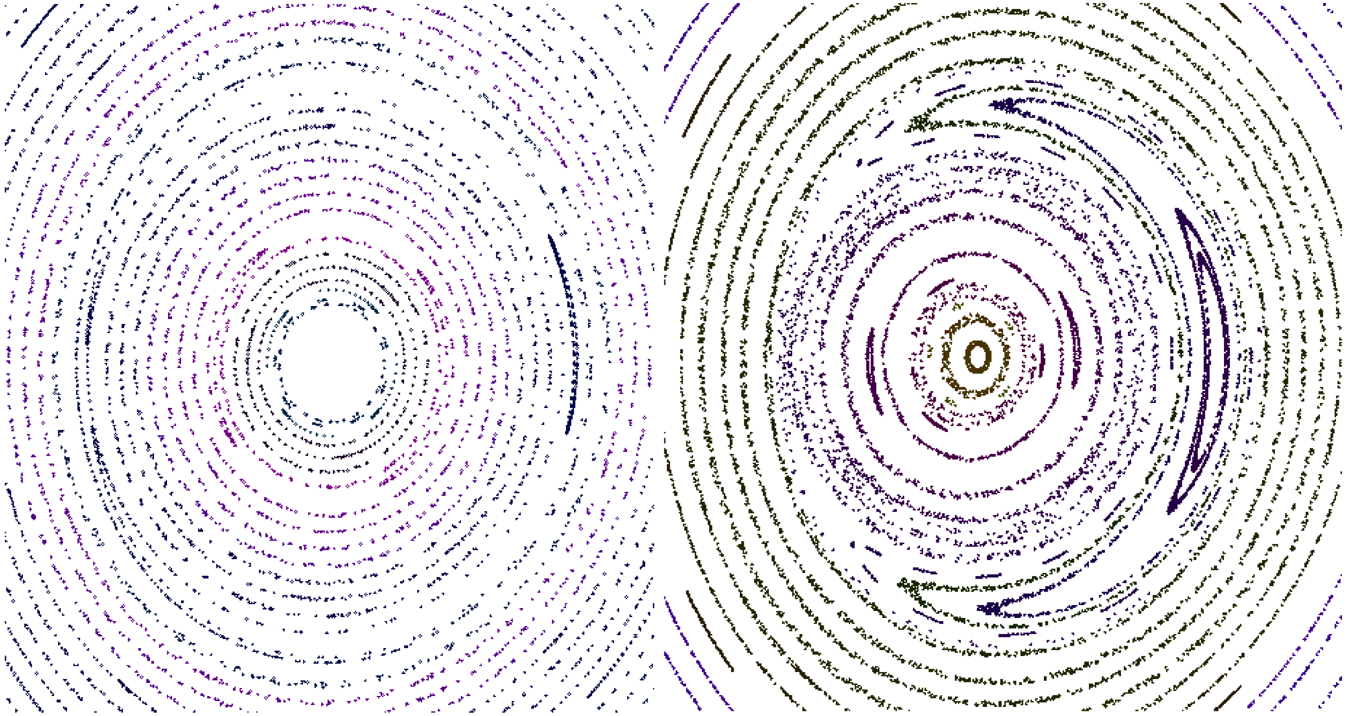
- Internal kink of the m=1, n=1 surface near the center of the plasma.





Sawtooth Instability

- Internal kink of the m=1, n=1 surface near the center of the plasma − magnetic field



Large island: Temperature and density

Time t=467.2 during fast crash, when the magnetic island width ≃ ‘hot’ core diameter ≃ r_1 (q=1 radius)

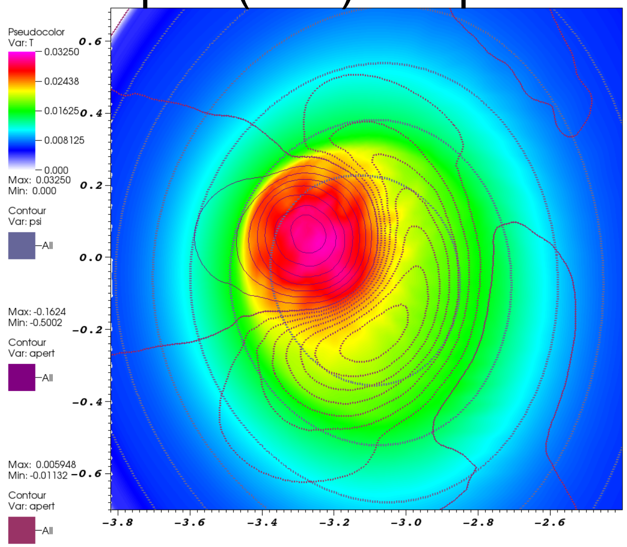
Single slice through the torus: angles φ=0,π

Expanded view of central region

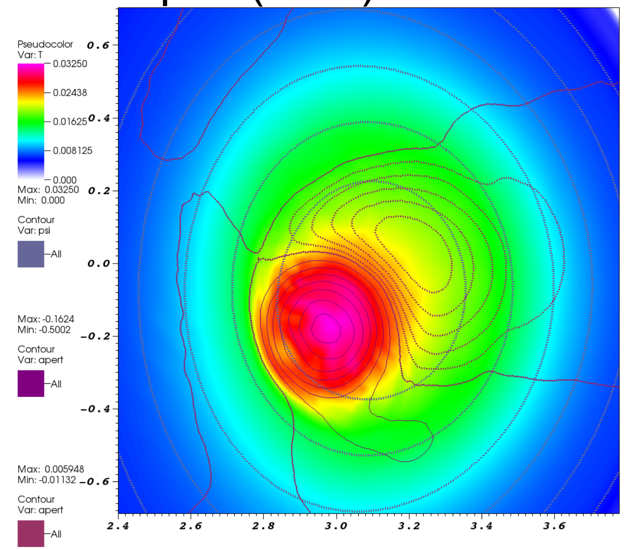
Alcator C-Mod-like plasma, low q_o near 0.5, Lundquist number $S=10^8$

From Sugiyama, Phys. Plasmas (2013))

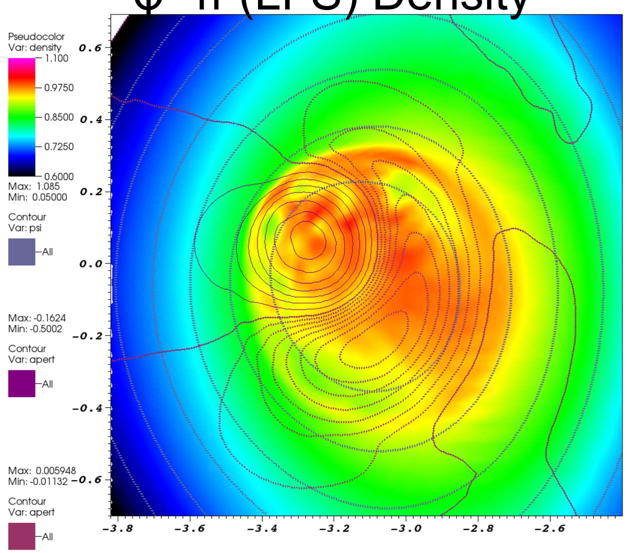
φ=π (LFS) Temperature



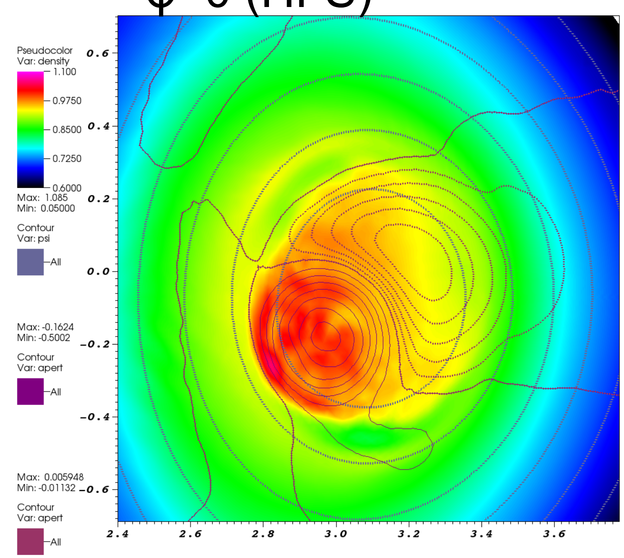
φ=0 (HFS)



φ=π (LFS) Density

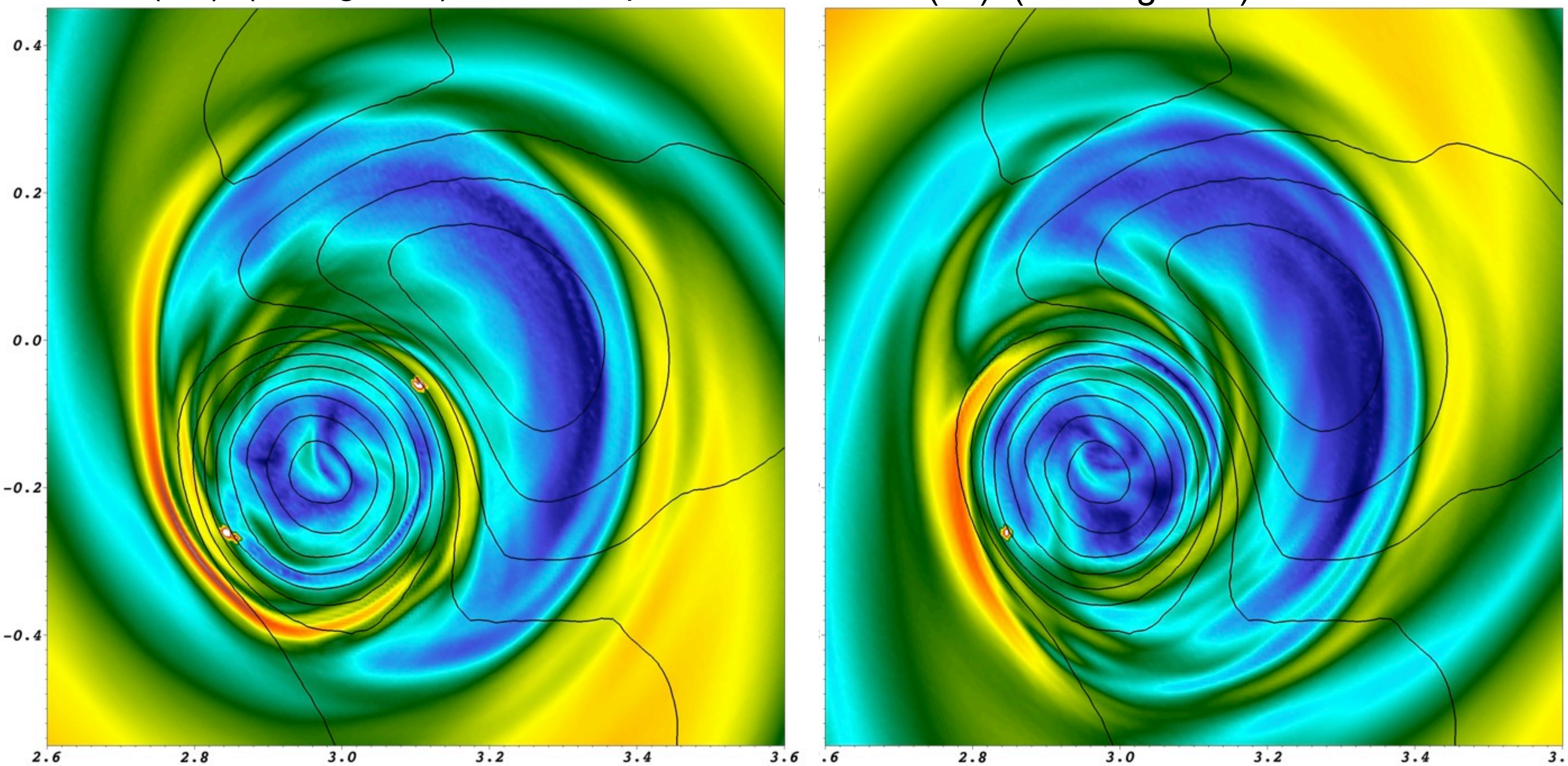


φ=0 (HFS)



Magnetic field FTLE: divergence and convergence

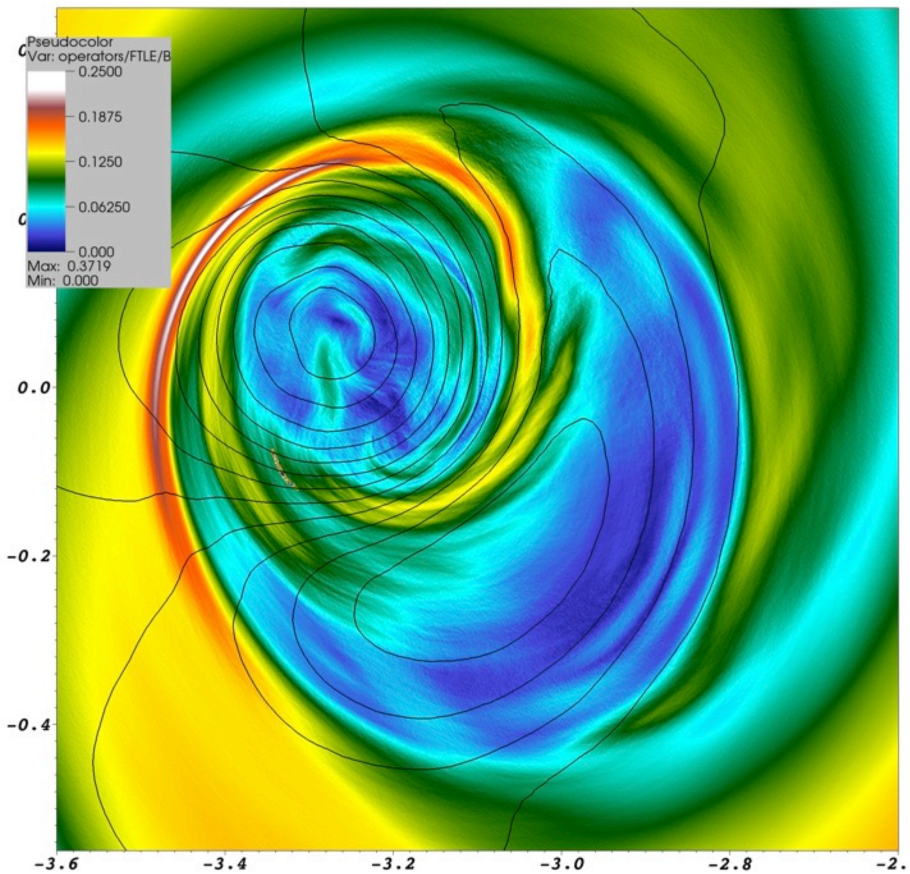
FTLE(+**B**) (divergence) at φ=0 | FTLE(-**B**) (convergence)



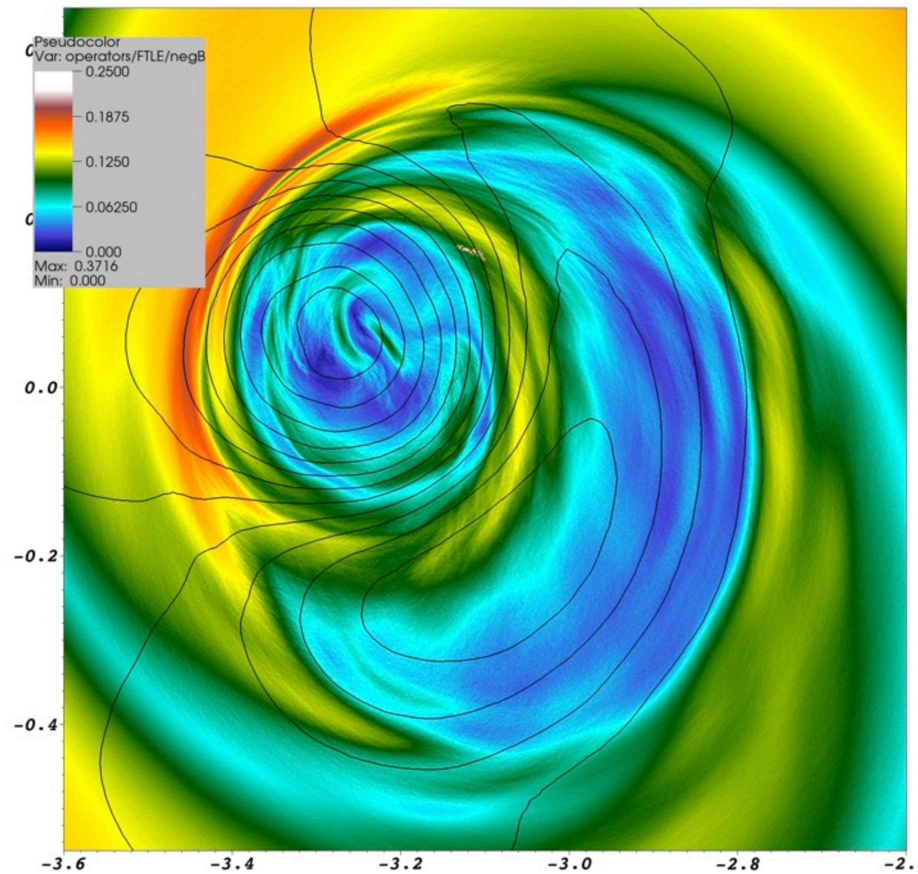
- FTLEs of total B (color) follow the perturbed poloidal magnetic flux $\tilde{\psi}$ (lines) where $\mathbf{B} = \nabla\psi\times\nabla\varphi + (1/R)\nabla_{\perp}F + I\nabla\varphi$.
- Diverging (large FTLE(+B)) and converging (large FTLE(-B)) field are different. Both are large near q=1 boundaries, reconnection X-point.

Magnetic field FTLEs at φ=π

FTLE(B), contour(apert)

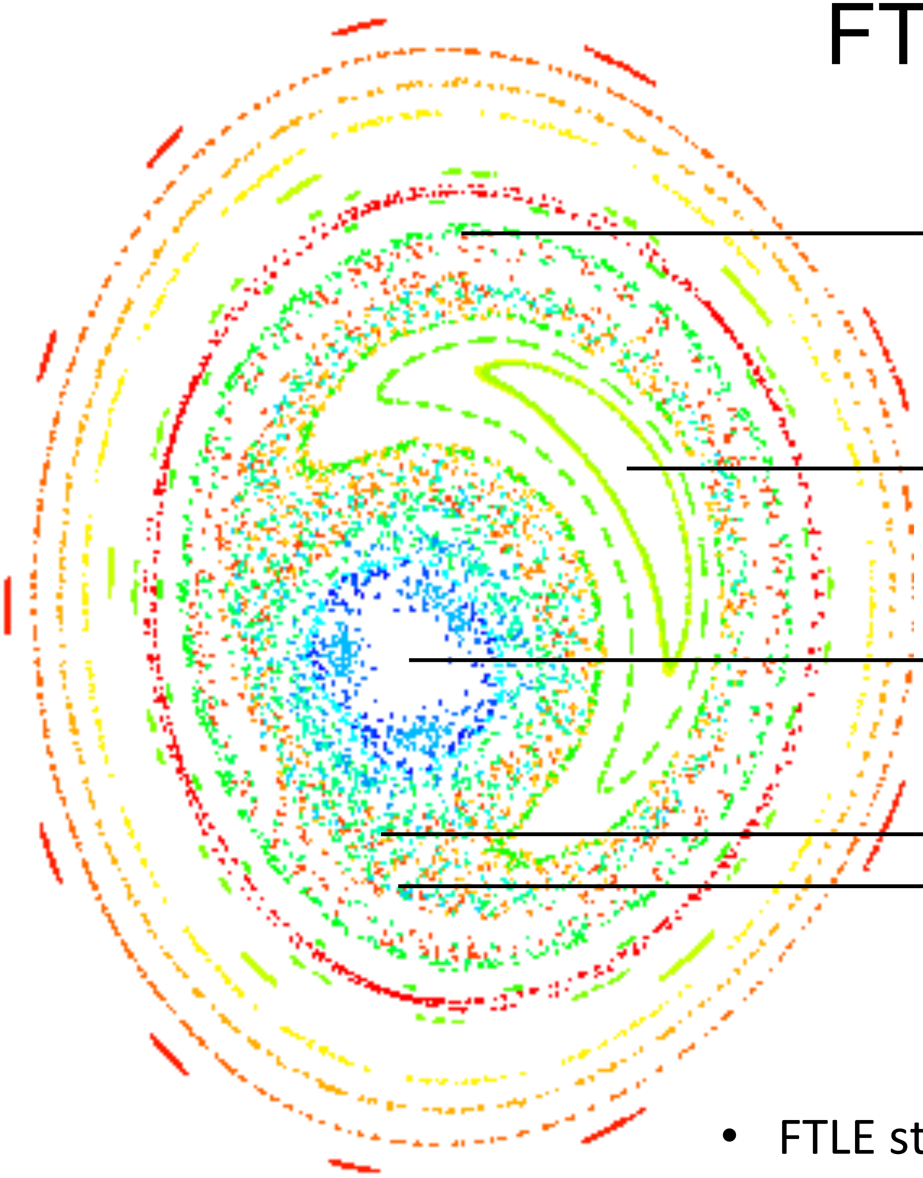


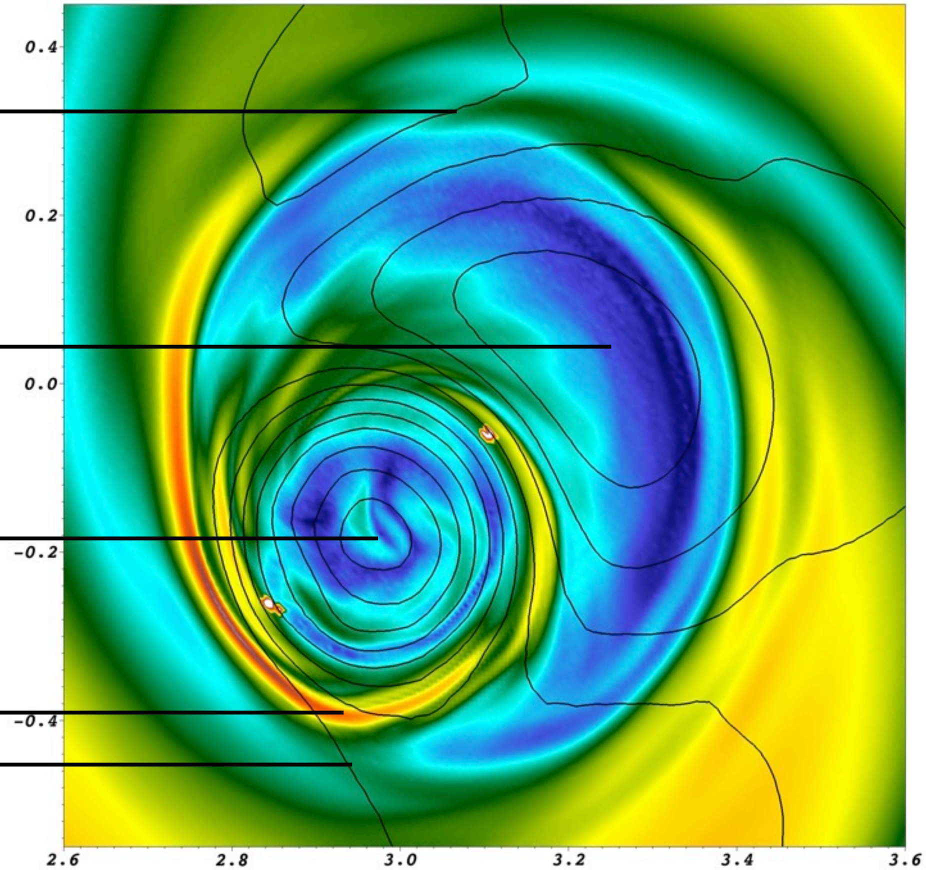
FTLE(-B), contour(apert)



- Outboard (LFS) X-point shows somewhat different shape than inboard
- Difference is larger at the peak of the crash (final fast temperature outflow from $q<1$)

FTLE(**B**) and puncture plot

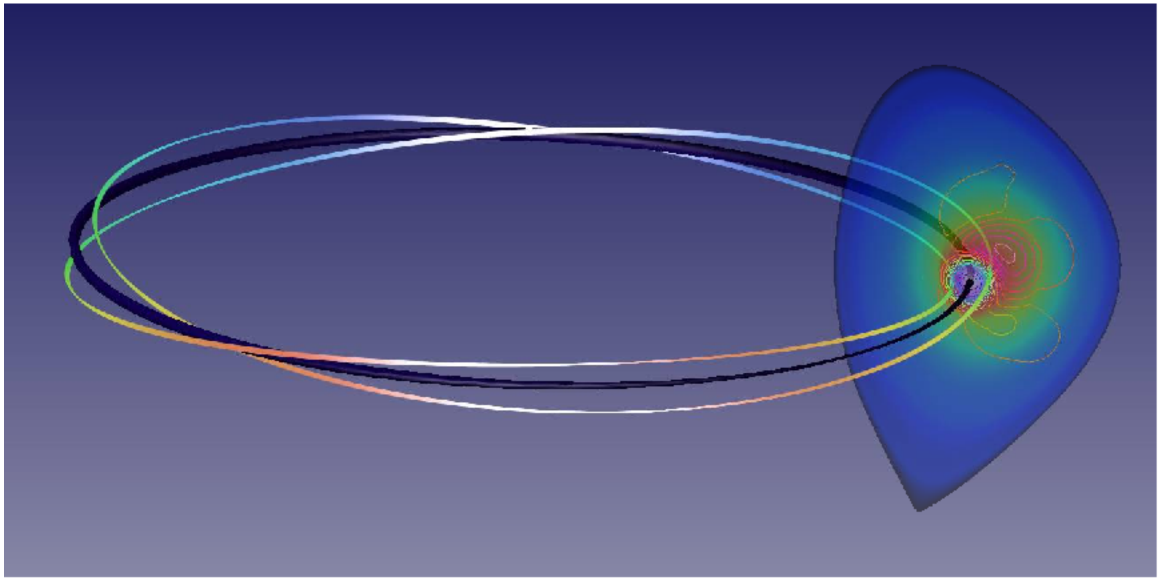


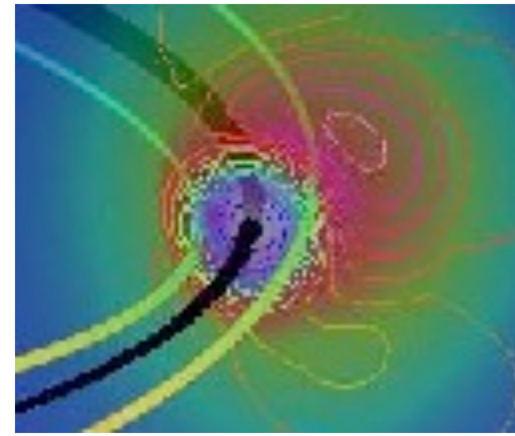


φ=0 (HFS) t=467.2

- FTLE structures are *not* aligned with flux surfaces.
- Low values in regions around magnetic O-points
- High values - boundaries between flux regions

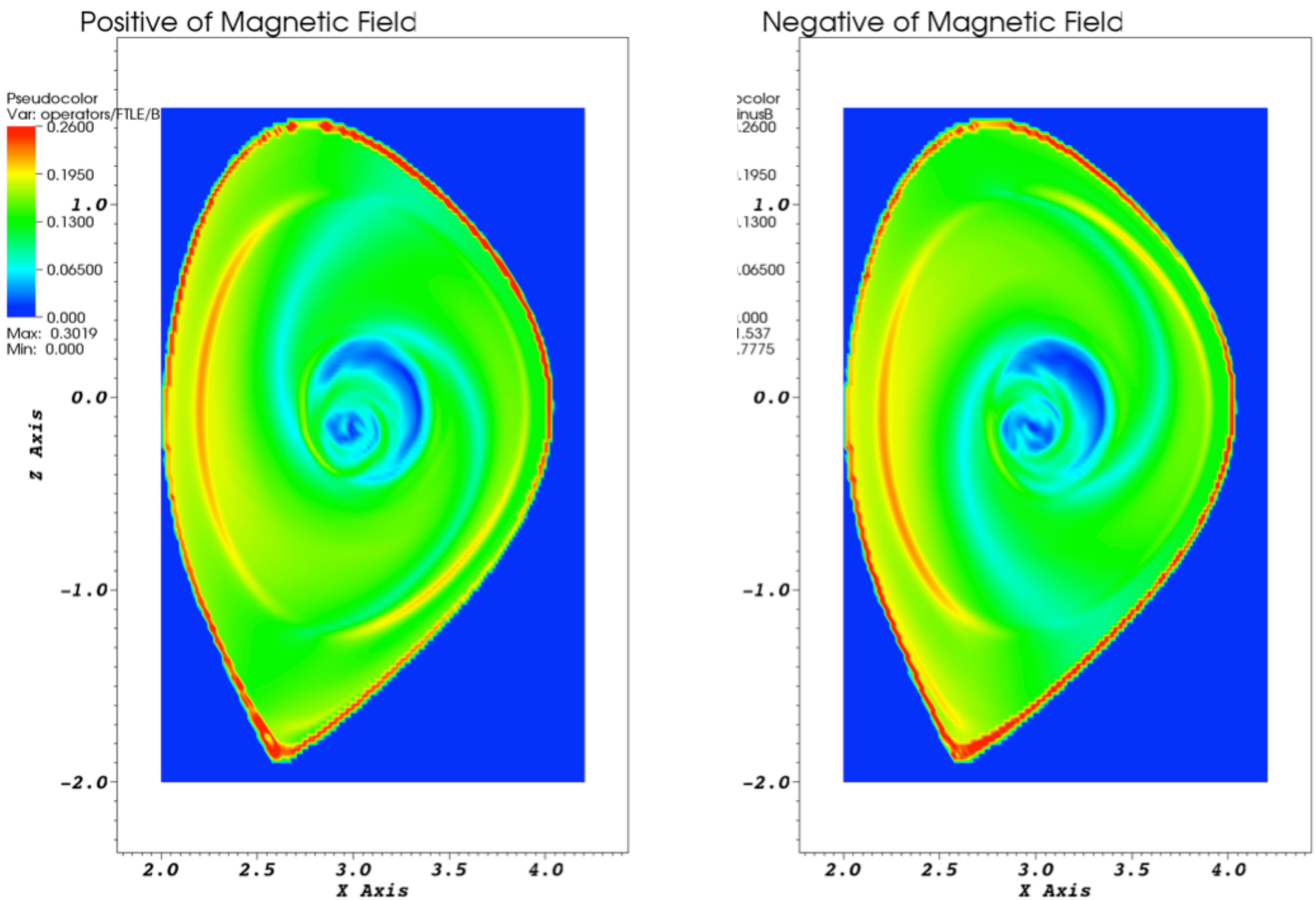
FTLE picks out 2/1 field line encircling the hot core





- Two spots have very high FTLE(B) values, one on each side of the hot core (near reconnection X-point at q=1 and opposite it) arise from an m=2/n=1 X-line encircling the hot core (unexpected).
- Linear internal kink mode has large 2/1 $\tilde{\psi}$ peaked at q=1 (Bussac 1975). In typical tokamak $\tilde{\psi}_{2,1}/\tilde{\psi}_{1,1}$=1/5-1/4. (Not in RMHD!)
- Nonlinearly this can lead to annulus around the hot core, where 2/1 X-line is surrounded by low m,n island chains; widens reconnection X-region (Seen in other full MHD simulations (Aydemir PoP 1989))

Full cross section: FTLE(±B)



- Full cross-section view of FTLEs (+**B**) and (-**B**), traced for same distance L/a=20, approximately one complete helical circuit at q=1
- Picks up 3/1 island at q=3 (largest values around X-points of islands).
- Global spirals due to flux surface geometry at fixed tracing distance

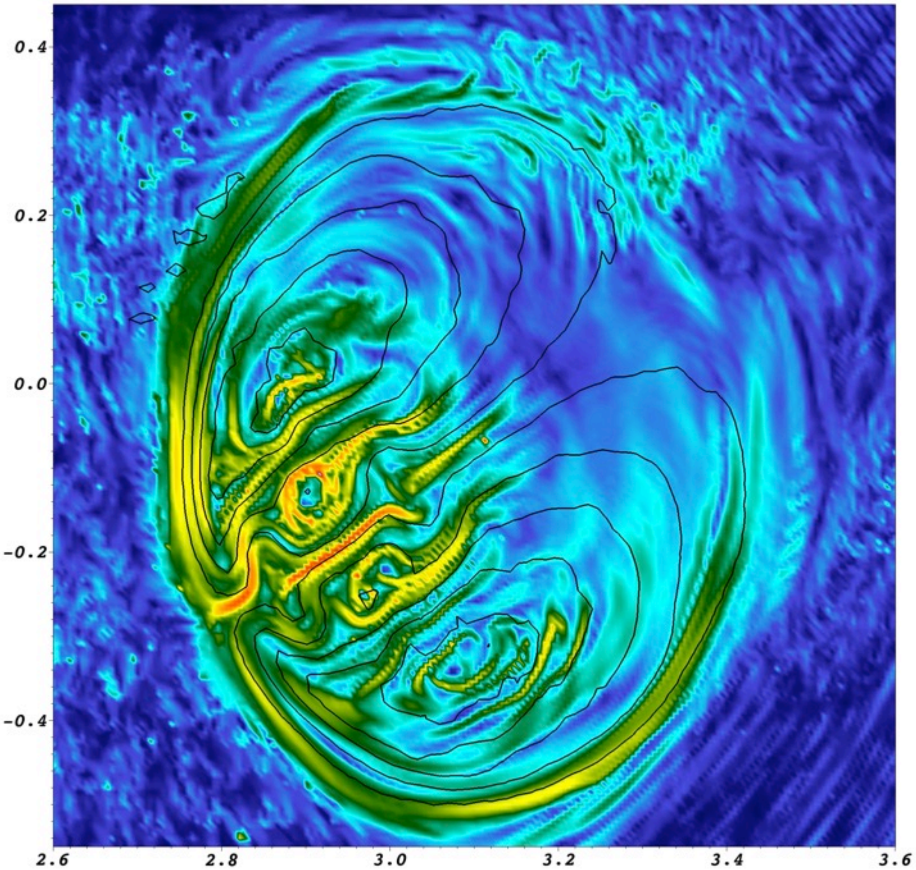
FTLEs for MHD plasma velocity

- FTLE(±**V**) structures follow the 1/1 internal kink convective cells
  - Traced for same "time" as **B** → much shorter spatial distance $L = tV \approx 2\pi RV/B$, since fluid velocity $|V|<|B|$
  - Overall, follows the poloidal stream function U contours, which are approximately the velocity flow lines in the 2D poloidal plane. $\mathbf{v} = \varepsilon R \nabla U \times \nabla \varphi + \nabla_{\perp}\chi + v_{\varphi} \boldsymbol{\varphi}$
  - Suggests MHD turbulence hidden in overall flows – clear signatures of finer scale vortices or eddies.
- ``Incompressiblity'' of the fully compressible simulation:

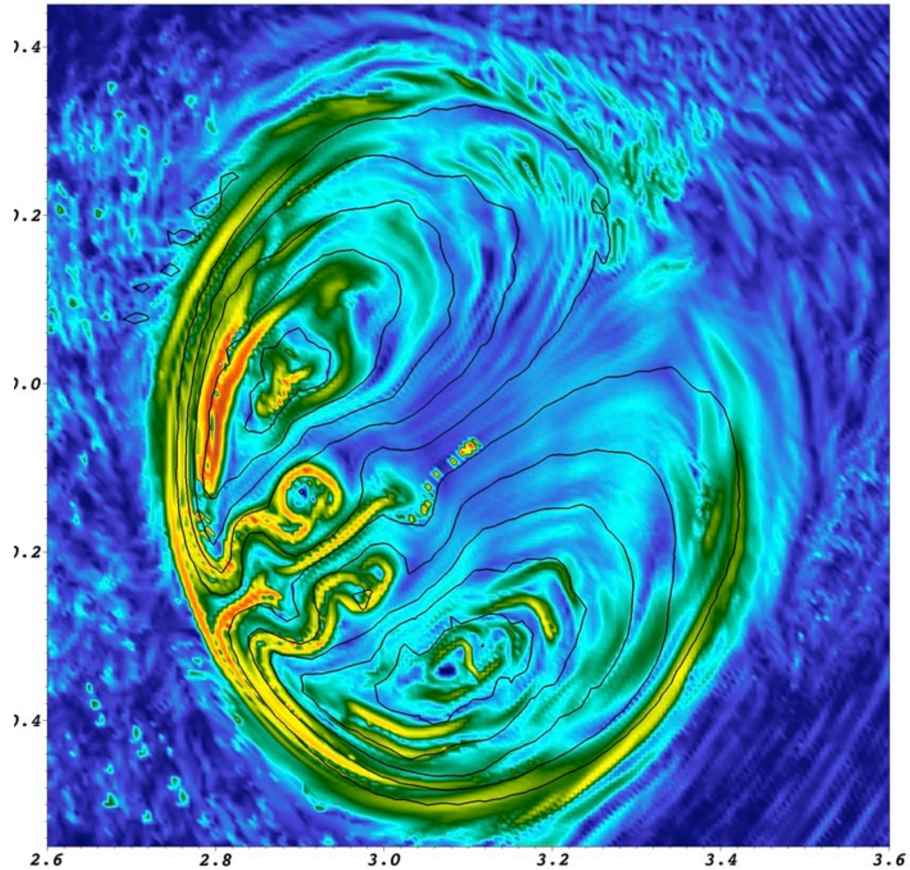
  FTLE(**V+B**) and (**V-B**) also have a 1/1 kink-like structure
  - Rotation of **B** structure by *smaller* **V**
  - Approximately follows composite poloidal flow $U \pm \tilde{\psi}/R_o$, $R_o \approxeq 3$

FTLE(MHD velocity)

FTLE(+**V**) with $\tilde{U}$ lines

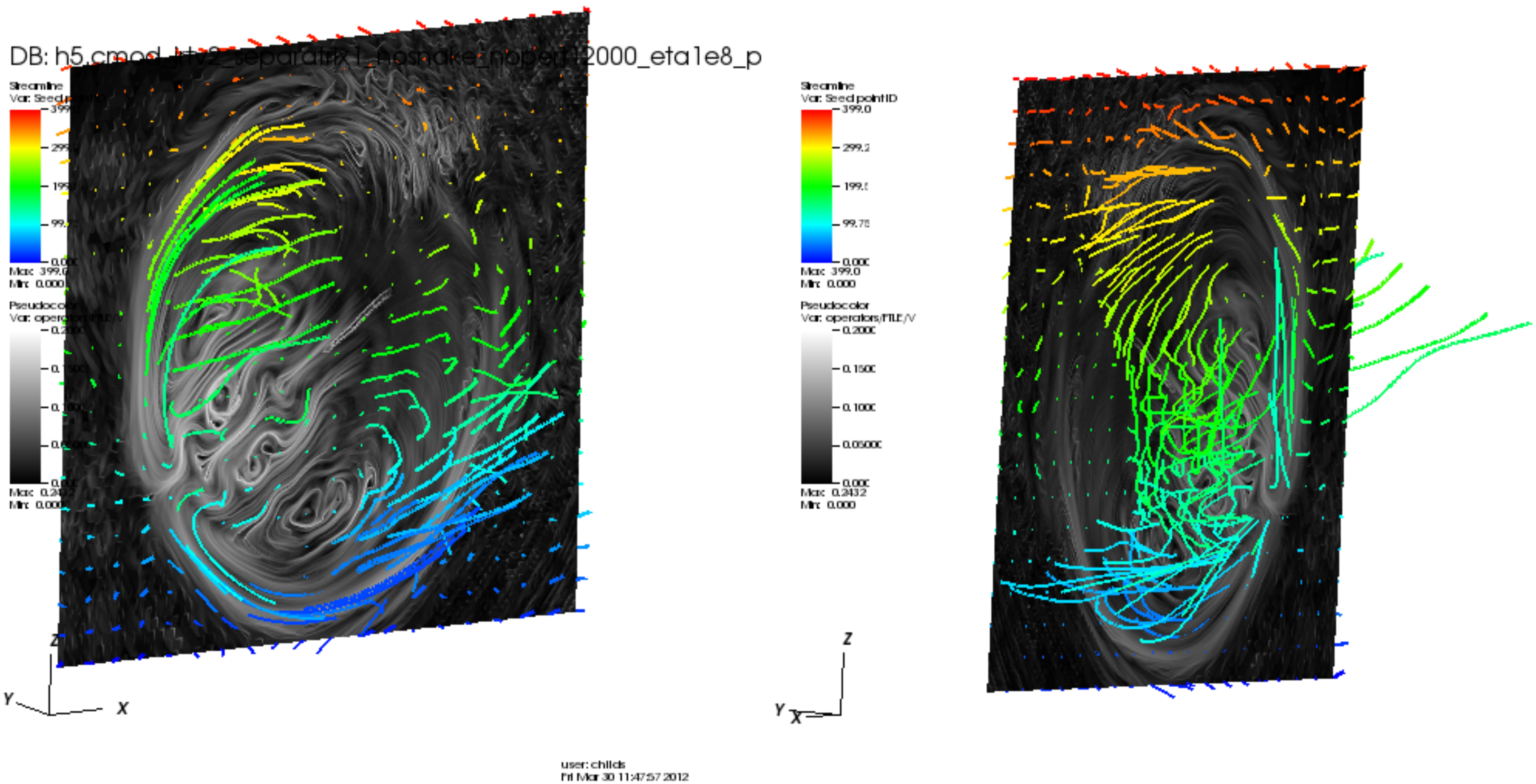


FTLE(-**V**)



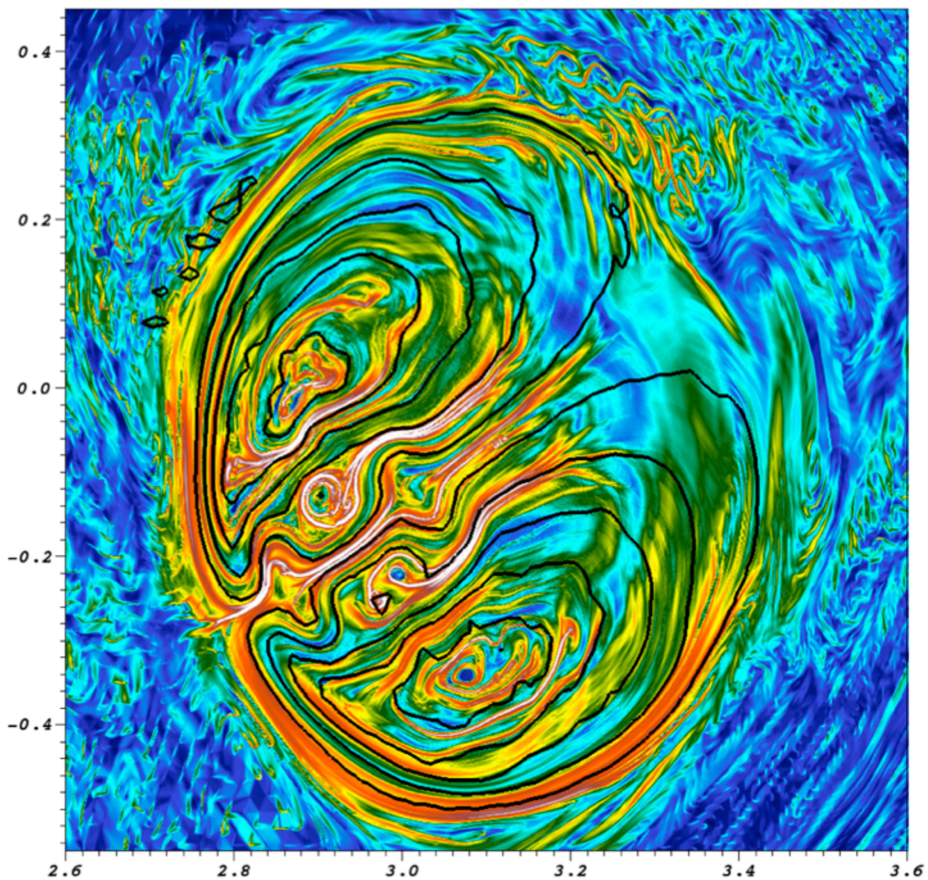
- Follows poloidal velocity stream function U (lines), i.e., 1/1 convective cells
- Vorticity ‘eddies’ at smaller scales, e.g., outflow regions into island
- Traced short distance (approx 1/10 of toroidal circumference)

Velocity vector streamlines cover ≲1/10 of torus

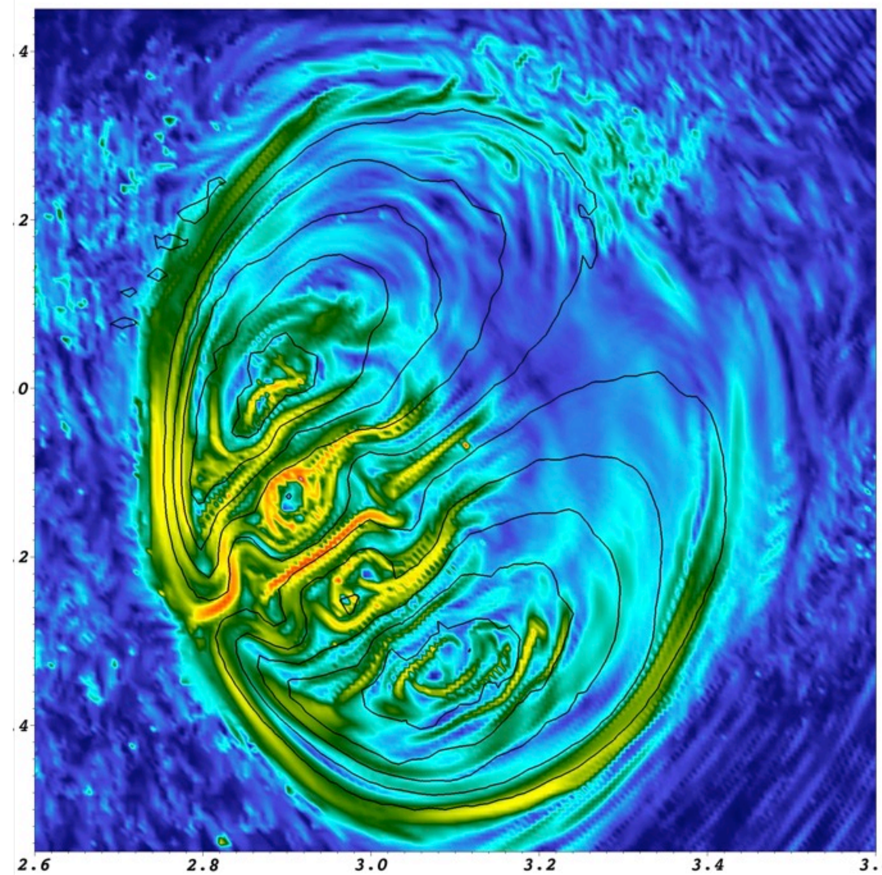


MHD turbulence – longer trace

FTLE(**V**) traced for t=42 (2X longer)



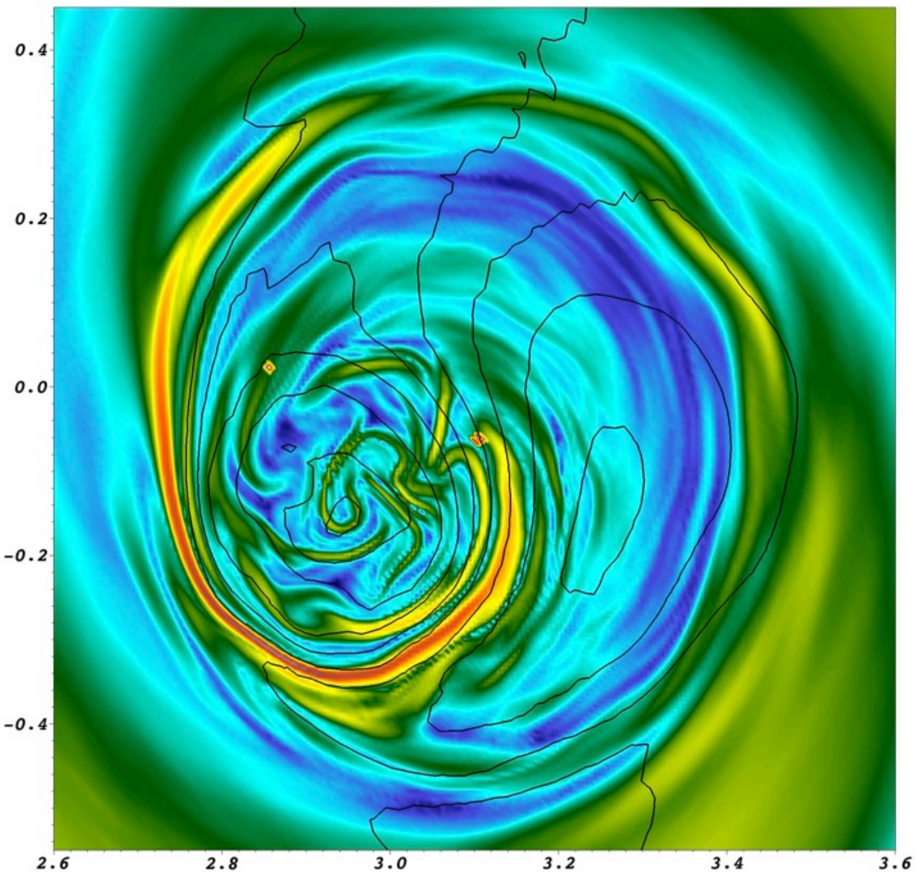
FTLE(**V**) traced for t=20



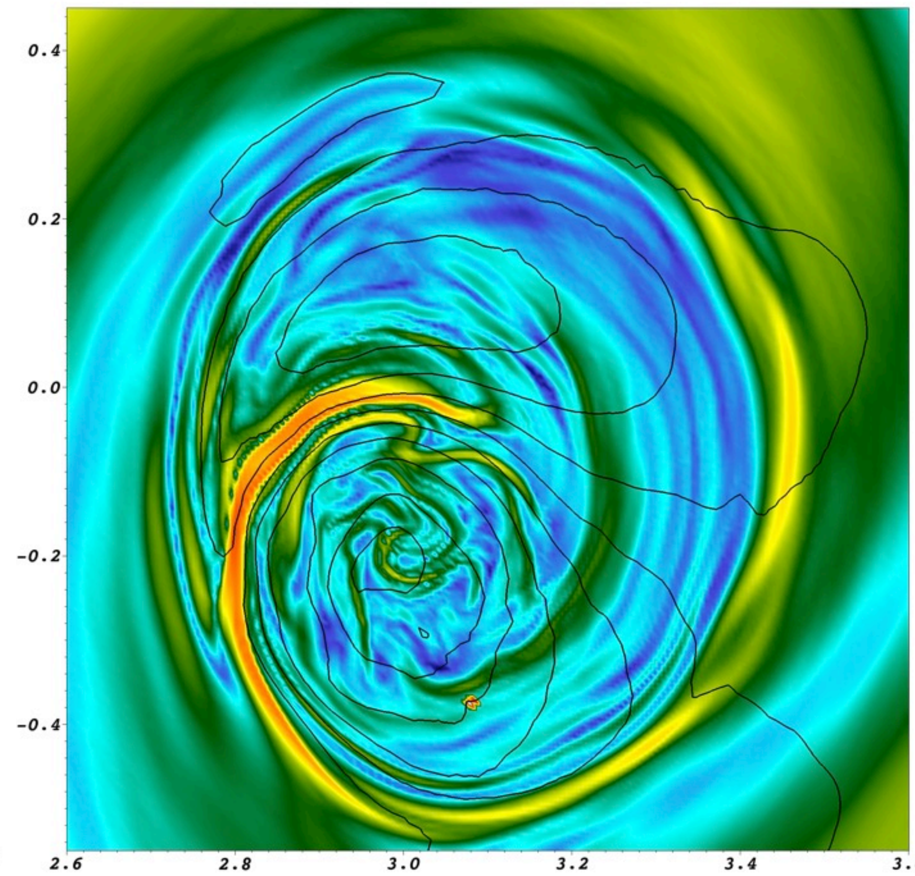
- Left: Tracing velocity FTLE for longer time/distance shows more and smaller vortices
- Still only 1/5 of torus circumference 2πR -> MHD turbulence evident at relatively large scales (n≤23 toroidal harmonics)

``Partially incompressible'': FTLE(**V**±**B**)

FTLE(**V**+**B**) with $U+\tilde{\psi}/R_o$ lines

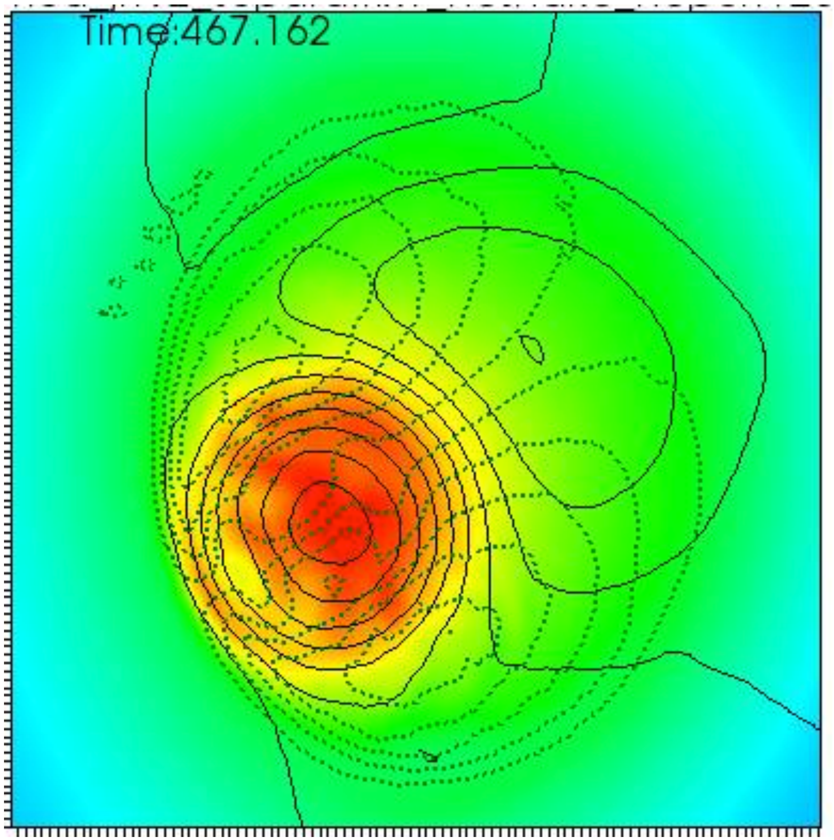


FTLE(**V**-**B**) with $U-\tilde{\psi}/R_o$



- **V±B** also has a 1/1 FTLE structure, somewhat related to the composite poloidal stream functions $U \pm \tilde{\psi}/R_o$
- Elsässer variables $\mathbf{z}^{\pm}=\mathbf{V}\pm\mathbf{B}$ are exactly equivalent to incompressible MHD (**V**,**B**)

Field and flow



FTLEs can address fundamental plasma questions

- What role does stochasticity of **B** and turbulence in **V** play in instabilities at low resistivity?
- Role of compressibility ($\nabla \cdot \mathbf{v} \neq 0$): the Elsässer variables $\mathbf{z}^{\pm} = \mathbf{V} \pm \mathbf{B}$ are exactly equivalent to (**V**, **B**) for incompressible MHD at uniform density (Elsässer (1950))
  - Nonlinear MHD simulations often appear "nearly incompressible," but require compressibility for accurate results – how to quantify?
- Relation between kinetic (particle motion) and MHD or fluid-based plasma models
  - FTLE computation is equivalent to tracing virtual particles along field lines. Particle models add particle drifts to this motion. A number of techniques developed for FTLE computation might speed up calc.
  - Gyrokinetic particle simulations reduce compressibility, since some motions are too fast relative to ion Larmor orbit averaging. Equivalent to reduced MHD when velocity moments taken.
- FTLE applies to any vector field – many interesting plasma fields!

New MHD models for next generation supercomputers

- Real time computation of FTLE's for MHD vector fields, simultaneously with the MHD solution, will allow computation of the underlying dynamic plasma structures (Lagrangian Coherent Structures)
- Way to extend MHD to next generation (exascale) computers
  - MHD is a global solution, uses only "few" processors, eg, few 1000's
  - FTLE/LCS computation requires strong parallelization to match MHD times – can take advantage of exascale processing
- Connection to particle dynamics and simulation models
  - Tracing the vector field lines for the FTLE is closely related to following virtual particles along field lines: kinetic information, new methods
- Currently under development in collaboration with H. Childs, R. Bliele, Univ. Oregon

Summary

- Modern Finite time Lyapunov Exponents are a powerful tool for studying 3D time-dependent vector fields in the presence of turbulence or stochasticity
  - Rigorously related to the underlying structures of vector fields (Lagrangian Coherent Structures) and their evolution
  - Rapidly improving methods, interpretation, applications in fluids
- First application to plasmas
  - Plasma has multiple vector fields: simplest are MHD **B**, **V**
  - High accuracy FTLE computation developed in VisIt open source visualization package
  - Sawtooth crash in M3D at a single time – new insights
- FTLE applies to general vector fields; many uses for plasmas
- Next step: time-dependent LCS for plasma
  - Extension of MHD simulation for next generation computers – FTLE/LCS computed simultaneously with MHD