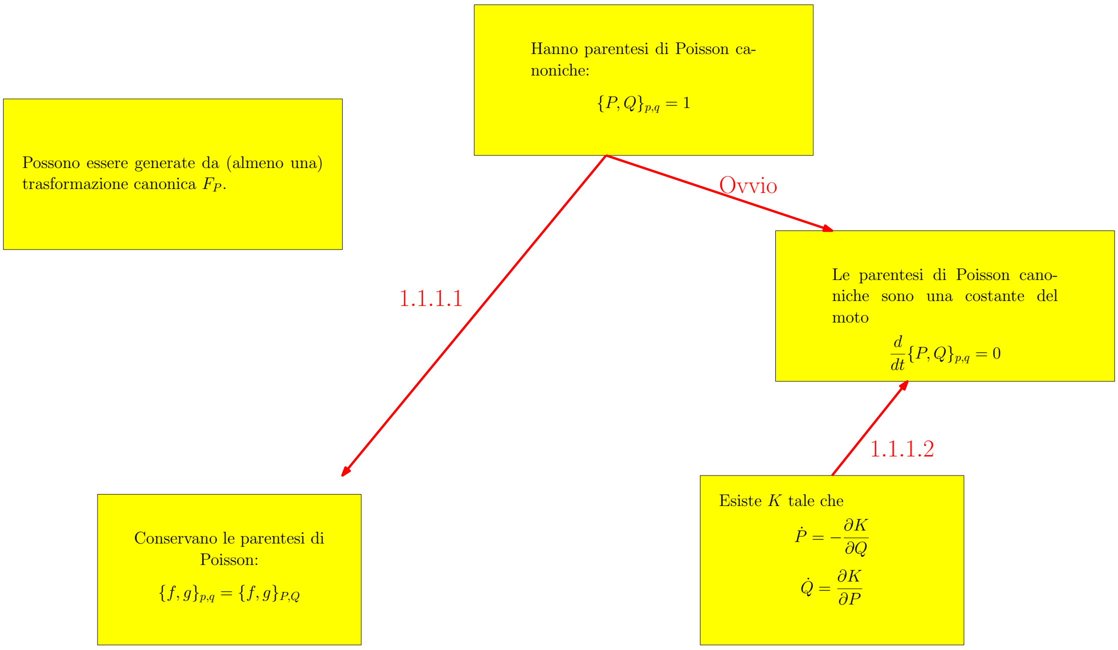

Figure 1.1.: Some relations between transformations.

1.1. Canonical transformations

1.1.1. Single degree of freedom

Una *trasformazione* canonica è definita come una trasformazione di coordinata e impulso

$$\begin{aligned} Q &= Q(q,p,t) \\ P &= P(q,p,t) \end{aligned}$$

che conserve le parentesi di Poisson canoniche, ossia

$$\{Q,P\}_{p,q} = 1 \tag{1.1.1}$$

1.1.1.1. Una trasformazione canonica conserva le parentesi di Poisson

Questo si può verificare direttamente:

$$\begin{aligned} \{f,g\}_{p,q} &\equiv \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} \\ &= \left(\frac{\partial f}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial f}{\partial P}\frac{\partial P}{\partial q}\right)\left(\frac{\partial g}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial g}{\partial P}\frac{\partial P}{\partial p}\right) - \left(\frac{\partial f}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial f}{\partial P}\frac{\partial P}{\partial p}\right)\left(\frac{\partial g}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial g}{\partial P}\frac{\partial P}{\partial q}\right) \\ &= \frac{\partial f}{\partial Q}\frac{\partial g}{\partial P}\left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right) + \frac{\partial f}{\partial P}\frac{\partial g}{\partial Q}\left(\frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q}\right) \\ &= \{f,g\}_{P,Q}\{Q,P\}_{p,q} \end{aligned} \tag{1.1.2}$$

1.1.1.2. Se P, Q conservano la forma Hamiltoniana delle equazioni del moto allora $\{Q,P\}_{p,q}$ è una costante del moto

Supponiamo che il sistema sia descritto da equazioni Hamiltoniane

$$\dot{q} = \frac{\partial H}{\partial p}$$
$$\dot{p} = -\frac{\partial H}{\partial q}$$

e che sia possibile introdurre una nuova Hamiltoniana $K(P,Q,t)$ e nuove coordinate $P(p,q,t)$, $Q(p,q,t)$ tali da soddisfare

$$\dot{Q} = \frac{\partial K}{\partial P}$$
$$\dot{P} = -\frac{\partial K}{\partial Q}$$

Valutando esplicitamente la derivata totale nel tempo di Q otteniamo

$$\begin{aligned}
\dot{Q} &= \frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial p}\dot{p} + \frac{\partial Q}{\partial t} \\
&= \frac{\partial Q}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial Q}{\partial t} \\
&= \{Q,H\}_{p,q} + \frac{\partial Q}{\partial t} \\
&= \frac{\partial Q}{\partial q}\left(\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial p}\right) - \frac{\partial Q}{\partial p}\left(\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial q}\right) + \frac{\partial Q}{\partial t} \\
&= \frac{\partial H}{\partial P}\{Q,P\} + \frac{\partial Q}{\partial t}
\end{aligned}$$

che permette di scrivere

$$\frac{\partial Q}{\partial t} = -\frac{\partial H}{\partial P}\{Q,P\} + \frac{\partial K}{\partial P} \tag{1.1.3}$$

Allo stesso modo la derivata totale di P vale

$$\begin{aligned}
\dot{P} &= \frac{\partial P}{\partial q}\dot{q} + \frac{\partial P}{\partial p}\dot{p} + \frac{\partial P}{\partial t} \\
&= \{P,H\}_{p,q} + \frac{\partial P}{\partial t} \\
&= \frac{\partial P}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial P}{\partial t} \\
&= \frac{\partial P}{\partial q}\left(\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial p}\right) - \frac{\partial P}{\partial p}\left(\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial q}\right) + \frac{\partial P}{\partial t} \\
&= -\frac{\partial H}{\partial Q}\{Q,P\} + \frac{\partial P}{\partial t}
\end{aligned}$$

e quindi

$$\frac{\partial P}{\partial t} = \frac{\partial H}{\partial Q}\{Q,P\} - \frac{\partial K}{\partial Q} \tag{1.1.4}$$

Vogliamo mostrare che le parentesi di Poisson delle nuove coordinate sono necessariamente costanti del moto. Abbiamo esplicitamente

$$\begin{aligned}
\frac{\partial}{\partial t}\{Q,P\}_{p,q} &= \frac{\partial}{\partial t}\left[\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}\right] \\
&= \frac{\partial}{\partial q}\left(\frac{\partial Q}{\partial t}\right)\frac{\partial P}{\partial p} + \frac{\partial Q}{\partial q}\frac{\partial}{\partial p}\left(\frac{\partial P}{\partial t}\right) - \frac{\partial}{\partial q}\left(\frac{\partial P}{\partial t}\right)\frac{\partial Q}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial}{\partial p}\left(\frac{\partial Q}{\partial t}\right) \\
&= \left(\frac{\partial P}{\partial p}\frac{\partial}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial}{\partial p}\right)\frac{\partial Q}{\partial t} + \left(\frac{\partial Q}{\partial q}\frac{\partial}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial}{\partial q}\right)\frac{\partial P}{\partial t}
\end{aligned}$$

Utilizzando la (1.1.3) e la (1.1.4) otteniamo

$$\begin{aligned}
\frac{\partial}{\partial t}\{Q,P\}_{p,q} &= \left(\frac{\partial P}{\partial p}\frac{\partial}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial}{\partial p}\right)\left(-\frac{\partial H}{\partial P}\{Q,P\} + \frac{\partial K}{\partial P}\right) + \left(\frac{\partial Q}{\partial q}\frac{\partial}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial}{\partial q}\right)\left(\frac{\partial H}{\partial Q}\{Q,P\} - \frac{\partial K}{\partial Q}\right) \\
&= \left(-\frac{\partial H}{\partial P}\frac{\partial P}{\partial p}\frac{\partial\{Q,P\}}{\partial q} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial q}\frac{\partial\{Q,P\}}{\partial p} + \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q}\frac{\partial\{Q,P\}}{\partial p} - \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p}\frac{\partial\{Q,P\}}{\partial q}\right) \\
&+ \{Q,P\}\left(-\frac{\partial P}{\partial p}\frac{\partial}{\partial q}\frac{\partial H}{\partial P} + \frac{\partial P}{\partial q}\frac{\partial}{\partial p}\frac{\partial H}{\partial P} + \frac{\partial Q}{\partial q}\frac{\partial}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial}{\partial q}\frac{\partial H}{\partial Q}\right) \\
&+ \frac{\partial P}{\partial p}\frac{\partial}{\partial q}\frac{\partial K}{\partial P} - \frac{\partial P}{\partial q}\frac{\partial}{\partial p}\frac{\partial K}{\partial P} - \frac{\partial Q}{\partial q}\frac{\partial}{\partial p}\frac{\partial K}{\partial Q} + \frac{\partial Q}{\partial p}\frac{\partial}{\partial q}\frac{\partial K}{\partial Q} \\
&= \frac{\partial\{Q,P\}}{\partial q}\left(-\frac{\partial H}{\partial P}\frac{\partial P}{\partial p} - \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p}\right) + \frac{\partial\{Q,P\}}{\partial p}\left(\frac{\partial H}{\partial P}\frac{\partial P}{\partial q} + \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q}\right) \\
&+ \{Q,P\}\left(\left\{P,\frac{\partial H}{\partial P}\right\} + \left\{Q,\frac{\partial H}{\partial Q}\right\}\right) \\
&- \left\{Q,\frac{\partial K}{\partial Q}\right\} - \left\{P,\frac{\partial K}{\partial P}\right\} \\
&= \frac{\partial\{Q,P\}}{\partial p}\frac{\partial H}{\partial q} - \frac{\partial\{Q,P\}}{\partial q}\frac{\partial H}{\partial p} \\
&+ \{Q,P\}\left(\left\{P,\frac{\partial H}{\partial P}\right\} + \left\{Q,\frac{\partial H}{\partial Q}\right\}\right) - \left(\left\{Q,\frac{\partial K}{\partial Q}\right\} + \left\{P,\frac{\partial K}{\partial P}\right\}\right)
\end{aligned}$$

Riordinando i termini abbiamo

$$\begin{aligned}
\frac{\partial}{\partial t}\{Q,P\}_{p,q} + \frac{\partial\{Q,P\}}{\partial p}\dot{p} + \frac{\partial\{Q,P\}}{\partial q}\dot{q} &= \{Q,P\}\left(\left\{P,\frac{\partial H}{\partial P}\right\} + \left\{Q,\frac{\partial H}{\partial Q}\right\}\right) \\
&- \left(\left\{Q,\frac{\partial K}{\partial Q}\right\} + \left\{P,\frac{\partial K}{\partial P}\right\}\right)
\end{aligned}$$

ossia

$$\frac{d}{dt}\{Q,P\}_{p,q} = \{Q,P\}\left(\left\{P,\frac{\partial H}{\partial P}\right\} + \left\{Q,\frac{\partial H}{\partial Q}\right\}\right) - \left(\left\{Q,\frac{\partial K}{\partial Q}\right\} + \left\{P,\frac{\partial K}{\partial P}\right\}\right) \tag{1.1.5}$$

Usando adesso l'identità (1.1.2) otteniamo

$$\begin{aligned}
\left\{P, \frac{\partial H}{\partial P}\right\}_{p,q} &= \left\{P, \frac{\partial H}{\partial P}\right\}_{P,Q} \{Q,P\}_{p,q} = -\{Q,P\}_{p,q} \frac{\partial^2 H}{\partial Q \partial P} \\
\left\{Q, \frac{\partial H}{\partial Q}\right\}_{p,q} &= \left\{Q, \frac{\partial H}{\partial Q}\right\}_{P,Q} \{Q,P\}_{p,q} = \{Q,P\}_{p,q} \frac{\partial^2 H}{\partial P \partial Q} \\
\left\{Q, \frac{\partial K}{\partial Q}\right\}_{p,q} &= \left\{Q, \frac{\partial K}{\partial Q}\right\}_{P,Q} \{Q,P\}_{p,q} = \{Q,P\}_{p,q} \frac{\partial^2 K}{\partial P \partial Q} \\
\left\{P, \frac{\partial K}{\partial P}\right\}_{p,q} &= \left\{P, \frac{\partial K}{\partial P}\right\}_{P,Q} \{Q,P\}_{p,q} = - - \{Q,P\}_{p,q} \frac{\partial^2 K}{\partial Q \partial P}
\end{aligned}$$

e sostituendo nella (1.1.5) troviamo

$$\frac{d}{dt} \{Q,P\}_{p,q} = 0 \tag{1.1.6}$$

che è quanto volevamo dimostrare.

1.1.1.3. Una trasformazione canonica conserva la forma delle equazioni del moto

Introduciamo la matrice

$$\mathbb{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

e la notazione

$$\boldsymbol{x} = \begin{pmatrix} q \\ p \end{pmatrix}$$

Le equazioni del moto di Hamilton si scrivono allora nella forma

$$\dot{\boldsymbol{x}} = \mathbb{J} \frac{\partial H}{\partial \boldsymbol{x}}$$

Introduciamo adesso nuove coordinate $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x}, t)$. Abbiamo

$$\begin{aligned}
\dot{y}_i &= \frac{\partial y_i}{\partial x_j} \dot{x}_j + \frac{\partial y_i}{\partial t} \\
&= \frac{\partial y_i}{\partial x_j} \mathbb{J}_{jk} \frac{\partial H}{\partial x_k} + \frac{\partial y_i}{\partial t} \\
&= \frac{\partial y_i}{\partial x_j} \mathbb{J}_{jk} \frac{\partial y_p}{\partial x_k} \frac{\partial H}{\partial y_p} + \frac{\partial y_i}{\partial t}
\end{aligned}$$

Calcoliamo esplicitamente la matrice

$$\frac{\partial y_i}{\partial x_j}\mathbb{J}_{jk}\frac{\partial y_p}{\partial x_k} = \left[\begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial P}{\partial p} \end{pmatrix}\right]_{ip}$$
$$= \left[\begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}\begin{pmatrix} \frac{\partial Q}{\partial p} & \frac{\partial P}{\partial p} \\ -\frac{\partial Q}{\partial q} & -\frac{\partial P}{\partial q} \end{pmatrix}\right]_{ip}$$
$$= \begin{pmatrix} \{Q,Q\}_{p,q} & \{Q,P\}_{p,q} \\ -\{Q,P\}_{p,q} & \{P,P\}_{p,q} \end{pmatrix}_{ip} = \mathbb{J}_{ip}$$

$$\mathbb{H}\mathbb{J}\mathbb{H}^T = \mathbb{J}$$

dove l'ultima uguaglianza segue dal fatto che le trasformazioni sono canoniche. Allora le equazioni del moto sono della forma

$$\dot{\boldsymbol{y}} = \mathbb{J}\frac{\partial H}{\partial \boldsymbol{y}} + \frac{\partial \boldsymbol{y}}{\partial t} = \mathbb{J}\frac{\partial K}{\partial \boldsymbol{y}}$$

dove

$$\frac{\partial K}{\partial \boldsymbol{y}} = \frac{\partial H}{\partial \boldsymbol{y}} - \mathbb{J}\frac{\partial \boldsymbol{y}}{\partial t}$$

Se la trasformazione di coordinate non dipende dal tempo abbiamo che K e H coincidono a meno di una funzione del solo tempo irrilevante. In caso contrario si può verificare che è possibile trovare una funzione K che soddisfa l'equazione precedente. Infatti la condizione di integrabilità

$$\frac{\partial}{\partial y_i}\left[\frac{\partial H}{\partial y_j} - \mathbb{J}_{jk}\frac{\partial y_k}{\partial t}\right] = \frac{\partial}{\partial y_j}\left[\frac{\partial H}{\partial y_i} - \mathbb{J}_{ik}\frac{\partial y_k}{\partial t}\right]$$

diviene

$$\frac{\partial x_p}{\partial y_i}\frac{\partial}{\partial x_p}\left[\mathbb{J}_{jk}\frac{\partial y_k}{\partial t}\right] = \frac{\partial x_p}{\partial y_j}\frac{\partial}{\partial x_p}\left[\mathbb{J}_{ik}\frac{\partial y_k}{\partial t}\right]$$

$$\left(\mathbb{J}\dot{\mathbb{H}}\mathbb{H}^{-1}\right)^T = \mathbb{J}\dot{\mathbb{H}}\mathbb{H}^{-1}$$
$$\left(\mathbb{J}\dot{\mathbb{H}}\mathbb{H}^{-1}\right)^T$$

che è automaticamente soddisfatta dato che possiamo riscriverla nella forma

$$\frac{\partial x_p}{\partial y_i}\frac{\partial}{\partial t}\mathbb{J}_{jk}\frac{\partial y_k}{\partial x_p} = \frac{\partial x_p}{\partial y_j}\frac{\partial}{\partial t}\mathbb{J}_{ik}\frac{\partial y_k}{\partial x_p}$$

mostrare che se e solo se l'Equazione (1.1.1) è verificata il differenziale

$$dF = PdQ - Kdt - pdq + Hdt$$

è esatto. Iniziamo esplicitando rispetto alle coordinate p e q

$$\begin{aligned} dF &= P\left(\frac{\partial Q}{\partial q}dq + \frac{\partial Q}{\partial p}dp + \frac{\partial Q}{\partial t}dt\right) - Kdt - pdq + Hdt \\ &= \left(P\frac{\partial Q}{\partial q} - p\right)dq + P\frac{\partial Q}{\partial p}dp + \left(P\frac{\partial Q}{\partial t} - K + H\right)dt \end{aligned}$$

e scriviamo le condizioni di integrabilità:

$$\frac{\partial}{\partial p}\left(P\frac{\partial Q}{\partial q} - p\right) = \frac{\partial}{\partial q}\left(P\frac{\partial Q}{\partial p}\right) \tag{1.1.7}$$

$$\frac{\partial}{\partial t}\left(P\frac{\partial Q}{\partial q} - p\right) = \frac{\partial}{\partial q}\left(P\frac{\partial Q}{\partial t} - K + H\right) \tag{1.1.8}$$

$$\frac{\partial}{\partial t}\left(P\frac{\partial Q}{\partial p}\right) = \frac{\partial}{\partial p}\left(P\frac{\partial Q}{\partial t} - K + H\right) \tag{1.1.9}$$

Calcolando le derivate nella (1.1.7) troviamo

$$\frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} + P\frac{\partial^2 Q}{\partial p\partial q} - 1 = \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} + P\frac{\partial^2 Q}{\partial q\partial p}$$

che è verificata se sono valide le parentesi di Poisson canoniche. Dalla (1.1.8) otteniamo

$$\frac{\partial P}{\partial t}\frac{\partial Q}{\partial q} + P\frac{\partial^2 Q}{\partial t\partial q} = \frac{\partial P}{\partial q}\frac{\partial Q}{\partial t} + P\frac{\partial^2 Q}{\partial q\partial t} + \frac{\partial}{\partial q}(H - K)$$

che, usando la (1.1.3) e la (1.1.4), diviene l'identità

$$\frac{\partial Q}{\partial q}\frac{\partial (H-K)}{\partial Q} + \frac{\partial P}{\partial q}\frac{\partial (H-K)}{\partial P} = \frac{\partial (H-K)}{\partial q}$$

Infine dalla (1.1.9) otteniamo

$$\frac{\partial P}{\partial t}\frac{\partial Q}{\partial p} + P\frac{\partial^2 Q}{\partial t\partial p} = \frac{\partial P}{\partial p}\frac{\partial Q}{\partial t} + P\frac{\partial^2 Q}{\partial p\partial t} + \frac{\partial}{\partial p}(H - K)$$

Usando nuovamente la (1.1.3) e la (1.1.4) otteniamo

$$\frac{\partial (H-K)}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial P}{\partial p}\frac{\partial (H-K)}{\partial P} = \frac{\partial}{\partial p}(H - K)$$

e quindi ancora una identità.