1.2. Canonical transformations

Let us suppose that a mechanical system is described by a set of canonical coordinates p_i, q^i and by an Hamiltonian $H\left(p_i, q^i, t\right)$ in such a way that the motion equations are given by

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}$$
$$\dot{q}^i = \frac{\partial H}{\partial p_i}$$

We want to introduce a new set of coordinates

$$\begin{aligned} P_i &= P_i\,(p, q, t) \\ Q^i &= Q^i\,(p, q, t) \end{aligned} \tag{1.2.1}$$

in such a way that

$$\left\{Q^i, Q^j\right\}_{p,q} = 0 \tag{1.2.2}$$
$$\left\{P_i, P_j\right\}_{p,q} = 0 \tag{1.2.3}$$
$$\left\{Q^i, P_j\right\}_{p,q} = \delta^i_j \tag{1.2.4}$$

where the Poisson brackets of two functions are defined by

$$\{A, B\}_{p,q} = \frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i}$$

and a sum over repeated indices is understood. We will call the (1.2) a canonical transformation.

1.2.1. A canonical transformation leaves the equation of motion in Hamiltonian form

Now we ask the following question: is it possible to introduce a new Hamiltonian function K in such a way that

$$\dot{P}_i = -\frac{\partial K}{\partial Q^i} \tag{1.2.5}$$
$$\dot{Q}^i = \frac{\partial K}{\partial P_i} \tag{1.2.6}$$

If this is true, we can expand the total time derivative in (1.2.6) obtaining

$$
\begin{aligned}
\frac{\partial K}{\partial P_i} &= \frac{\partial Q^i}{\partial q^j}\dot{q}^j + \frac{\partial Q^i}{\partial p_j}\dot{p}_j + \frac{\partial Q^i}{\partial t} \\
&= \frac{\partial Q^i}{\partial q^j}\frac{\partial H}{\partial p_j} - \frac{\partial Q^i}{\partial p_j}\frac{\partial H}{\partial q^j} + \frac{\partial Q^i}{\partial t} \\
&= \frac{\partial Q^i}{\partial q^j}\left(\frac{\partial H}{\partial Q^k}\frac{\partial Q^k}{\partial p_j} + \frac{\partial H}{\partial P_k}\frac{\partial P_k}{\partial p_j}\right) - \frac{\partial Q^i}{\partial p_j}\left(\frac{\partial H}{\partial Q^k}\frac{\partial Q^k}{\partial q^j} + \frac{\partial H}{\partial P_k}\frac{\partial P_k}{\partial q^j}\right) + \frac{\partial Q^i}{\partial t} \\
&= \frac{\partial H}{\partial Q^k}\left(\frac{\partial Q^i}{\partial q^j}\frac{\partial Q^k}{\partial p_j} - \frac{\partial Q^i}{\partial p_j}\frac{\partial Q^k}{\partial q^j}\right) + \frac{\partial H}{\partial P_k}\left(\frac{\partial Q^i}{\partial q^j}\frac{\partial P_k}{\partial p_j} - \frac{\partial Q^i}{\partial p_j}\frac{\partial P_k}{\partial q^j}\right) + \frac{\partial Q^i}{\partial t} \\
&= \frac{\partial H}{\partial Q^k}\left\{Q^i, Q^k\right\}_{p,q} + \frac{\partial H}{\partial P_k}\left\{Q^i, P_k\right\}_{p,q} + \frac{\partial Q^i}{\partial t} \\
&= \frac{\partial H}{\partial P_i} + \frac{\partial Q^i}{\partial t}
\end{aligned}
$$

and in the same way, expanding (1.2.5),

$$
\begin{aligned}
-\frac{\partial K}{\partial Q^i} &= \frac{\partial P_i}{\partial p_j}\dot{p}_j + \frac{\partial P_i}{\partial q^j}\dot{q}_j + \frac{\partial P_i}{\partial t} \\
&= -\frac{\partial P_i}{\partial p_j}\frac{\partial H}{\partial q^j} + \frac{\partial P_i}{\partial q^j}\frac{\partial H}{\partial p_j} + \frac{\partial P_i}{\partial t} \\
&= -\frac{\partial P_i}{\partial p_j}\left(\frac{\partial H}{\partial Q^k}\frac{\partial Q^k}{\partial q^j} + \frac{\partial H}{\partial P_k}\frac{\partial P_k}{\partial q^j}\right) + \frac{\partial P_i}{\partial q^j}\left(\frac{\partial H}{\partial Q^k}\frac{\partial Q^k}{\partial p_j} + \frac{\partial H}{\partial P_k}\frac{\partial P_k}{\partial p_j}\right) + \frac{\partial P_i}{\partial t} \\
&= \frac{\partial H}{\partial Q^k}\left(\frac{\partial P_i}{\partial q^j}\frac{\partial Q^k}{\partial p_j} - \frac{\partial Q^k}{\partial q^j}\frac{\partial P_i}{\partial p_j}\right) + \frac{\partial H}{\partial P_k}\left(\frac{\partial P_i}{\partial q^j}\frac{\partial P_k}{\partial p_j} - \frac{\partial P_i}{\partial p_j}\frac{\partial P_k}{\partial q^j}\right) + \frac{\partial P_i}{\partial t} \\
&= \frac{\partial H}{\partial Q^k}\left\{P_i, Q^k\right\}_{p,q} + \frac{\partial H}{\partial P_k}\left\{P_i, P_k\right\}_{p,q} + \frac{\partial P_i}{\partial t} \\
&= -\frac{\partial H}{\partial Q^i} + \frac{\partial P_i}{\partial t}
\end{aligned}
$$

It follows that

$$
\frac{\partial Q^i}{\partial t} = -\frac{\partial\left(H-K\right)}{\partial P_i} \tag{1.2.7}
$$

$$
\frac{\partial P_i}{\partial t} = \frac{\partial\left(H-K\right)}{\partial Q^i} \tag{1.2.8}
$$

Now we want to show that the differential

$$
dF = P_i dQ^i - K dt - p_i dq^i + H dt \tag{1.2.9}
$$

is exact if and only if the transformation is canonical. We start by expressing dF in terms of the differentials dp_i, dq^i and dt only:

$$\begin{aligned} dF &= P_i\left(\frac{\partial Q^i}{\partial q^j}dq^j + \frac{\partial Q^i}{\partial p_j}dp_j + \frac{\partial Q^i}{\partial t}dt\right) - K dt - p_i dq^i + H dt \\ &= \left(P_i\frac{\partial Q^i}{\partial q^j} - p_j\right)dq^j + P_i\frac{\partial Q^i}{\partial p_j}dp_j + \left(P_i\frac{\partial Q^i}{\partial t} + H - K\right)dt \end{aligned}$$

The necessary and sufficient condition is the equality of mixed derivatives, namely it must be

$$\frac{\partial}{\partial p_k}\left(P_i\frac{\partial Q^i}{\partial q^j} - p_j\right) = \frac{\partial}{\partial q^j}\left(P_i\frac{\partial Q^i}{\partial p_k}\right) \tag{1.2.10}$$

$$\frac{\partial}{\partial t}\left(P_i\frac{\partial Q^i}{\partial q^j} - p_j\right) = \frac{\partial}{\partial q^j}\left(P_i\frac{\partial Q^i}{\partial t} + H - K\right) \tag{1.2.11}$$

$$\frac{\partial}{\partial t}\left(P_i\frac{\partial Q^i}{\partial p^j}\right) = \frac{\partial}{\partial p_j}\left(P_i\frac{\partial Q^i}{\partial t} + H - K\right) \tag{1.2.12}$$

Let us check that these relations are verified for a canonical transformation. The identity (1.2.10) can be expanded in the following way:

$$\frac{\partial P_i}{\partial p_k}\frac{\partial Q^i}{\partial q^j} + P_i\frac{\partial^2 Q^i}{\partial p_k \partial q^j} - \delta^k_j = \frac{\partial P_i}{\partial q^j}\frac{\partial Q^i}{\partial p_k} + P_i\frac{\partial^2 Q^i}{\partial q^j \partial p_k}$$

or

$$\frac{\partial P_i}{\partial p_k}\frac{\partial Q^i}{\partial q^j} - \frac{\partial P_i}{\partial q^j}\frac{\partial Q^i}{\partial p_k} - \delta^k_j = P_i\left(\frac{\partial^2 Q^i}{\partial q^j \partial p_k} - \frac{\partial^2 Q^i}{\partial p_k \partial q^j}\right)$$

The right member is obviously zero. The left one is also zero, as a consequence of Equations (1.2.2), (1.2.3) and (1.2.4). In fact we can write

$$\ldots$$

The identity (1.2.11) can be rewritten as

$$\frac{\partial P_i}{\partial t}\frac{\partial Q^i}{\partial q^j} = \frac{\partial P_i}{\partial q^j}\frac{\partial Q^i}{\partial t} + \frac{\partial}{\partial q^j}(H - K)$$

and using Eq. (1.2.7) and Eq. (1.2.8) we get

$$\left(\frac{\partial Q^i}{\partial q^j}\frac{\partial}{\partial Q^i} + \frac{\partial P_i}{\partial q^j}\frac{\partial}{\partial P_i}\right)(H - K) = \frac{\partial}{\partial q^j}(H - K)$$

which is obviously verified. In the same way the identity (1.2.12) gives

$$\frac{\partial P_i}{\partial t}\frac{\partial Q^i}{\partial p^j} = \frac{\partial P_i}{\partial p_j}\frac{\partial Q^i}{\partial t} + \frac{\partial}{\partial p_j}(H - K)$$

and using again Eq. (1.2.7) and Eq. (1.2.8)

$$\left(\frac{\partial Q^i}{\partial p^j}\frac{\partial}{\partial Q^i}+\frac{\partial P_i}{\partial p_j}\frac{\partial}{\partial P_i}\right)(H-K)=\frac{\partial}{\partial p_j}(H-K)$$

which is also verified.

We proved that dF is an exact differential. This give us an algorithm to generate the canonical transformation. For each i let us choose two functionally independent coordinates, one in the set $\left\{p_i, q^i\right\}$ and one in the set $\left\{P_i, Q^i\right\}$. To give an example, let us suppose that it is possible to choose all the q^i's and all the Q^i's. In this we can write (1.2.9) as

$$\frac{\partial F}{\partial Q^i}dQ^i+\frac{\partial F}{\partial q^i}dq^i+\frac{\partial F}{\partial t}=P_i dQ^i-Kdt-p_i dq^i+Hdt$$

getting

$$\begin{aligned}P_i&=\frac{\partial F}{\partial Q^i}\\ p_i&=-\frac{\partial F}{\partial q^i}\\ K&=H-\frac{\partial F}{\partial t}\end{aligned}$$