Personal Notes

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Part I.

Classical Mechanics

0.1. Problema a tre corpi ristretto

Consideriamo due stelle di massa M_1 ed M_2 in orbita attorno al centro di massa comune. La velocità angolare del sistema si trova facilmente dall'equazione del moto radiale per il moto relativo:

$$\mu a \omega^2 = \frac{G M_1 M_2}{a^2}$$

da cui

$$\omega^2 = \frac{G\left(M_1 + M_2\right)}{a^3}$$

Nel sistema rotante il potenziale gravitazionale delle due stelle si può scrivere

$$\phi\left(x, y, z\right) = -\frac{G M_1}{\sqrt{\left(x - \frac{\mu a}{M_1}\right)^2 + y^2 + z^2}} - \frac{G M_2}{\sqrt{\left(x + \frac{\mu a}{M_2}\right)^2 + y^2 + z^2}}$$

Con una traslazione

$$x' = x + \frac{\mu a}{M_2}$$

$$\phi\left(x, y, z\right) = -\frac{G M_1}{\sqrt{\left(x' - a\right)^2 + y^2 + z^2}} - \frac{G M_2}{\sqrt{x'^2 + y^2 + z^2}}$$

Consideriamo adesso una particella di prova. La sua energia cinetica in un sistema inerziale sarà

$$T = \frac{1}{2} m \frac{d\vec{R}}{dt} \cdot \frac{d\vec{R}}{dt}$$

Il vettore $\vec{R}$ è legato alle coordinate inerziali da

$$\vec{R} = \begin{pmatrix} X \\ Y \\ X \end{pmatrix}$$

ed è legato da

$$\vec{r} = \mathbb{R}\vec{R}$$

alle coordinate nel sistema rotante

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

dove $\mathbb{R}$ è una matrice di rotazione della forma

$$\mathbb{R} = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Possiamo allora scrivere

$$\vec{v} = \frac{d\mathbb{R}}{dt}\vec{R} + \mathbb{R}\vec{V}$$

e quindi

$$\vec{V} = \mathbb{R}^T\vec{v} - \mathbb{R}^T\frac{d\mathbb{R}}{dt}\mathbb{R}^T\vec{r}$$

Ora,

$$\frac{d\mathbb{R}}{dt}\mathbb{R}^T\vec{r} = \vec{\omega} \wedge \vec{r}$$

e quindi

$$\vec{V} = \mathbb{R}^T\vec{v} - \mathbb{R}^T\vec{\omega} \wedge \vec{r}$$

Sostituendo nell'energia cinetica troviamo

$$T = \frac{1}{2}m\left[v^2 + (\vec{\omega} \wedge \vec{r})^2 - 2\vec{v} \cdot (\vec{\omega} \wedge \vec{r})\right]$$

La Lagrangiana per la particella di prova nel sistema rotante sarà dunque

$$\begin{aligned}\mathcal{L}_m &= \frac{1}{2}m\left[v^2 - 2\vec{v} \cdot (\vec{\omega} \wedge \vec{r})\right] - m\phi(\vec{r}) + \frac{1}{2}m(\vec{\omega} \wedge \vec{r})^2 \\ &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - m\omega(\dot{x}y - \dot{y}x) - m\phi(\vec{r}) + \frac{1}{2}m\omega^2\left(x^2 + y^2\right)\end{aligned}$$

Con la traslazione precedente il potenziale centrifugo vale

$$\frac{1}{2}m\omega^2\left[\left(x' - \frac{\mu a}{M_2}\right)^2 + y^2\right]$$

Il secondo termine è legato alla forza di Coriolis, in quarto a quella centrifuga.

Determiniamo nel piano xy i punti stazionari del potenziale efficace, che corrispondono a posizioni di equilibrio per la massa di test. Consideriamo il solo caso $M_1 = M_2$

$$\begin{aligned}\phi_{eff} &= GM\left\{-\frac{1}{2}K\left(x^2 + y^2\right) + \frac{1}{\sqrt{\left(x - \frac{a}{2}\right)^2 + y^2}} + \frac{1}{\sqrt{\left(x + \frac{a}{2}\right)^2 + y^2}}\right\} \\ K &= \frac{\omega^2}{GM}\end{aligned}$$

$$\begin{aligned}\frac{\partial\phi_{eff}}{\partial x} &= GM\left[Kx + \frac{x - \frac{a}{2}}{\left[\left(x - \frac{a}{2}\right)^2 + y^2\right]^{3/2}} + \frac{x + \frac{a}{2}}{\left[\left(x + \frac{a}{2}\right)^2 + y^2\right]^{3/2}}\right] \\ \frac{\partial\phi_{eff}}{\partial x} &= GM\left[Ky + \frac{y}{\left[\left(x - \frac{a}{2}\right)^2 + y^2\right]^{3/2}} + \frac{y}{\left[\left(x + \frac{a}{2}\right)^2 + y^2\right]^{3/2}}\right]\end{aligned}$$

Sull'asse $y = 0$ la seconda derivata si annulla identicamente, la prima se

$$Kx + \frac{x - \frac{a}{2}}{\left|x - \frac{a}{2}\right|^3} + \frac{x + \frac{a}{2}}{\left|x + \frac{a}{2}\right|^3} = 0$$

Se $x > a/2$

$$Kx + \frac{1}{\left(x - \frac{a}{2}\right)^2} + \frac{1}{\left(x + \frac{a}{2}\right)^2} = 0$$

per $-a/2 < x < a/2$

$$Kx - \frac{1}{\left(x - \frac{a}{2}\right)^2} + \frac{1}{\left(x + \frac{a}{2}\right)^2} = 0$$

e per $x < -a/2$

$$Kx - \frac{1}{\left(x - \frac{a}{2}\right)^2} - \frac{1}{\left(x + \frac{a}{2}\right)^2} = 0$$

1. Hamiltonian formalism

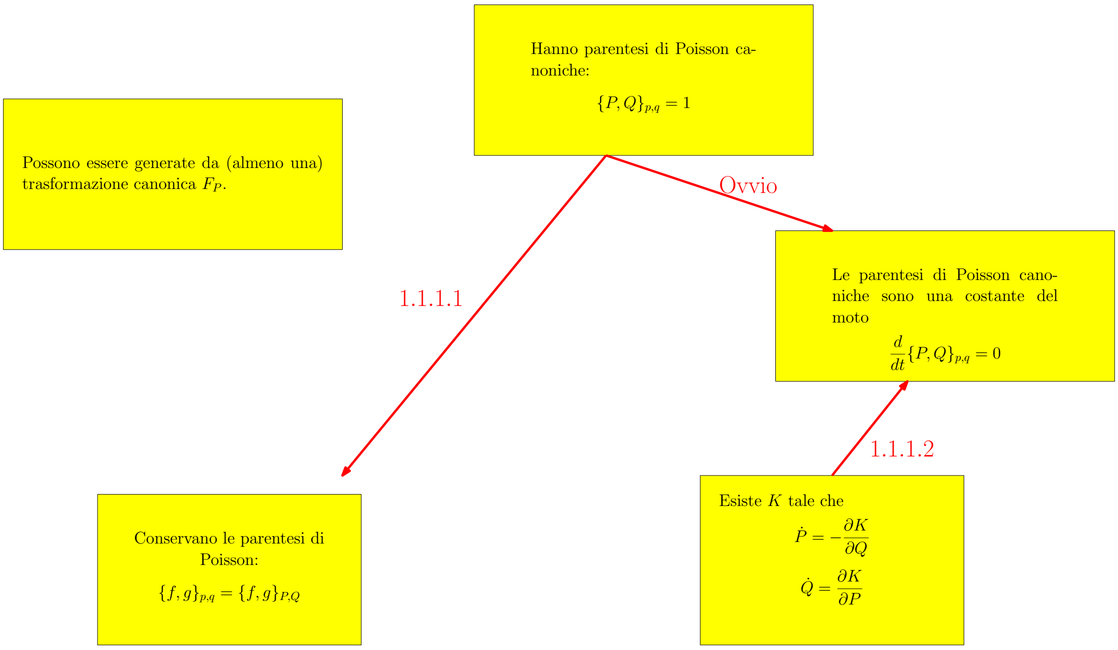

Figure 1.1.: Some relations between transformations.

1.1. Canonical transformations

1.1.1. Single degree of freedom

Una *trasformazione* canonica è definita come una trasformazione di coordinata e impulso

$$\begin{aligned} Q &= Q(q,p,t) \\ P &= P(q,p,t) \end{aligned}$$

che conserve le parentesi di Poisson canoniche, ossia

$$\{Q,P\}_{p,q} = 1 \tag{1.1.1}$$

1.1.1.1. Una trasformazione canonica conserva le parentesi di Poisson

Questo si può verificare direttamente:

$$\begin{aligned} \{f,g\}_{p,q} &\equiv \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} \\ &= \left(\frac{\partial f}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial f}{\partial P}\frac{\partial P}{\partial q}\right)\left(\frac{\partial g}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial g}{\partial P}\frac{\partial P}{\partial p}\right) - \left(\frac{\partial f}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial f}{\partial P}\frac{\partial P}{\partial p}\right)\left(\frac{\partial g}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial g}{\partial P}\frac{\partial P}{\partial q}\right) \\ &= \frac{\partial f}{\partial Q}\frac{\partial g}{\partial P}\left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right) + \frac{\partial f}{\partial P}\frac{\partial g}{\partial Q}\left(\frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial Q}{\partial q}\right) \\ &= \{f,g\}_{P,Q}\{Q,P\}_{p,q} \end{aligned} \tag{1.1.2}$$

1.1.1.2. Se P, Q conservano la forma Hamiltoniana delle equazioni del moto allora $\{Q,P\}_{p,q}$ è una costante del moto

Supponiamo che il sistema sia descritto da equazioni Hamiltoniane

$$\begin{aligned}\dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q}\end{aligned}$$

e che sia possibile introdurre una nuova Hamiltoniana $K(P,Q,t)$ e nuove coordinate $P(p,q,t)$, $Q(p,q,t)$ tali da soddisfare

$$\begin{aligned}\dot{Q} &= \frac{\partial K}{\partial P} \\ \dot{P} &= -\frac{\partial K}{\partial Q}\end{aligned}$$

Valutando esplicitamente la derivata totale nel tempo di Q otteniamo

$$\begin{aligned}\dot{Q} &= \frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial p}\dot{p} + \frac{\partial Q}{\partial t} \\ &= \frac{\partial Q}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial Q}{\partial t} \\ &= \{Q,H\}_{p,q} + \frac{\partial Q}{\partial t} \\ &= \frac{\partial Q}{\partial q}\left(\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial p}\right) - \frac{\partial Q}{\partial p}\left(\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial q}\right) + \frac{\partial Q}{\partial t} \\ &= \frac{\partial H}{\partial P}\{Q,P\} + \frac{\partial Q}{\partial t}\end{aligned}$$

che permette di scrivere

$$\frac{\partial Q}{\partial t} = -\frac{\partial H}{\partial P}\{Q,P\} + \frac{\partial K}{\partial P} \tag{1.1.3}$$

Allo stesso modo la derivata totale di P vale

$$\begin{aligned}\dot{P} &= \frac{\partial P}{\partial q}\dot{q} + \frac{\partial P}{\partial p}\dot{p} + \frac{\partial P}{\partial t} \\ &= \{P,H\}_{p,q} + \frac{\partial P}{\partial t} \\ &= \frac{\partial P}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial P}{\partial t} \\ &= \frac{\partial P}{\partial q}\left(\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial p}\right) - \frac{\partial P}{\partial p}\left(\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial q}\right) + \frac{\partial P}{\partial t} \\ &= -\frac{\partial H}{\partial Q}\{Q,P\} + \frac{\partial P}{\partial t}\end{aligned}$$

e quindi

$$\frac{\partial P}{\partial t} = \frac{\partial H}{\partial Q}\{Q,P\} - \frac{\partial K}{\partial Q} \tag{1.1.4}$$

Vogliamo mostrare che le parentesi di Poisson delle nuove coordinate sono necessariamente costanti del moto. Abbiamo esplicitamente

$$\begin{aligned}
\frac{\partial}{\partial t}\{Q,P\}_{p,q} &= \frac{\partial}{\partial t}\left[\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p}\right] \\
&= \frac{\partial}{\partial q}\left(\frac{\partial Q}{\partial t}\right)\frac{\partial P}{\partial p} + \frac{\partial Q}{\partial q}\frac{\partial}{\partial p}\left(\frac{\partial P}{\partial t}\right) - \frac{\partial}{\partial q}\left(\frac{\partial P}{\partial t}\right)\frac{\partial Q}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial}{\partial p}\left(\frac{\partial Q}{\partial t}\right) \\
&= \left(\frac{\partial P}{\partial p}\frac{\partial}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial}{\partial p}\right)\frac{\partial Q}{\partial t} + \left(\frac{\partial Q}{\partial q}\frac{\partial}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial}{\partial q}\right)\frac{\partial P}{\partial t}
\end{aligned}$$

Utilizzando la (1.1.3) e la (1.1.4) otteniamo

$$\begin{aligned}
\frac{\partial}{\partial t}\{Q,P\}_{p,q} &= \left(\frac{\partial P}{\partial p}\frac{\partial}{\partial q} - \frac{\partial P}{\partial q}\frac{\partial}{\partial p}\right)\left(-\frac{\partial H}{\partial P}\{Q,P\} + \frac{\partial K}{\partial P}\right) + \left(\frac{\partial Q}{\partial q}\frac{\partial}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial}{\partial q}\right)\left(\frac{\partial H}{\partial Q}\{Q,P\} - \frac{\partial K}{\partial Q}\right) \\
&= \left(-\frac{\partial H}{\partial P}\frac{\partial P}{\partial p}\frac{\partial\{Q,P\}}{\partial q} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial q}\frac{\partial\{Q,P\}}{\partial p} + \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q}\frac{\partial\{Q,P\}}{\partial p} - \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p}\frac{\partial\{Q,P\}}{\partial q}\right) \\
&+ \{Q,P\}\left(-\frac{\partial P}{\partial p}\frac{\partial}{\partial q}\frac{\partial H}{\partial P} + \frac{\partial P}{\partial q}\frac{\partial}{\partial p}\frac{\partial H}{\partial P} + \frac{\partial Q}{\partial q}\frac{\partial}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial}{\partial q}\frac{\partial H}{\partial Q}\right) \\
&+ \frac{\partial P}{\partial p}\frac{\partial}{\partial q}\frac{\partial K}{\partial P} - \frac{\partial P}{\partial q}\frac{\partial}{\partial p}\frac{\partial K}{\partial P} - \frac{\partial Q}{\partial q}\frac{\partial}{\partial p}\frac{\partial K}{\partial Q} + \frac{\partial Q}{\partial p}\frac{\partial}{\partial q}\frac{\partial K}{\partial Q} \\
&= \frac{\partial\{Q,P\}}{\partial q}\left(-\frac{\partial H}{\partial P}\frac{\partial P}{\partial p} - \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p}\right) + \frac{\partial\{Q,P\}}{\partial p}\left(\frac{\partial H}{\partial P}\frac{\partial P}{\partial q} + \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q}\right) \\
&+ \{Q,P\}\left(\left\{P,\frac{\partial H}{\partial P}\right\} + \left\{Q,\frac{\partial H}{\partial Q}\right\}\right) \\
&- \left\{Q,\frac{\partial K}{\partial Q}\right\} - \left\{P,\frac{\partial K}{\partial P}\right\} \\
&= \frac{\partial\{Q,P\}}{\partial p}\frac{\partial H}{\partial q} - \frac{\partial\{Q,P\}}{\partial q}\frac{\partial H}{\partial p} \\
&+ \{Q,P\}\left(\left\{P,\frac{\partial H}{\partial P}\right\} + \left\{Q,\frac{\partial H}{\partial Q}\right\}\right) - \left(\left\{Q,\frac{\partial K}{\partial Q}\right\} + \left\{P,\frac{\partial K}{\partial P}\right\}\right)
\end{aligned}$$

Riordinando i termini abbiamo

$$\begin{aligned}
\frac{\partial}{\partial t}\{Q,P\}_{p,q} + \frac{\partial\{Q,P\}}{\partial p}\dot{p} + \frac{\partial\{Q,P\}}{\partial q}\dot{q} &= \{Q,P\}\left(\left\{P,\frac{\partial H}{\partial P}\right\} + \left\{Q,\frac{\partial H}{\partial Q}\right\}\right) \\
&- \left(\left\{Q,\frac{\partial K}{\partial Q}\right\} + \left\{P,\frac{\partial K}{\partial P}\right\}\right)
\end{aligned}$$

ossia

$$\frac{d}{dt}\{Q,P\}_{p,q} = \{Q,P\}\left(\left\{P,\frac{\partial H}{\partial P}\right\} + \left\{Q,\frac{\partial H}{\partial Q}\right\}\right) - \left(\left\{Q,\frac{\partial K}{\partial Q}\right\} + \left\{P,\frac{\partial K}{\partial P}\right\}\right) \tag{1.1.5}$$

Usando adesso l'identità (1.1.2) otteniamo

$$\begin{aligned}
\left\{P,\frac{\partial H}{\partial P}\right\}_{p,q} &= \left\{P,\frac{\partial H}{\partial P}\right\}_{P,Q}\{Q,P\}_{p,q} = -\{Q,P\}_{p,q}\frac{\partial^2 H}{\partial Q\partial P}\\
\left\{Q,\frac{\partial H}{\partial Q}\right\}_{p,q} &= \left\{Q,\frac{\partial H}{\partial Q}\right\}_{P,Q}\{Q,P\}_{p,q} = \{Q,P\}_{p,q}\frac{\partial^2 H}{\partial P\partial Q}\\
\left\{Q,\frac{\partial K}{\partial Q}\right\}_{p,q} &= \left\{Q,\frac{\partial K}{\partial Q}\right\}_{P,Q}\{Q,P\}_{p,q} = \{Q,P\}_{p,q}\frac{\partial^2 K}{\partial P\partial Q}\\
\left\{P,\frac{\partial K}{\partial P}\right\}_{p,q} &= \left\{P,\frac{\partial K}{\partial P}\right\}_{P,Q}\{Q,P\}_{p,q} = --\{Q,P\}_{p,q}\frac{\partial^2 K}{\partial Q\partial P}
\end{aligned}$$

e sostituendo nella (1.1.5) troviamo

$$\frac{d}{dt}\{Q,P\}_{p,q} = 0 \tag{1.1.6}$$

che è quanto volevamo dimostrare.

1.1.1.3. Una trasformazione canonica conserva la forma delle equazioni del moto

Introduciamo la matrice

$$\mathbb{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

e la notazione

$$\boldsymbol{x} = \begin{pmatrix} q \\ p \end{pmatrix}$$

Le equazioni del moto di Hamilton si scrivono allora nella forma

$$\dot{\boldsymbol{x}} = \mathbb{J}\frac{\partial H}{\partial \boldsymbol{x}}$$

Introduciamo adesso nuove coordinate $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x}, t)$. Abbiamo

$$\begin{aligned}
\dot{y}_i &= \frac{\partial y_i}{\partial x_j}\dot{x}_j + \frac{\partial y_i}{\partial t}\\
&= \frac{\partial y_i}{\partial x_j}\mathbb{J}_{jk}\frac{\partial H}{\partial x_k} + \frac{\partial y_i}{\partial t}\\
&= \frac{\partial y_i}{\partial x_j}\mathbb{J}_{jk}\frac{\partial y_p}{\partial x_k}\frac{\partial H}{\partial y_p} + \frac{\partial y_i}{\partial t}
\end{aligned}$$

Calcoliamo esplicitamente la matrice

$$\begin{aligned}
\frac{\partial y_i}{\partial x_j}\mathbb{J}_{jk}\frac{\partial y_p}{\partial x_k} &= \left[\begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial P}{\partial p} \end{pmatrix}\right]_{ip} \\
&= \left[\begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix}\begin{pmatrix} \frac{\partial Q}{\partial p} & \frac{\partial P}{\partial p} \\ -\frac{\partial Q}{\partial q} & -\frac{\partial P}{\partial q} \end{pmatrix}\right]_{ip} \\
&= \begin{pmatrix} \{Q,Q\}_{p,q} & \{Q,P\}_{p,q} \\ -\{Q,P\}_{p,q} & \{P,P\}_{p,q} \end{pmatrix}_{ip} = \mathbb{J}_{ip}
\end{aligned}$$

$$\mathbb{H}\mathbb{J}\mathbb{H}^T = \mathbb{J}$$

dove l'ultima uguaglianza segue dal fatto che le trasformazioni sono canoniche. Allora le equazioni del moto sono della forma

$$\dot{\boldsymbol{y}} = \mathbb{J}\frac{\partial H}{\partial \boldsymbol{y}} + \frac{\partial \boldsymbol{y}}{\partial t} = \mathbb{J}\frac{\partial K}{\partial \boldsymbol{y}}$$

dove

$$\frac{\partial K}{\partial \boldsymbol{y}} = \frac{\partial H}{\partial \boldsymbol{y}} - \mathbb{J}\frac{\partial \boldsymbol{y}}{\partial t}$$

Se la trasformazione di coordinate non dipende dal tempo abbiamo che K e H coincidono a meno di una funzione del solo tempo irrilevante. In caso contrario si può verificare che è possibile trovare una funzione K che soddisfa l'equazione precedente. Infatti la condizione di integrabilità

$$\frac{\partial}{\partial y_i}\left[\frac{\partial H}{\partial y_j} - \mathbb{J}_{jk}\frac{\partial y_k}{\partial t}\right] = \frac{\partial}{\partial y_j}\left[\frac{\partial H}{\partial y_i} - \mathbb{J}_{ik}\frac{\partial y_k}{\partial t}\right]$$

diviene

$$\frac{\partial x_p}{\partial y_i}\frac{\partial}{\partial x_p}\left[\mathbb{J}_{jk}\frac{\partial y_k}{\partial t}\right] = \frac{\partial x_p}{\partial y_j}\frac{\partial}{\partial x_p}\left[\mathbb{J}_{ik}\frac{\partial y_k}{\partial t}\right]$$

$$\begin{aligned}
\left(\mathbb{J}\dot{\mathbb{H}}\mathbb{H}^{-1}\right)^T &= \mathbb{J}\dot{\mathbb{H}}\mathbb{H}^{-1} \\
\left(\mathbb{J}\dot{\mathbb{H}}\mathbb{H}^{-1}\right)^T &
\end{aligned}$$

che è automaticamente soddisfatta dato che possiamo riscriverla nella forma

$$\frac{\partial x_p}{\partial y_i}\frac{\partial}{\partial t}\mathbb{J}_{jk}\frac{\partial y_k}{\partial x_p} = \frac{\partial x_p}{\partial y_j}\frac{\partial}{\partial t}\mathbb{J}_{ik}\frac{\partial y_k}{\partial x_p}$$

mostrare che se e solo se l'Equazione (1.1.1) è verificata il differenziale

$$dF = PdQ - Kdt - pdq + Hdt$$

è esatto. Iniziamo esplicitando rispetto alle coordinate p e q

$$
\begin{aligned}
dF &= P\left(\frac{\partial Q}{\partial q}dq + \frac{\partial Q}{\partial p}dp + \frac{\partial Q}{\partial t}dt\right) - K dt - pdq + H dt \\
&= \left(P\frac{\partial Q}{\partial q} - p\right)dq + P\frac{\partial Q}{\partial p}dp + \left(P\frac{\partial Q}{\partial t} - K + H\right)dt
\end{aligned}
$$

e scriviamo le condizioni di integrabilità:

$$
\frac{\partial}{\partial p}\left(P\frac{\partial Q}{\partial q} - p\right) = \frac{\partial}{\partial q}\left(P\frac{\partial Q}{\partial p}\right) \tag{1.1.7}
$$

$$
\frac{\partial}{\partial t}\left(P\frac{\partial Q}{\partial q} - p\right) = \frac{\partial}{\partial q}\left(P\frac{\partial Q}{\partial t} - K + H\right) \tag{1.1.8}
$$

$$
\frac{\partial}{\partial t}\left(P\frac{\partial Q}{\partial p}\right) = \frac{\partial}{\partial p}\left(P\frac{\partial Q}{\partial t} - K + H\right) \tag{1.1.9}
$$

Calcolando le derivate nella (1.1.7) troviamo

$$
\frac{\partial P}{\partial p}\frac{\partial Q}{\partial q} + P\frac{\partial^2 Q}{\partial p\partial q} - 1 = \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} + P\frac{\partial^2 Q}{\partial q\partial p}
$$

che è verificata se sono valide le parentesi di Poisson canoniche. Dalla (1.1.8) otteniamo

$$
\frac{\partial P}{\partial t}\frac{\partial Q}{\partial q} + P\frac{\partial^2 Q}{\partial t\partial q} = \frac{\partial P}{\partial q}\frac{\partial Q}{\partial t} + P\frac{\partial^2 Q}{\partial q\partial t} + \frac{\partial}{\partial q}(H - K)
$$

che, usando la (1.1.3) e la (1.1.4), diviene l'identità

$$
\frac{\partial Q}{\partial q}\frac{\partial (H-K)}{\partial Q} + \frac{\partial P}{\partial q}\frac{\partial (H-K)}{\partial P} = \frac{\partial (H-K)}{\partial q}
$$

Infine dalla (1.1.9) otteniamo

$$
\frac{\partial P}{\partial t}\frac{\partial Q}{\partial p} + P\frac{\partial^2 Q}{\partial t\partial p} = \frac{\partial P}{\partial p}\frac{\partial Q}{\partial t} + P\frac{\partial^2 Q}{\partial p\partial t} + \frac{\partial}{\partial p}(H - K)
$$

Usando nuovamente la (1.1.3) e la (1.1.4) otteniamo

$$
\frac{\partial (H-K)}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial P}{\partial p}\frac{\partial (H-K)}{\partial P} = \frac{\partial}{\partial p}(H - K)
$$

e quindi ancora una identità.

1.2. Canonical transformations

Let us suppose that a mechanical system is described by a set of canonical coordinates p_i, q^i and by an Hamiltonian $H\left(p_i, q^i, t\right)$ in such a way that the motion equations are given by

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}$$
$$\dot{q}^i = \frac{\partial H}{\partial p_i}$$

We want to introduce a new set of coordinates

$$\begin{aligned} P_i &= P_i\left(p, q, t\right) \\ Q^i &= Q^i\left(p, q, t\right) \end{aligned} \tag{1.2.1}$$

in such a way that

$$\left\{Q^i, Q^j\right\}_{p,q} = 0 \tag{1.2.2}$$
$$\left\{P_i, P_j\right\}_{p,q} = 0 \tag{1.2.3}$$
$$\left\{Q^i, P_j\right\}_{p,q} = \delta^i_j \tag{1.2.4}$$

where the Poisson brackets of two functions are defined by

$$\{A, B\}_{p,q} = \frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i}$$

and a sum over repeated indices is understood. We will call the (1.2) a canonical transformation.

1.2.1. A canonical transformation leaves the equation of motion in Hamiltonian form

Now we ask the following question: is it possible to introduce a new Hamiltonian function K in such a way that

$$\dot{P}_i = -\frac{\partial K}{\partial Q^i} \tag{1.2.5}$$
$$\dot{Q}^i = \frac{\partial K}{\partial P_i} \tag{1.2.6}$$

If this is true, we can expand the total time derivative in (1.2.6) obtaining

$$\begin{aligned}
\frac{\partial K}{\partial P_i} &= \frac{\partial Q^i}{\partial q^j}\dot{q}^j + \frac{\partial Q^i}{\partial p_j}\dot{p}_j + \frac{\partial Q^i}{\partial t} \\
&= \frac{\partial Q^i}{\partial q^j}\frac{\partial H}{\partial p_j} - \frac{\partial Q^i}{\partial p_j}\frac{\partial H}{\partial q^j} + \frac{\partial Q^i}{\partial t} \\
&= \frac{\partial Q^i}{\partial q^j}\left(\frac{\partial H}{\partial Q^k}\frac{\partial Q^k}{\partial p_j} + \frac{\partial H}{\partial P_k}\frac{\partial P_k}{\partial p_j}\right) - \frac{\partial Q^i}{\partial p_j}\left(\frac{\partial H}{\partial Q^k}\frac{\partial Q^k}{\partial q^j} + \frac{\partial H}{\partial P_k}\frac{\partial P_k}{\partial q^j}\right) + \frac{\partial Q^i}{\partial t} \\
&= \frac{\partial H}{\partial Q^k}\left(\frac{\partial Q^i}{\partial q^j}\frac{\partial Q^k}{\partial p_j} - \frac{\partial Q^i}{\partial p_j}\frac{\partial Q^k}{\partial q^j}\right) + \frac{\partial H}{\partial P_k}\left(\frac{\partial Q^i}{\partial q^j}\frac{\partial P_k}{\partial p_j} - \frac{\partial Q^i}{\partial p_j}\frac{\partial P_k}{\partial q^j}\right) + \frac{\partial Q^i}{\partial t} \\
&= \frac{\partial H}{\partial Q^k}\left\{Q^i, Q^k\right\}_{p,q} + \frac{\partial H}{\partial P_k}\left\{Q^i, P_k\right\}_{p,q} + \frac{\partial Q^i}{\partial t} \\
&= \frac{\partial H}{\partial P_i} + \frac{\partial Q^i}{\partial t}
\end{aligned}$$

and in the same way, expanding (1.2.5),

$$\begin{aligned}
-\frac{\partial K}{\partial Q^i} &= \frac{\partial P_i}{\partial p_j}\dot{p}_j + \frac{\partial P_i}{\partial q^j}\dot{q}_j + \frac{\partial P_i}{\partial t} \\
&= -\frac{\partial P_i}{\partial p_j}\frac{\partial H}{\partial q^j} + \frac{\partial P_i}{\partial q^j}\frac{\partial H}{\partial p_j} + \frac{\partial P_i}{\partial t} \\
&= -\frac{\partial P_i}{\partial p_j}\left(\frac{\partial H}{\partial Q^k}\frac{\partial Q^k}{\partial q^j} + \frac{\partial H}{\partial P_k}\frac{\partial P_k}{\partial q^j}\right) + \frac{\partial P_i}{\partial q^j}\left(\frac{\partial H}{\partial Q^k}\frac{\partial Q^k}{\partial p_j} + \frac{\partial H}{\partial P_k}\frac{\partial P_k}{\partial p_j}\right) + \frac{\partial P_i}{\partial t} \\
&= \frac{\partial H}{\partial Q^k}\left(\frac{\partial P_i}{\partial q^j}\frac{\partial Q^k}{\partial p_j} - \frac{\partial Q^k}{\partial q^j}\frac{\partial P_i}{\partial p_j}\right) + \frac{\partial H}{\partial P_k}\left(\frac{\partial P_i}{\partial q^j}\frac{\partial P_k}{\partial p_j} - \frac{\partial P_i}{\partial p_j}\frac{\partial P_k}{\partial q^j}\right) + \frac{\partial P_i}{\partial t} \\
&= \frac{\partial H}{\partial Q^k}\left\{P_i, Q^k\right\}_{p,q} + \frac{\partial H}{\partial P_k}\left\{P_i, P_k\right\}_{p,q} + \frac{\partial P_i}{\partial t} \\
&= -\frac{\partial H}{\partial Q^i} + \frac{\partial P_i}{\partial t}
\end{aligned}$$

It follows that

$$\frac{\partial Q^i}{\partial t} = -\frac{\partial\left(H-K\right)}{\partial P_i} \tag{1.2.7}$$

$$\frac{\partial P_i}{\partial t} = \frac{\partial\left(H-K\right)}{\partial Q^i} \tag{1.2.8}$$

Now we want to show that the differential

$$dF = P_i dQ^i - K dt - p_i dq^i + H dt \tag{1.2.9}$$

is exact if and only if the transformation is canonical. We start by expressing dF in terms of the differentials dp_i, dq^i and dt only:

$$
\begin{aligned}
dF &= P_i\left(\frac{\partial Q^i}{\partial q^j}dq^j + \frac{\partial Q^i}{\partial p_j}dp_j + \frac{\partial Q^i}{\partial t}dt\right) - K dt - p_i dq^i + H dt \\
&= \left(P_i\frac{\partial Q^i}{\partial q^j} - p_j\right)dq^j + P_i\frac{\partial Q^i}{\partial p_j}dp_j + \left(P_i\frac{\partial Q^i}{\partial t} + H - K\right)dt
\end{aligned}
$$

The necessary and sufficient condition is the equality of mixed derivatives, namely it must be

$$\frac{\partial}{\partial p_k}\left(P_i\frac{\partial Q^i}{\partial q^j} - p_j\right) = \frac{\partial}{\partial q^j}\left(P_i\frac{\partial Q^i}{\partial p_k}\right) \tag{1.2.10}$$

$$\frac{\partial}{\partial t}\left(P_i\frac{\partial Q^i}{\partial q^j} - p_j\right) = \frac{\partial}{\partial q^j}\left(P_i\frac{\partial Q^i}{\partial t} + H - K\right) \tag{1.2.11}$$

$$\frac{\partial}{\partial t}\left(P_i\frac{\partial Q^i}{\partial p^j}\right) = \frac{\partial}{\partial p_j}\left(P_i\frac{\partial Q^i}{\partial t} + H - K\right) \tag{1.2.12}$$

Let us check that these relations are verified for a canonical transformation. The identity (1.2.10) can be expanded in the following way:

$$\frac{\partial P_i}{\partial p_k}\frac{\partial Q^i}{\partial q^j} + P_i\frac{\partial^2 Q^i}{\partial p_k \partial q^j} - \delta^k_j = \frac{\partial P_i}{\partial q^j}\frac{\partial Q^i}{\partial p_k} + P_i\frac{\partial^2 Q^i}{\partial q^j \partial p_k}$$

or

$$\frac{\partial P_i}{\partial p_k}\frac{\partial Q^i}{\partial q^j} - \frac{\partial P_i}{\partial q^j}\frac{\partial Q^i}{\partial p_k} - \delta^k_j = P_i\left(\frac{\partial^2 Q^i}{\partial q^j \partial p_k} - \frac{\partial^2 Q^i}{\partial p_k \partial q^j}\right)$$

The right member is obviously zero. The left one is also zero, as a consequence of Equations (1.2.2), (1.2.3) and (1.2.4). In fact we can write

...

The identity (1.2.11) can be rewritten as

$$\frac{\partial P_i}{\partial t}\frac{\partial Q^i}{\partial q^j} = \frac{\partial P_i}{\partial q^j}\frac{\partial Q^i}{\partial t} + \frac{\partial}{\partial q^j}(H - K)$$

and using Eq. (1.2.7) and Eq. (1.2.8) we get

$$\left(\frac{\partial Q^i}{\partial q^j}\frac{\partial}{\partial Q^i} + \frac{\partial P_i}{\partial q^j}\frac{\partial}{\partial P_i}\right)(H - K) = \frac{\partial}{\partial q^j}(H - K)$$

which is obviously verified. In the same way the identity (1.2.12) gives

$$\frac{\partial P_i}{\partial t}\frac{\partial Q^i}{\partial p^j} = \frac{\partial P_i}{\partial p_j}\frac{\partial Q^i}{\partial t} + \frac{\partial}{\partial p_j}(H - K)$$

and using again Eq. (1.2.7) and Eq. (1.2.8)

$$\left(\frac{\partial Q^i}{\partial p^j}\frac{\partial}{\partial Q^i}+\frac{\partial P_i}{\partial p_j}\frac{\partial}{\partial P_i}\right)(H-K)=\frac{\partial}{\partial p_j}(H-K)$$

which is also verified.

We proved that dF is an exact differential. This give us an algorithm to generate the canonical transformation. For each i let us choose two functionally independent coordinates, one in the set $\left\{p_i, q^i\right\}$ and one in the set $\left\{P_i, Q^i\right\}$. To give an example, let us suppose that it is possible to choose all the q^i's and all the Q^i's. In this we can write (1.2.9) as

$$\frac{\partial F}{\partial Q^i}dQ^i+\frac{\partial F}{\partial q^i}dq^i+\frac{\partial F}{\partial t}=P_i dQ^i-K dt-p_i dq^i+H dt$$

getting

$$P_i=\frac{\partial F}{\partial Q^i}$$
$$p_i=-\frac{\partial F}{\partial q^i}$$
$$K=H-\frac{\partial F}{\partial t}$$

Part II.

General Relativity

2. Black holes

2.1. Kerr metric

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3. Friedmann Robertson Walker metric

Studiamo le geodetiche della metrica

$$g_{\mu\nu} = a^2\eta_{\mu\nu}$$

dove a dipende solo dalla coordinate temporale (il tempo conforme).

$$\begin{aligned}
\Gamma^\mu_{\nu\rho} &= \frac{1}{2}g^{\mu\lambda}\left[g_{\lambda\nu,\rho} + g_{\lambda\rho,\nu} - g_{\nu\rho,\lambda}\right] \\
&= \frac{1}{2a^2}\eta^{\mu\lambda}\left[2aa_{,\rho}\eta_{\lambda\nu} + 2aa_{,\nu}\eta_{\lambda\rho} - 2aa_{,\lambda}\eta_{\nu\rho}\right] \\
&= \left(\frac{a_{,\rho}}{a}\delta^\mu_\nu + \frac{a_{,\nu}}{a}\delta^\mu_\rho - \frac{a_{,\lambda}}{a}\eta^{\mu\lambda}\eta_{\nu\rho}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{du^\mu}{d\tau} + \left(\frac{a_{,\rho}}{a}\delta^\mu_\nu + \frac{a_{,\nu}}{a}\delta^\mu_\rho - \eta^{\mu\lambda}\frac{a_{,\lambda}}{a}\eta_{\nu\rho}\right)u^\nu u^\rho &= 0 \\
\frac{du^\mu}{d\tau} + \left(2\frac{a_{,\rho}}{a}u^\rho u^\mu - \eta^{\mu\lambda}\frac{a_{,\lambda}}{a^3}\right) &= 0 \\
\frac{du^\mu}{d\tau} + \frac{a'}{a}\left(2u^\mu u^0 - \eta^{\mu 0}\frac{1}{a^2}\right) &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{du^0}{d\tau} + \frac{a'}{a}\left(2u^0u^0 - \frac{1}{a^2}\right) &= 0 \\
\frac{du^1}{d\tau} + \frac{2a'}{a}u^1u^0 &= 0
\end{aligned}$$

$$\begin{aligned}
u^0 &= \frac{1}{a}\cosh\theta \\
u^1 &= \frac{1}{a}\sinh\theta
\end{aligned}$$

$$\begin{aligned}
\frac{d\theta}{d\tau}\sinh\theta - \frac{a'}{a}\frac{d\eta}{d\tau}\cosh\theta + \frac{a'}{a^2}\left(2\cosh^2\theta - 1\right) &= 0 \\
\frac{d\theta}{d\tau}\cosh\theta - \frac{a'}{a}\frac{d\eta}{d\tau}\sinh\theta + \frac{a'}{a^2}2\cosh\theta\sinh\theta &= 0 \\
\frac{d\eta}{d\tau} &= \frac{1}{a}\cosh\theta \\
\frac{d\theta}{d\tau} &= -\frac{a'}{a^2}\sinh\theta
\end{aligned}$$

$$\frac{d\theta}{d\eta} = -\frac{a'}{a}\tanh\theta$$

$$\frac{d\theta}{\tanh\theta} = -\frac{a'}{a}d\eta = -d\log a$$

$$\log\sinh\theta = -\log a + C$$

$$\begin{aligned}
\sinh\theta &= \frac{k}{a} \\
u^0 &= \frac{1}{a}\sqrt{1+\frac{k^2}{a^2}} \simeq \frac{1}{a} \\
u^1 &= \frac{k}{a^2} \simeq 0
\end{aligned}$$

$$\frac{d\eta}{d\tau} \simeq \frac{1}{a} \to \tau \simeq t$$

3.1. Light cones in FRW

Exercise 1. Find the universe lines of massless particles for a flat Friedmann Robertson Walker universe using the cosmic time and the space coordinates $u = ax$.

The interval is of the form

$$ds^2 = dt^2 - a^2(t)dx^2$$

and the light cones are solution of

$$dt = \pm a dx$$

Now

$$du = adx + xda = adx + u\frac{da}{a}$$

which gives

$$dt = \pm du \mp u\frac{dt}{a}\frac{da}{dt}$$

and

$$\frac{du}{dt} - Hu = \pm 1$$

where $H = \dot{a}/a$. The complete solution can be found writing

$$\left(\frac{du}{dt} - Hu\right) e^{-\int_{t_0}^{t} H(t')dt'} = \pm e^{-\int_{t_0}^{t} H(t')dt'}$$

or

$$\frac{d}{dt}\left(ue^{-\int_{t_0}^{t} H(t')dt'}\right) = \pm e^{-\int_{t_0}^{t} H(t')dt'}$$

$$u(t) = u(t_0)e^{\int_{t_0}^{t} H(t')dt'} \pm e^{\int_{t_0}^{t} H(t')dt'} \int_{t_0}^{t} dt' e^{-\int_{t_0}^{t'} H(t'')dt''}$$

As a first example we consider the case of a constant Hubble parameter H. We obtain

$$\begin{aligned} u(t) &= u(t_0)e^{H(t-t_0)} \pm e^{H(t-t_0)} \int_{t_0}^{t} dt' e^{-Ht'} \\ &= u(0)e^{H(t-t_0)} \mp H^{-1}\left(1 - e^{H(t-t_0)}\right) \end{aligned}$$

In the case considered

$$a = e^{H(t-t_0)}$$

and the u coordinate of a comoving object is

$$u = e^{H(t-t_0)}x_0$$

Now let us consider the case

$$a = \left(\frac{t}{t_0}\right)^{\alpha}$$

which means

$$H = \frac{\alpha}{t}$$

and

$$e^{\int_{t_0}^{t} H(t')dt'} = e^{\int_{t_0}^{t} \frac{\alpha}{t'}dt'} = e^{\alpha \log \frac{t}{t_0}} = \left(\frac{t}{t_0}\right)^{\alpha}$$

$$\begin{aligned}
u(t) &= u(t_0)\left(\frac{t}{t_0}\right)^{\alpha} \pm \left(\frac{t}{t_0}\right)^{\alpha} \int_{t_0}^{t} dt' \left(\frac{t'}{t_0}\right)^{-\alpha} \\
&= u(t_0)\left(\frac{t}{t_0}\right)^{\alpha} \pm t^{\alpha}\frac{1}{1-\alpha}\left(t^{1-\alpha} - t_0^{1-\alpha}\right) \\
&= u(t_0)\left(\frac{t}{t_0}\right)^{\alpha} \pm \frac{t_0}{1-\alpha}\left[\left(\frac{t}{t_0}\right) - \left(\frac{t_0}{t}\right)^{\alpha}\right] \\
u(t) &= \pm\frac{t_0}{1-\alpha}\left[\left(\frac{t}{t_0}\right) - \left(\frac{t_0}{t}\right)^{\alpha}\right]
\end{aligned}$$

Part III.

Quantum Optics

4. Wigner function

4.1. Wigner function for a two mode squeezed state

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Part IV.

Field Theory

5. Quantum fields in curved spaces

5.1. The Rindler metric

In this section we will study the Minkowski space from the point of view of an observer in accelerated motion. The plan is the following:

1. find the universe line of an uniformly accelerated observer;
2. determine the time and space coordinates of the other events (if possible) using a syncronization procedure based on the exchange of light signals;
3. write the metric as a function of these coordinates.

5.1.1. Constant acceleration

In special relativity an uniform acceleration is defined by the requirement that the acceleration in the reference frame of the accelerated object is constant. If we consider the quadrivector

$$\frac{du^\nu}{d\tau}$$

we have in the rest frame

$$\frac{du^\nu}{d\tau} \stackrel{*}{=} \begin{pmatrix} 0 \\ \vec{a} \end{pmatrix}$$

where we choose the x axis in the acceleration direction. With a Lorentz transformation we can rewrite this in the inertial frame

$$\frac{du^\nu}{d\tau} = \frac{d}{d\tau}\begin{pmatrix} \gamma \\ \gamma\vec{v} \end{pmatrix} = \begin{pmatrix} \gamma\vec{v}\cdot\vec{a} \\ \vec{a} + (\gamma-1)\,\frac{\vec{v}\cdot\vec{a}}{v^2}\vec{v} \end{pmatrix}$$

If we suppose that initially the object has zero velocity, we see that $\vec{v}$ will be always in the same direction of $\vec{a}$. This means that we can restrict our equations to the acceleration direction only, obtaining

$$\frac{d}{d\tau}\begin{pmatrix} \gamma \\ \gamma v \end{pmatrix} = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}\begin{pmatrix} \gamma \\ \gamma v \end{pmatrix}$$

which can be solved as

$$\begin{pmatrix} \gamma \\ \gamma v \end{pmatrix} = \exp\left[\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}\tau\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cosh a\tau \\ \sinh a\tau \end{pmatrix}$$

With another integration we get the universe line of the object

$$\begin{aligned} \gamma &= \frac{dt}{d\tau} = \cosh a\tau \\ \gamma v &= \frac{dx}{d\tau} = \sinh a\tau \end{aligned}$$

which gives

$$\begin{aligned} x_o^t &= \frac{1}{a}\sinh a\tau \\ x_o^x &= \frac{1}{a}\left(\cosh a\tau - 1\right) + x_0 \end{aligned}$$

For future convenience we choose x_0 in such a way that asymptotically $x = t$, which means that $x_0 = a^{-1}$ and

$$x_o^t = \frac{1}{a}\sinh a\tau$$
$$x_o^x = \frac{1}{a}\cosh a\tau$$

5.1.2. Observer coordinates

The accelerated observer will consider an event to happen at his time τ, if a light signal emitted at $\tau - \Delta$ and reflected at the event will be received at the symmetric time $\tau + \Delta$. This means that the τ coordinate of an event will be given by

$$\tau\left(x^\mu\right) = \frac{\tau_+ + \tau_-}{2}$$

where τ_+ and τ_- are the two solutions of

$$\left(x_o^t\left(\tau\right) - x^t\right)^2 - \left(x_o^x\left(\tau\right) - x^x\right)^2 = \left(x^y\right)^2 + \left(x^z\right)^2$$

The same observer will assign a distance δ to the event given by

$$\delta\left(x^\mu\right) = \frac{\tau_+ - \tau_-}{2}$$

Explicitly, it must be

$$\left(\frac{1}{a}\sinh a\tau_+ - x^t\right)^2 - \left(\frac{1}{a}\cosh a\tau_+ - x^x\right)^2 = \left(x^y\right)^2 + \left(x^z\right)^2$$
$$\left(\frac{1}{a}\sinh a\tau_- - x^t\right)^2 - \left(\frac{1}{a}\cosh a\tau_- - x^x\right)^2 = \left(x^y\right)^2 + \left(x^z\right)^2$$

Expanding we get

$$\begin{pmatrix}\cosh a\tau_+ & -\sinh a\tau_+\\ \cosh a\tau_- & -\sinh a\tau_-\end{pmatrix}\begin{pmatrix}x^x\\ x^t\end{pmatrix} = \frac{a}{2}\begin{pmatrix}1\\ 1\end{pmatrix}\left[\frac{1}{a^2} - \left(x^t\right)^2 + \left(x^x\right)^2 + \left(x^y\right)^2 + \left(x^z\right)^2\right]$$

or

$$\begin{pmatrix}\cosh a\left(\tau+\delta\right) & -\sinh a\left(\tau+\delta\right)\\ \cosh a\left(\tau-\delta\right) & -\sinh a\left(\tau-\delta\right)\end{pmatrix}\begin{pmatrix}x^x\\ x^t\end{pmatrix} = \frac{a}{2}\begin{pmatrix}1\\ 1\end{pmatrix}\left[\frac{1}{a^2} - \left(x^t\right)^2 + \left(x^x\right)^2 + \left(x^y\right)^2 + \left(x^z\right)^2\right]$$

It must be

$$x^x\left[\cosh a\left(\tau+\delta\right) - \cosh a\left(\tau-\delta\right)\right] = x^t\left[\sinh a\left(\tau+\delta\right) - \sinh a\left(\tau-\delta\right)\right]$$

or

$$x^x \sinh a\tau = x^t \cosh a\tau$$

which means that

$$\tanh a\tau = \frac{x^t}{x^x}$$

In other words, we will assign the same time coordinate to all the events in the hyperplane

$$x^t = x^x \tanh a\tau$$

Substituting we obtain the distance from the event

$$\cosh a\delta \quad = \tfrac{1}{2a} \quad \frac{1}{2\sqrt{(x^x)^2 - (x^t)^2}} \left\{ \frac{1}{a^2} - \left[\left(x^t\right)^2 - (x^x)^2 - (x^y)^2 - (x^z)^2 \right] \right\}$$

A set of observers:

$$x^t_{\vec{a}} = \frac{1}{a} \sinh a\tau$$
$$x^i_{\vec{a}} = \frac{a^i}{a^2} \cosh a\tau$$

5.2. The Unruh effect

Part V.

Statistical Mechanics

6. Fluctuation dissipation theorem

Part VI.

Data analysis

7. Bayesian methods

7.1. Dummy