

# Purifications and fidelity for Gaussian states

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There has recently been a great deal of interest in communication theory for giving quantitative measures of the *fidelity* with which signals are transmitted within a quantum channel. If at least one of the input and output states were pure, fidelity is conveniently described by the transition probability

$$F(\rho_1, \rho_2) = \text{Tr}(\rho_1 \rho_2). \quad (1)$$

However, Eq. (1) is unsatisfactory as a measure of fidelity, when both states are mixed. An obvious candidate for defining fidelity between two mixed quantum states is the distance  $\mathcal{D}_B$  between the corresponding density operators discovered by Bures [1] and interpreted by Uhlmann [2] as a measure of the transition probability  $\mathcal{P}$  via the relation

$$\mathcal{D}_B^2(\rho_1, \rho_2) = 2[1 - \sqrt{\mathcal{P}(\rho_1, \rho_2)}]. \quad (2)$$

The explicit formula for the Bures-Uhlmann fidelity is

$$\mathcal{P}(\rho_1, \rho_2) = \left\{ \text{Tr}[(\sqrt{\rho_1} \rho_2 \sqrt{\rho_1})^{1/2}] \right\}^2. \quad (3)$$

If  $\rho_1$  describes a pure state, this fidelity simplifies to the transition probability (1).

Explicit expressions for the transition probability between mixed quantum states were first written down using Eq. (3) for finite-dimensional cases. Evaluation of the fidelity in an infinite-dimensional Hilbert space of quantum states is now of great importance due to the experimental interest in quantum information processing of radiation field states [3]. All the recent results concerning the fidelity between single- and multimode Gaussian states have been obtained by exploiting the Bures-Uhlmann formula, Eq. (3).

Recall that the density operator of a single-mode Gaussian state (DSTS) [4] is completely described by squeezing and displacement unitary actions on the thermal density operator  $\rho_T$ .  $\rho_T$  has the nondegenerate and positive eigenvalues

$$\lambda_j = \frac{1}{\bar{n} + 1} \left( \frac{\bar{n}}{\bar{n} + 1} \right)^j. \quad (4)$$

The corresponding eigenvectors form the Fock basis of the Hilbert space. Therefore, the orthonormal set of eigenvectors of a Gaussian density operator is  $\{|\Psi_n\rangle\} := \{D(\alpha)S(\zeta)|n\rangle\}$ .

Here we have introduced the Weyl operator  $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$  and the squeeze operator  $S(\zeta) = \exp(\zeta(a^\dagger)^2 - \zeta^* a^2)$ .

The present work gives an alternative derivation of the fidelity between two single-mode Gaussian states, without using Eq. (3). The steps of the derivation are as follows.

- We use the Schmidt polar form to build purifications lying in an extended (two-mode) Hilbert space  $\mathcal{H}_e = \mathcal{H} \otimes \mathcal{H}$ , where  $\mathcal{H}$  is the (single-mode) Hilbert space to which  $\rho_{DST}^{(1)}$  and  $\rho_{DST}^{(2)}$  belong. Therefore, a state in  $\mathcal{H}_e$  is a two-mode pure state whose reduced subsystems (modes) are described by single-mode DSTS's. Let  $|\Phi_1\rangle$  be a specified purification of  $\rho_{DST}^{(1)}$ ,

$$|\Phi_1\rangle = \sum \sqrt{\lambda_n^{(1)}} D(\alpha_1) S(\zeta_1) |n\rangle \otimes |n\rangle. \quad (5)$$

$|\Phi_1\rangle$  has the reduced mode I described by  $\rho_{DST}^{(1)}$  and the reduced mode II in a thermal state at temperature  $T_1$ . The most general form of a purification for  $\rho_{DST}^{(2)}$  is

$$|\Phi_2\rangle = \sum \sqrt{\lambda_n^{(2)}} D(\alpha_2) S(\zeta_2) |n\rangle \otimes D(\beta) S(\xi) |n\rangle, \quad (6)$$

where the coherent amplitude  $\beta$  and the complex squeeze parameter  $\xi$  are arbitrary.

- We evaluate the transition probability between the purifications (5) and (6). Explicitly, we have to perform a double summation involving nondiagonal matrix elements of products  $D(\alpha)S(\zeta)$  in the Fock basis.
- Recall that the transition probability between the mixed states described by  $\rho_{DST}^{(1)}$  and  $\rho_{DST}^{(2)}$  is given by the maximum of the transition probability between the corresponding purifications (5) and (6).

Considerable analytic work is needed to perform the above mentioned double summation and then to maximize the transition probability with respect to the parameters  $\beta$  and  $\xi$ . We have found that the displacement parameter  $\beta_m$  giving this maximum depends on both the coherent amplitudes  $\alpha_1, \alpha_2$  and the temperatures  $T_1$  and  $T_2$ . Similarly, the squeeze parameter  $\xi_m$  corresponding to the maximum of  $|\langle \Phi_1 | \Phi_2 \rangle|^2$  is determined by both temperatures and squeeze parameters of the two single-mode states.

In this way we have determined not only the fidelity but also *the most parallel entangled states which satisfy the requirement of generating the mixed states  $\rho_{DST}^{(1)}$  and  $\rho_{DST}^{(2)}$  as reduced states*. To the best of our knowledge this is a new result which seems to deserve further analysis.

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