Factorized representations of Wigner D-functions

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Recently [1] we have found a factorized form of the "parity-projected" Wigner $\mathbf{d}^{j\lambda_p}(\beta)$-matrices as a product of two triangular matrices, each composed of Gegenbauer polynomials. Parity-projected $\mathbf{d}^{j\lambda_p}$-matrices naturally appear in three-body problems having definite parity. Factorized representations allow one to reduce the number of polynomials to be calculated when composing the $\mathbf{d}^{j\lambda_p}$-matrix. The analysis in [1] is based on the use of so-called invariant (i.e., having explicit tensor form) representations of finite rotation matrices (FRM), $R^j_{km}(\Omega)$, [2]. All results obtained in [1, 2] are valid only for FRM and $\mathbf{d}^{j\lambda_p}(\beta)$ matrices of integer rank j. Below we present the standard (not the parity-projected) $\mathbf{d}^j(\beta)$-matrix with an arbitrary rank j (either integer or half-integer) as a product of two triangular matrices composed of various powers of $\cos(\beta/2)$ and $\sin(\beta/2)$. We use standard definitions of the theory of angular momentum [3]. The Wigner functions $\mathbf{D}^j(\alpha, \beta, \gamma)$ (i.e., the parametrization of the FRM $\mathbf{R}^j(\Omega)$ in terms of Euler's angles: $\Omega \equiv \{\alpha, \beta, \gamma\}$) describe the transformation of an irreducible tensor T_{jm} under the rotation from an "old", K, to a "new", $\tilde{K}$, coordinate frame

$$T'_{jm} = \sum_{k=-j}^{j} T_{jk} D^j_{km}(\alpha\beta\gamma). \tag{1}$$

The nontrivial part of the $\mathbf{D}^j$-function is the $\mathbf{d}^j(\beta)$-matrix:

$$D^j_{km}(\alpha\beta\gamma) = \exp(-ik\alpha)\, d^j_{km}(\beta)\, \exp(-im\gamma). \tag{2}$$

We use below the letter χ for a spinor of rank $1/2$, while its components are denoted by χ_μ, $\mu = \pm 1/2$. An arbitrary spinor can be expanded in the spinor basis, $\beta^{(\pm 1/2)}$, as follows:

$$\chi = \chi_{1/2}\beta^{(-1/2)} - \chi_{-1/2}\beta^{(1/2)}, \quad \chi_{\pm 1/2} = (\chi \cdot \beta^{(\pm 1/2)}), \tag{3}$$

where the scalar product of two spinors, χ and ϕ, is $(\chi \cdot \phi) = \chi_{1/2}\phi_{-1/2} - \chi_{-1/2}\phi_{1/2}$. The properties of the basis spinors $\beta^{(\pm 1/2)}$ are

$$(\beta^{(\mu)} \cdot \beta^{(\nu)}) = (-1)^{1/2+\mu}\delta_{\mu,-\nu}, \quad \beta^{(\nu)}_p = (-1)^{1/2-p}\delta_{p,-\nu}, \quad p = \pm 1/2. \tag{4}$$

An analysis of the FRM, similar to that in [2] for the case of integer j, allows one to find the following invariant spinor representation of the FRM for an arbitrary rank j:

$$R^j_{km}(\Omega) = (-1)^{j-k}\sqrt{\frac{(2j)!}{(j+k)!\,(j-k)!}}\,\{\{\beta^{(-1/2)}\}_{\frac{j+k}{2}} \otimes \{\beta^{(1/2)}\}_{\frac{j-k}{2}}\}_{jm}, \tag{5}$$

where $\{\beta^{(1/2)}\}_{a\alpha}$ is the tensor product of $2a$ spinors $\beta^{(1/2)}$, and where the components of $\beta^{(1/2)}$ should be calculated in the (new, or rotated) coordinate frame $\tilde{K}$. Taking into account that the identity (3) can be rewritten as

$$\beta^{(-1/2)} = \frac{1}{b}(\chi + a\beta^{(1/2)}), \tag{6}$$

we obtain another representation of the FRM containing free parameters, which are the components of the spinor χ:

$$R^j_{km}(\Omega) = (-1)^{2k}\left(-\frac{a}{b}\right)^{j+k}\sqrt{\frac{(2j)!}{(j-k)!\,(j+k)!}}\sum_{n=0}^{j+k}\binom{j+k}{n}\frac{1}{a^n}\left\{\{\chi\}_{n/2}\otimes\{\beta^{(1/2)}\}_{j-n/2}\right\}_{jm}. \tag{7}$$

In order to obtain a factorized representation for the $\mathbf{d}^j$-matrix we assume the spinor χ in (7) to be the basis spinor $\tilde{\beta}^{(1/2)}$ of the frame $\tilde{K}$. In this case we have:

$$\begin{aligned} a &= (\tilde{\beta}^{(1/2)}\cdot\beta^{(1/2)}) = -D^{1/2}_{1/2\,1/2}(\alpha\beta\gamma) = -e^{-i(\alpha+\gamma)/2}\cos(\beta/2), \\ b &= (\tilde{\beta}^{(1/2)}\cdot\beta^{(-1/2)}) = D^{1/2}_{-1/2\,1/2}(\alpha\beta\gamma) = e^{i(\alpha-\gamma)/2}\sin(\beta/2). \end{aligned} \tag{8}$$

Direct calculation of the tensor product on the r.h.s. of Eq. (7) gives:

$$\left\{\{\tilde{\beta}^{(1/2)}\}_{n/2}\otimes\{\beta^{(1/2)}\}_{j-n/2}\right\}_{jm} = (-1)^n\, C^{jm}_{n/2\,-n/2\;j-n/2\,m+n/2}\left(\beta^{(1/2)}_{1/2}\right)^{j+m}\left(\beta^{(1/2)}_{-1/2}\right)^{j-m-n}. \tag{9}$$

For a simpler representation, it is convenient to introduce the matrix $\overline{\mathbf{d}}^j$ instead of $\mathbf{d}^j$ as follows:

$$d^j_{p-j-1\,q-j-1}(\beta) = (\sin\beta/2)^{q-p}\sqrt{\frac{(p-1)!\,(2j+1-q)!}{(q-1)!\,(2j+1+p)!}}\,\overline{d}^j_{km}(\beta), \quad p,q = 1,2\ldots 2j+1. \tag{10}$$

Taking into account Eqs. (8) - (10) and omitting the trivial dependence of the Wigner functions $D^j_{km}(\alpha\beta\gamma)$ on the angles α and γ, we obtain from Eq. (7) the final matrix identity:

$$\overline{\mathbf{d}}^j(\beta) = \mathbf{C}(\beta)\cdot\mathbf{A}(\beta) \quad \text{or} \quad \overline{d}^j_{pq}(\beta) = \sum_{n=1}^{2j+1} C_{pn}(\beta)A_{nq}(\beta), \tag{11}$$

where the matrix elements $A_{nq}(\beta)$ and $C_{pn}(\beta)$ are simple powers of $\cos(\beta/2)$:

$$\begin{aligned} \mathbf{A}(\beta) \equiv A_{nq}(\beta) &= \frac{(2j+1-n)!}{(2j+2-n-q)!}(\cos\beta/2)^{2j+2-n-q}, \\ \mathbf{C}(\beta) \equiv C_{pn}(\beta) &= \frac{(-1)^{n+1}}{(n-1)!\,(p-n)!}(\cos\beta/2)^{p-n}. \end{aligned} \tag{12}$$

These equations show that $\mathbf{A}$ is an upper-left triangular matrix, while $\mathbf{C}$ is lower-left triangular; diagonal elements of both matrices equal unity. Note that the matrix elements (12) do not depend on the rank j: it enters Eq. (12) only in the combination $2j+1-n$, which determines the matrix dimensions. Consequently, the $\mathbf{A}^{j+1}$ (or $\mathbf{C}^{j+1}$) matrix can be calculated by simply adding the next highest (or lowest) row to $\mathbf{A}^j$ (or to $\mathbf{C}^j$).

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