Kater's Pendulum

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1 Historical Background

An accurate value of g, the acceleration due to gravity, is needed to calculate an object's motion in the earth's gravitational field or to measure the mass of the earth using the universal gravitational constant, G, such as in Cavendish's original experiment described in the Gravitational Torsion Balance experiment. Local variations in g are important for the study of geological formations, and thus for locating mineral deposits. Theoretically, one could determine g from the measurement of the period of a simple pendulum. In practice, however, it is physically impossible to make a point-mass pendulum with a weightless support. In the early nineteenth century, Henry Kater[1] devised another method, by constructing a compound pendulum, which he oscillated about a knife-edge, then turned upside down and oscillated about a knife-edge on the other side of the center of mass (cms). (See Fig 1.) If the two periods are made equal by adjusting the weights on the pendulum, g can be determined from only the period and the distance between the two knife edges. In essence, use of the parallel axis theorem for the moment of inertia of a rigid body allows us to avoid approximating a point mass on a massless string.

Kater used this device to measure the acceleration of gravity at various locations in England. He also determined the length of what was called the second's pendulum, i.e., a pendulum whose half-period is one second. One can show that this length is nearly one meter, and this length of the one second pendulum was proposed as the standard for the meter in the 1790s, but was not adopted.

2 Kater's Pendulum

The torque, τ, that gravity exerts on a pendulum is given by the expression

$$\tau = -Mgd\sin\theta, \tag{1}$$

where M is the mass of the object, θ is the deviation from the vertical, and d is the distance from the center of mass to the point about which it is oscillated. The negative sign comes from the fact that in this case gravity is a restoring force. The torque is also equal to the angular acceleration, α, times the moment of inertia about the point of oscillation, I_0; $\tau = I_0\alpha$. Since the angular acceleration is equal to the second derivative of the angular position,

$$d^2\theta/dt^2 + (Mgd/I_0)\sin\theta = 0. \tag{2}$$

This differential equation can be solved in the small angle approximation ($\sin\theta = \theta$)

$$d^2\theta/dt^2 + (Mgd/I_0)\,\theta = 0 \tag{3}$$

[1]Henry Kater, 1777-1825, English army captain; participated in the great triangulation of India and has a peninsula on the east coast of Baffin Island named for him.

and has the solution

$$\theta = \theta_{max} \sin \sqrt{\frac{Mgd}{I_0}} t. \tag{4}$$

The period for an oscillator satisfying this equation is

$$T = 2\pi \sqrt{I_0/Mgd} \tag{5}$$

Comparing this period to that of a simple pendulum, T $= 2\pi\sqrt{l/g}$, one finds that a physical pendulum

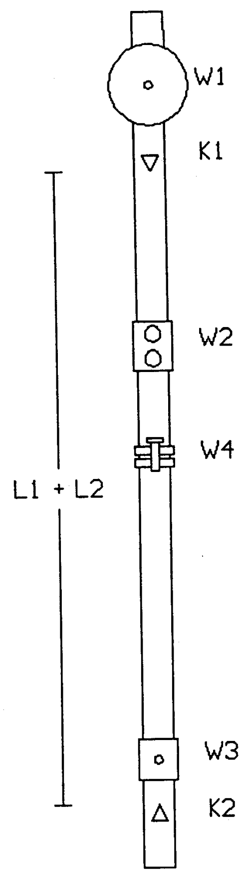

Figure 1: Diagram of the Kater pendulum. The symbols L_1 and L_2 on the figure are ℓ_1 and ℓ_2 in the text. L $= \ell_1 + \ell_2$ is the distance between fulcrum points. The distances of most of the weights from each fulcrum are adjustable. Note that W2 is between W4 and W3, rather than as shown.

has the same period as a simple pendulum of length l $= I_0/Md$. This is called the reduced length of the physical pendulum.

From the parallel axis theorem, we know that the moment of inertia of an object about an axis parallel to an axis through its center of mass is related to the moment of inertia about the center of mass, I_c, by

$$I_0 = I_c + MR^2, \tag{6}$$

where R is the distance from the axis of I_0 to the center of mass. Designating I_1 as the moment of inertia about P1 and I_2 as the moment of inertia about P2, one can substitute Eq 6 into Eq 5 to obtain

$$T_1 = 2\pi\sqrt{\{(I_c + M\ell_1^2)/Mg\ell_1\}} \tag{7}$$

and

$$T_2 = 2\pi\sqrt{\{(I_c + M\ell_2^2)/Mg\ell_2\}} \tag{8}$$

If the weights on the pendulum are adjusted so that $T_1 = T_2$, then

$$(I_c + M\ell_1^2)\ell_2 = (I_c + M\ell_2^2)\ell_1. \tag{9}$$

and

$$I_c(\ell_2 - \ell_1) = M\ell_1\ell_2(\ell_2 - \ell_1), \tag{10}$$

or

$$I_c = M\ell_1\ell_2, \tag{11}$$

provided that ℓ_1 differs significantly from ℓ_2, that is, that the pendulum is quite asymmetric. If Eq 11 is substituted into Eq 7 or Eq 8, the result is

$$T = 2\pi\sqrt{[(\ell_1 + \ell_2)/g]} = 2\pi\sqrt{L/g} \tag{12}$$

Note that ℓ_1 and ℓ_2 individually have disappeared and only the sum occurs in the equation. We therefore conjecture that if the masses are adjusted so that the two periods are very nearly the same, then g will be determined primarily by the sum of the periods and the distance between the two knife-edges with weak dependence on the differences in the lengths and periods. If this conjecture can be proved correct a much easier way to measure g will be apparent because $L = l_1 + l_2$, the distance between the two knife-edges, can be measured to a fraction of a millimeter, while the measurement of the individual ℓs depends on locating the center of gravity. It is difficult to locate the center of gravity more accurately than a few millimeters. Thus we are motivated to manipulate Eqs. 7 and 8 to emphasize terms in the sum and difference of T_1 and T_2 in order to address the practical case in which the periods are nearly, but not exactly, equal,

$$g = \frac{8\pi^2}{\frac{T_1^2+T_2^2}{l_1+l_2} + \frac{T_1^2-T_2^2}{l_1-l_2}} \approx \frac{4\pi^2 L}{\left(\frac{(T_1+T_2)}{2}\right)^2}\left[1 + 2\left(\frac{T_1 - T_2}{T_1 + T_2}\ \frac{L}{L - 2l_1}\right)\right], \tag{13}$$

where the first expression is exact and the second is obtained by changing to variables $\Delta T = T_1 - T_2$ and $T = (T_1 + T_2)/2$ and keeping up to linear terms in ΔT. This second form is probably due to Bessel and is convenient for showing that the expression is insensitive to the location of the center of mass for approximately equal periods. The second expression is an adequate approximation if $T_1 - T_2 \ll T_1 + T_2$ and the pendulum is sufficiently asymmetric that ℓ_1 differs appreciably from L/2. Note that the term which depends on ℓ_1 is needed only as a small correction and thus need not to be measured with great precision.

For error analysis it is essential to use the measured variables: T_1, T_2, L, and ℓ_1. Note that ℓ_2 is not measured but is L - ℓ_1. Returning to the first form of Eq. 13 and eliminating ℓ_2 the derivatives needed for error analysis are:

$$\frac{\partial g}{\partial T_1} = \frac{g^2}{8\pi^2}\ \frac{4T_1\ell_1}{L(L - 2\ell_1)} \sim 2.1 \tag{14}$$

$$\frac{\partial g}{\partial T_2} = -\frac{g^2}{8\pi^2}\ \frac{4(L - \ell_1)T_2}{L(L - 2\ell_1)} \sim -12.$$

$$\frac{\partial \mathrm{g}}{\partial \mathrm{L}} = \frac{\mathrm{g}^2}{8\pi^2}\left(\frac{\mathrm{T}_1^2+\mathrm{T}_2^2}{\mathrm{L}^2} - \frac{\mathrm{T}_1^2-\mathrm{T}_2^2}{(\mathrm{L}-2\ell_1)^2}\right) \sim 9.8$$

$$\frac{\partial \mathrm{g}}{\partial \ell_1} = \frac{\mathrm{g}^2}{8\pi^2}\,\frac{2(\mathrm{T}_1^2-\mathrm{T}_2^2)}{(\mathrm{L}-2\ell_1)^2} \sim 20.\ (\mathrm{T}_1-\mathrm{T}_2)$$

Approximate values $\mathrm{T}_1 = \mathrm{T}_2 = 2,\ \mathrm{L} = 1,$ and $\ell_1 = 0.15$ have been used except in the difference between the two periods.

The theory of Kater's Pendulum was worked out in 1826 by Bessel.[2] The study of the pendulum without the small angle approximation requires elliptic integrals. It is covered in most intermediate mechanics texts and shows that

$$T_\alpha = T_0\left(1 + (\tfrac{1}{2})^2\sin^2(\alpha/2) + (\tfrac{1\cdot 3}{2\cdot 4})^2\sin^4(\alpha/2) + \cdots\right) \tag{15}$$

for oscillation with amplitude α radians and a period T(0). Kater's pendulum is about 1 meter long so an oscillation with amplitude 1 cm is 10^{-2} radians and a this equation shows that is a 6 in a million correction. ((1/2)sin($10^{-2}/2$ radians) = $10^{-2}/4$ and the square is $6/10^6$)

3 Apparatus

Reversible Kater pendulum and wall mount
Pasco ME 9206A photogate timer
balance block

4 Procedure

1. Hang the pendulum from the knife-edge K_1. Be careful to position the knife-edges on the pendulum on the steel insert in the supporting brass block. Be gentle when lowering the pendulum onto its support, or you may damage the knife-edges. The L discussed above is really the distance between the effective rotation points, but the measurement of L is the distance between the knife edges. If a knife edge is dull, the rotation point is below the rounded edge. Then a) the measurement of L is wrong and b) the motion of the pendulum is a complicated combination of rotation and translation back and forth.

2. Turn on the counter/timer. Set the switch to "pend". In this mode the timer will stop or start every other interruption, so that it measures a period, not a half period. Press the reset button. The display should show all zeroes. Press the start button.

3. Clamp the photogate in the mount at the bottom of the Kater Pendulum support with the photogate positioned as an inverted U, so that the end of the pendulum will interrupt the infrared beam. Be careful that the photogate is not hit by the pendulum. Since the height of the bottom of the pendulum bar depends upon which knife edge is used, you will have to adjust the height of the photogate every time you flip the bar, and you need to be sure that the bar will not hit the gate before you flip the pendulum.

4. Start the pendulum swinging. Measuring the period accurately requires that the edge of the bar move quickly across the infrared beam and high speed means large amplitude oscillations. But the small angle approximation breaks down for large oscillations and the period must be calculated

[2] Friedrich Wilhelm Bessel, 1784-1846, Prussian astronomer. Worked out, *inter alia*, the theory of instrumental errors.

from Eq. 15. Use the leading term of Eq. 15 to see how large an oscillation is permitted while keeping this correction less than 1/10,000. That is, find the amplitude which makes the leading term 1/10,000. Start the pendulum swinging with an acceptable amplitude and time the period of the oscillation, T_1. The counter/timer should begin counting and then pause while displaying a period of about 2 s. After this pause, the counting should resume, to pause again at about 4 s. Press the reset button to clear the display and start timing again. Record the period T_1.

5. First lower the photogate all the way down and then turn the pendulum upside down and time the period of oscillation about the knife-edge K_2.

6. The two period measurements probably differ by an unacceptable amount and the next step is to adjust the position of W2 to make the two periods equal. This should be done in a systematic way using linear interpolation or extrapolation, or Newton's method for solving equations. The steps and estimates should be recorded in your data sheet. Measure the distance from W2 to some reference on the bar, loosen the screw on W2 and move it a couple of cm. Measure both periods. How much has the difference in the periods changed?. Assume the change in period difference is proportional to the amount W2 was moved and estimate how much W2 should be moved to make the difference zero. Record in data sheet. Move W2 and iterate until the difference is acceptably small. Use either the second form of Eq. 13 or the error comparisons of Eq. 14 to decide what acceptably small means and explain on your data sheet.

7. Without moving the weights, time nine swings about each knife-edge. Record these values of T_1 and T_2 and the amplitude of the oscillation.

8. Repeat the last step a few times about each knife-edge to gather more data for statistical use. Note that the readout goes up to only 20 s, so accumulating for more than nine swings overflows the readout.

9. Locate the center of gravity by balancing the pendulum horizontally across the balance block, and measure either l_1. Estimate the error in this measurement.

10. Check the scale factor written on your timer. If it is different from unity, correct the two times you have measured with it.

11. Calculate the acceleration of gravity using your choice of the two alternatives shown in Eq. 13. Use the value $L = \ell_1 + \ell_2 = 1.000903 \pm 0.000012$ m. Using your estimates of the other uncertainties and the derivatives given in Eq. 14, calculate the error on g. Your write up should show the contribution of each of the four terms individually. Treat the uncertainties in all measurements as uncorrelated.

5 Results

1. What are the dominant sources of uncertainty in your calculations of g and the corrections to it?

2. The stated value of g in Nashville is 9.79822 m/s^2. How does this compare with your average value of g? Is the difference within the uncertainty of your value?

3. How is the accuracy of this experiment dependent on the size of the pendulum? What would you have to do to get results of the same quality for a pendulum 1/10 the length?

4. Calculate one of the four partial derivatives needed for the error analysis.

6 References

1. Hugh D. Young, *Fundamentals of Mechanics and Heat* ,(New York:McGraw-Hill, 1964)
2. Central Scientific Co., *Selective Experiments in Physics* leaflets: "Kater's Pendulum", 1940

7 Addendum

Addendum to Kater's Pendulum Write Up
Med Webster
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This experiment measures g, the acceleration of gravity at this location. Maintaining sharpness of the knife-edges is essential if the pendulum is to behave as a simple rotational pendulum without additional complication due to a small translation if the edges are not sharp. The mount has steel knife-edges resting on a steel insert in the brass support plate. All measurements should be made with the knife-edge on the steel insert. Be careful not to drop the pendulum when you hang it on the knife-edges. The brass support plate pushes to the left and then swings out to permit removing the pendulum from the mount. Climb up on a ladder or stool so that you can see how the support is constructed before you attempt to remove the pendulum for adjustment or measurement. You will need to move the white metal weight along the bar to adjust the periods, but you should not move any of the other weights on the bar.

The obvious point of this experiment is to measure g but the more significant purpose as a teaching laboratory is to demonstrate that precise measurements usually involve a strategy which requires one to measure precisely only the things that can be well measured and avoids the requirement of high precision for difficult measurements. Error analysis plays a double role in physics: it enables us to assign an error after we have done a measurement and it also enables us to choose measurement strategies which will give better errors.

In this experiment the distance from each knife-edge to the center of gravity of the pendulum is needed, but by making the two periods nearly equal and measuring them, the determination of g depends sensitively on only the sum of the distances and very weakly on the difference. The distance between the knife-edges is readily measured to better than 0.001 inch or 0.03 mm but the location of the cms is difficult to determine more accurately than a few millimeters, a factor of 100. The error analysis is rarely done satisfactorily in other lab writeups I have seen. It is moderately complicated and I admit that I got a few surprises when I did it. I propose to emphasize it more and to provide more assistance. In addition, after doing this analysis carefully, I think it is necessary to reduce the expectation from 1 part in 10000 to 5 in 10000.

8 Comments on History and Calibrations:

According to the note on the support board, this Kater's Pendulum is the one purchased by Chancellor Garland in 1875 as part of the original acquisition of equipment for the science departments at Vanderbilt. It is Item 224 described on p. 166 *The Garland Collection of Classical Physics Apparatus at Vanderbilt University* by Robert T. Lagemann and published in 1983 by Folio Publishers. It is identified by the makers mark on the long bar of the pendulum: "Deleuil à Paris." in script. Near the end of the fall 02 semester the pendulum apparently was damaged. One knife edge was twisted enough so that the pendulum hung on only one side.

Bob Patchin removed (they were held in by friction or force fit in mechanics parlance) the knife edges and ground then. He put them back and dented ("corked" to a machinist) the iron of the bar so that they are as accurately perpendicular to the bar as he could make them. Webster and Patchin then measured the distance between knife edges on both sides of the bar: 39.4055 and 39.4058 inches where the last digit is uncertain by at least 3 (I used .0005 inches below). Averaging and converting to meters gives 1.000903 $\pm$ 0.000012 m. Carlton 1983 got 39.403 inches; Webster and Wikswo 1990 got 39.401 $\pm$ 0.001 inches = 1.00079 $\pm$ 0.00003 m. Since the knife edges were reset, it is not surprising that the distance has increased a couple thousandths of an inch.

We are pushing the accuracy of the calibration of the Pasco timers. I bought the timer in 2001 and the catalog said 0.01 % of reading. The literature distributed with the instrument said 1 %. I think it would be hard to buy a crystal oscillator as bad as 1 % today and believe the catalog is correct. (A $15 watch from Radio Shack or Walgreens is good to a couple of minutes per year, about 0.001 %. This timer cost $300; would Pasco put a worse time reference in a $300 photogate than is used in a $15 watch?) I (Webster) called Pasco and their technical adviser has promised to get back to me when he understands the discrepancy between the catalog and the literature distributed with the hardware. In the meantime I have checked the calibration of the oscillator in each timer and put the correction factor on each unit. I used the frequency counter we use with the oscilloscope and electrical oscillation experiments. That counter agrees to better accuracy than is needed here with a pulser Will Johns bought recently, so I believe we are limited by readout significance rather than by internal errors.

A final plea: We admit that this is only a convention and an unwarranted infringement of your freedom to name things as you please, but we do request that you use the subscript 1 for lengths and periods relating to the knife-edge near the disk. On the other hand, we insist that you make your arbitrary choice of naming very clear if you refuse to follow our arbitrary choice.