Group Theory for Physics

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Abstract

Group theory and theory of Lie algebras are presented in a manner useful for understanding their numerous applications in physics, having in mind the readers who are undergraduate or graduate students and teachers in physics.

Contents

Part I

Introduction

It is the aim of this book to present the theory of Groups and Lie algebras in a way suitable for undergraduate and graduate students in physics. Accordingly, the style is dictated, not by the request of mathematical rigor, but by the need of facilitating their applications in physics, which are numerous and which cover a vast range of fields, including areas even beyond the arenas of physics.

What is a "group"? The short but precise answer is the following. It is a particular kind of a set, with elements

$$a, b, \ldots \in G, \tag{0.1}$$

having the following four properties (known as the "group axioms"):

- For each pair of elements $(a,\, b)$ of G, there is an element called their "product",

$$a, b \longrightarrow a \circ b \in G\ ; \tag{0.2}$$

- Such a product operation can obviously be iterated: it is required that they obey the associativity:

$$a \circ (b \circ c) = (a \circ b) \circ c\ ; \tag{0.3}$$

- G contains an element called unit element, e, such that

$$e \circ a = a\ , \qquad \forall a\ . \tag{0.4}$$

- For each element a there is an inverse, called a^{-1}, such that

$$a^{-1} \circ a = e\ . \tag{0.5}$$

That is all. It is quite amazing that these simple rules defining a "group" allow such a vast number of different types of interesting groups, with rich varieties of characteristics (see below for some), and lead to a formidable branch of mathematics, overlapping with geometries, algebra, functional analysis, number theory, topology, theory of differential equations, matrix theory, etc. The reader will have more than one occasions to get a glimpse of such a beauty of group theory world, already in this modest introductory book.

In physics, the concept of groups is closely associated with the idea of symmetries, and more generally, of that of transformations. A crucial question of interest in physics is whether or not certain things, such as the equation of motion of a system or the shapes or distributions of objects, *remain invariant* under changes of variables (e.g., the choice of coordinates, of the state vector basis in quantum mechanics, etc.), or if they do not, *how they transform.* Indeed, the group theory studies the notion of transformations in their purest form, classifying and studying possible types of transformation actions and their infinitesimal generators, the kind of the spaces on which they act, the properties of subclasses of transformations, the relations among different types of groups, etc., while discarding all other attributes (e.g., physical or chemical properties) of objects being transformed. This is the reason for their universality and for the omnipresence in physics, and, in particular, for the fact that the representation theory plays the central role in the whole subject.

1 Groups: definitions, examples and basic concepts

In this section the basic definitions and notions are introduced.

1.1 Set theory basics

The notion of sets underlies the group theory in all its aspects.

(i) Elements a_i are elements of a set M,

$$a_i \in M \ , \qquad M = \{a_1, a_2 \ldots\} \ ; \tag{1.1}$$

(ii) A set M is a subset of N, if all its elements belong also to N,

$$M \subset N \ , \qquad \forall a \in M \longrightarrow a \in N \ . \tag{1.2}$$

(iii) The union of two sets is made up of elements which belong either to one or the other, or to both, of the two sets,

$$a \in M \cup N \ ; \qquad \text{if} \quad a \in M \quad \text{or} \quad a \in N \ ; \tag{1.3}$$

(iv) The intersection is composed by the elements which belong to both sets

$$a \in M \cap N \ ; \qquad \text{if} \quad a \in M \quad \text{and} \quad a \in N \ ; \tag{1.4}$$

(v) The difference,

$$N \setminus M \ ; \tag{1.5}$$

is a set composed of elements

$$a \in N \ , \qquad a \notin M \ ; \tag{1.6}$$

(vi) If each element in M has a unique image in N

$$\forall x \in M \longrightarrow y = f(x) \in N \ , \tag{1.7}$$

in which two distinct elements in M are mapped to distinct images in N, it is an injective map $M \to N$ or a map from M "into" N.

(vii) If $\forall y \in N$ has at least one inverse image x, i.e, .

$$\exists x \in M \quad \text{such that} \quad f(x) = y \ , \tag{1.8}$$

the map $M \to N$ is surjective or "onto".

(viii) If in an injective map $M \to N$ each $y \in N$ possesses precisely one inverse image $x \in M$, i.e.,

$$\forall y \in N \ , \quad \exists x(\text{unique}) \in M \quad \text{such that} \quad f(x) = y \ , \tag{1.9}$$

the map is a "bijection", "injective and onto", or a "one-to-one map". In this case $x \equiv f^{-1}(y)$. f^{-1} defines the inverse map.

(ix) In any set, *equivalence relations* among its members may be defined in some way. An equivalence relation must satisfy three properties

1. Reflexivity: $a \sim a$
2. Symmetry: if $a \sim b$, then $b \sim a$,
3. Transitivity: if $a \sim b$ and $b \sim c$, then $a \sim c$.

Once such an equivalence relation is defined, the elements of the set is separated into disjoint classes of equivalent elements. Each element belongs to only one class.

1.2 Group axioms

A set G, with elements $g_i \in G$, $i = 1, 2, \ldots$, is said to form a group if the following properties ("the group axioms") are satisfied:

(1) Given each pair of elements $g_1, g_2 \in G$ their "product" is defined in G,

$$g_3 = g_1 \circ g_2 \in G \ ; \tag{1.10}$$

(2) The group product satisfies the associativity

$$g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3 \ ; \tag{1.11}$$

(3) In G there exists a unit element e,

$$\exists e \in G \ , \quad \text{such that} \quad \forall g \in G, \quad e \circ g = g \ ; \tag{1.12}$$

(4) Each element g possesses an inverse,

$$\forall g \ , \quad \exists g^{-1} \quad \text{such that} \quad g^{-1} \circ g = e \ . \tag{1.13}$$

Definition

Note that in general the order in which two elements appear in their product, (1.10), is significant. A group G is commutative (Abelian) if

$$g \circ h = h \circ g \ , \qquad \forall g, \ \forall h \in G \ ; \tag{1.14}$$

otherwise, the group is noncommutative or nonAbelian.

Definition

A group G may contain a finite numbers of elements: it is called a finite group in this case. Otherwise, it is an infinite group.

Definition

If G as a set contains a continuous infinity of elements, it is a continuous group. Otherwise, it is a discrete group.

Definition

The number of the elements is known as the order of the group.

Observation 1.

It follows from the axioms (1)$\sim$(4) that the "left unit element" of (3) and the "left inverse element" g^{-1} appearing in (4) are automatically the right unit element and the right inverse element, respectively. It follows also that the unit element is unique, and given an element g, its inverse g^{-1} is also unique. (Exercise: prove these).

Observation 2.

A particular manner its elements are related to each other by the product rule (1) and other axioms, characterize (i.e., defines) a given group.

1.3 Examples

Here are several examples of groups of interest.

(1) Additive group $\mathbb{Z}$.

The set of integers $\mathbb{Z}$ forms a group under addition:

$$m \circ n \equiv m + n \ , \qquad e \equiv 0 \ ; \qquad m^{-1} = -m \ . \tag{1.15}$$

It is infinite, commutative (Abelian) and discrete.

(2) Additive group $\mathbb{R}$.

The set of real numbers $\mathbb{R}$ forms a group under addition:

$$x \circ y \equiv x + y \ , \qquad e \equiv 0 \ ; \qquad x^{-1} = -x \ . \tag{1.16}$$

It is continuous (hence necessarily infinite) and commutative (Abelian).

(3) The set of real numbers $\mathbb{R}$ do not form a group under ordinary multiplication, as the inverse of 0 does not exist. The set $\mathbb{R} \setminus \{0\}$ forms a multiplicative group $\mathbb{R}^+$, with

$$x \circ y \equiv xy \ , \qquad e \equiv 1 \ ; \qquad x^{-1} = \frac{1}{x} \ . \tag{1.17}$$

(4) S_N is a group of permutations of N objects. For $N = 3$, S_3 consists of the following 6 elements

$$\begin{array}{ll} e: \quad (ABC) \to (ABC) \ ; & (12): \quad (ABC) \to (BAC) \ ; \\ (31): \quad (ABC) \to (CBA) \ ; & (23): \quad (ABC) \to (ACB) \ ; \\ (123): \quad (ABC) \to (CAB) \ ; & (321): \quad (ABC) \to (BCA) \ . \end{array}$$

Note that the multiplication rules are $(12) \circ (23) = (123)$, $(23) \circ (12) = (321)$, etc. As these examples show it is a noncommutative (nonAbelian) group. (Exercise: complete the multiplication table for S_3)

It is a finite group (the number of the elements being 6); it is discrete and noncommutative. Thus the *order* of the group S_3 is six. Analogously the order of S_N is $N!$.

(5) Triangular matrices of the type

$$T = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \ , \qquad a, b, c \in \mathbf{R}, \quad a > 0, \quad c > 0 \right\} \ , \tag{1.18}$$

form a group, under the standard matrix multiplication rule. (Exercise: find the unit element and the inverse element for each of them.)

(6) The set of complex (vice versa, real) regular $N \times N$ matrices form a group called the general linear group, $GL(N, \mathbb{C})$ or $GL(N, \mathbb{R})$, under the standard matrix multiplication. Special linear groups $SL(N, \mathbb{C})$ or $SL(N, \mathbb{R})$ are groups of regular $N \times N$

complex or real matrices M of unit determinant, $\det M = 1$. For instance,

$$SL(N, \mathbb{C})) = \{M_{N\times N} \mid \ \det M = 1 \} . \tag{1.19}$$

(7) The set of unitary $N \times N$ matrices form the unitary group $U(N)$ or the special unitary group $SU(N)$,

$$U(N) = \{U_{N\times N} \mid U^\dagger U = \mathbb{1}\} ; \qquad SU(N) = \{U_{N\times N} \mid U^\dagger U = \mathbb{1} \ , \ \det U = 1\} . \tag{1.20}$$

The unitary groups are groups of linear transformations acting on N-component complex vectors z, such that the inner product

$$w^\dagger \cdot z = \sum_n w_n^* z_n \tag{1.21}$$

is invariant under the action of U:

$$z \to U\, z \ , \qquad w \to U\, w \ . \tag{1.22}$$

(8) The (real or complex) orthogonal groups $O(N)$ or special orthogonal groups are formed by the set of orthogonal matrices

$$O(N) = \{\{O_{N\times N}\} \mid O^T O = \mathbb{1}\} ; \qquad SO(N) = \{\{O_{N\times N}\} \mid O^T O = \mathbb{1} \ , \ \det O = 1\} . \tag{1.23}$$

The orthogonal group of transformations leave the inner products of vectors defined by

$$x \cdot y = x^T \cdot y = x_i \delta_{ij} y_j \tag{1.24}$$

invariant. In other words they leave the metric $\mathbb{1}_{ij} = \delta_{ij}$ invariant,

$$O^T \mathbb{1} O = \mathbb{1} \ . \tag{1.25}$$

Elements of the familiar two-dimensional rotational group $SO(2)$ have the form,

$$x \to O\, x' \ , \qquad O = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \ , \qquad 0 \le \theta < 2\pi \ . \tag{1.26}$$

(9) The real or complex symplectic groups, $Sp(2N;\mathbb{R})$ or $Sp(2N;\mathbb{C})$ are groups of $2N \times 2N$

matrices, which transform $2N$-component vectors,

$$\begin{pmatrix} q_1 \\ \vdots \\ q_N \\ p_1 \\ \vdots \\ p_N \end{pmatrix} \to M \begin{pmatrix} q_1 \\ \vdots \\ q_N \\ p_1 \\ \vdots \\ p_N \end{pmatrix} \tag{1.27}$$

such that the metic

$$\mathbf{J} = \begin{pmatrix} \mathbf{0}_N & \mathbb{1}_N \\ -\mathbb{1}_N & \mathbf{0}_N \end{pmatrix} \tag{1.28}$$

is left invariant,

$$M^T \mathbf{J} M = \mathbf{J} \ . \tag{1.29}$$

In other words, the symplectic product of the two vectors

$$\begin{pmatrix} q^T & p^T \end{pmatrix}^{(1)} \cdot \mathbf{J} \cdot \begin{pmatrix} q \\ p \end{pmatrix}^{(2)} = \sum_{i=1}^{N} \left(q_i^{(1)} p_i^{(2)} - p_i^{(1)} q_i^{(2)} \right) \tag{1.30}$$

is kept invariant under $Sp(2N;\mathbb{R})$. These structures are familiar from classical mechanics in the canonical formalism: the Hamilton (canonical) equations, Poisson brackets, etc.

(10) The Euclidean groups, E_n, are defined as the group of rotations and translations in n dimensional Euclidean space, respectively. For instance, E_2 is the group of transformations,

$$x'^1 = x^1 \cos\theta + x^2 \sin\theta + b_1 \ ; \qquad x'^2 = x^2 \cos\theta - x^1 \sin\theta + b_2 \ . \tag{1.31}$$

(11) The Lorentz group $SO(3,1)$ is a group of transformations which leave invariant the Minkowski metric

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \tag{1.32}$$

i.e.,

$$g \to \Lambda^T g \Lambda = g \ . \tag{1.33}$$

Four vectors

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{1.34}$$

tranform as

$$x \to \Lambda x, \tag{1.35}$$

such that the scalar product

$$x \cdot y \equiv x^\mu y_\mu = x^\mu g_{\mu\nu} y^\nu = x^0 y^0 - \mathbf{x} \cdot \mathbf{y} \tag{1.36}$$

remains invariant.

(12) The Poincaré group is a group acting on four vectors in Minkowski space: its elements consist of a Lorentz transformation followed by a spacetime translation,

$$x \to \Lambda x + b \ . \tag{1.37}$$

The Poincaré group does not leave invariant a four vector squared, $x^\mu x_\mu$, but their difference squared,

$$(x - y)^\mu (x - y)_\mu \ , \tag{1.38}$$

and thus the geodesic element

$$dx^\mu dx_\mu = g_{\mu\nu}\, dx^\mu dx^\nu \ . \tag{1.39}$$

1.4 Rings and fields

A notion closely related to the groups is that of a ring. A commutative group R, with the group product defined by the addition rule,

$$x \circ y \equiv x + y \ , \tag{1.40}$$

in which also the product of two elements $xy \in R$ are defined, with the following properties,

$$(xy)z = x(yz) \ , \qquad (x + y)z = xz + yz \ , \tag{1.41}$$

is known as a ring. The unit element of the ring is the zero element 0 of the addition group. If the all the elements other than 0 form a group under the multiplication, then R is a division ring (or a division algebra).

In general, the multiplication rule is not commutative. If it is commutative, then the division ring R is called a *field*. In the case of continuous division rings of Abelian multiplication rule, there are only three known such fields: the rational numbers $\mathbb{Q}$, the real numbers, $\mathbb{R}$, and the complex numbers, $\mathbb{C}$.

1.5 Subgroups and homomorphisms

The structure of subgroups and mapping between the groups provide us with powerful tools for analyzing the properties of groups.

(a) A subset of group elements in G,

$$h_1, h_2, \ldots \in H \subset G \ , \tag{1.42}$$

is said to constitute a *subgroup* of G, if they form themselves a group,

$$h_i \circ h_j \in H \tag{1.43}$$

under the multiplication rule of G. The unit element itself always forms a subgroup, $I \subset G$. The whole group G is also a special "subgroup".

(b) A subgroupf of G, H, is called an *invariant subgroup*, or a *normal subgroup*, if

$$g\, h\, g^{-1} \in H \ , \qquad \forall h \in H \ , \quad \forall g \in G \ . \tag{1.44}$$

A group without invariant subgroups other than I and G, is called *simple.* A group without Abelian invariant subgroup other than I is called *semi-simple.*

(c) A map from a group G to another group G^*, $G \to G^*$,

$$g \in G \longrightarrow f(g) \in G^* \ , \tag{1.45}$$

such that

$$g_1 g_2 \longrightarrow f(g_1 g_2) = f(g_1) f(g_2) \ , \tag{1.46}$$

namely, the group product rules are respected, is called a *homomorphism* from G to G^*. Note that $f(e) = e^*$ is the unit element of G^* and the inverse element of G maps to the inverse element of G^*, i.e.,

$$f(g^{-1}) = (f(g))^{-1} \ . \tag{1.47}$$

The subset

$$k \in K \subset G, \qquad f(k) = e^* \ , \tag{1.48}$$

or, the set of all inverse images of the unit element of G^*, $f^{-1}(e^*) \in G$, forms the *kernel* of the homomorphism, $G \to G^*$.

The kernel K is necessarily an invariant subgroup of G, i.e.,

$$\forall g \in G, \qquad gKg^{-1} \longrightarrow f(gKg^{-1}) = f(g)e^* f(g)^{-1} = e^* \ . \tag{1.49}$$

(d) If the map $g \to h(g)$ above from G to G^* is one to one (bijective), the map is an *isomorphism.*

(e) An isomorphism from a group to itself, $G \to G$, is known as an *automorphism.* Two automorphisms can be multiplied: the identity element is the trivial one:

$$g_i \to g_i \ , \quad \forall g_i \in G \ , \tag{1.50}$$

the inverse can be defined obviously as the map is one to one. The ensemble of automorphisms $G \to G$ form a group, $Aut[G]$.

f) A particular type of an automorphism is given by using a fixed element $g \in G$ and defining the map,

$$\forall x \in G \ , \quad x \to x' = gxg^{-1} \in G \ . \tag{1.51}$$

This is known as an *inner automorphism.*

(g) If $N \subset G$ is a subgroup of G, then gN - the set of all elements of the form,

$$g\, n_i, \qquad g \in G, \quad n_i \in N \subset G, \tag{1.52}$$

collectively regarded as an element - form a set called *left coset* and denoted by G/N. The right coset can be defined analogously.

If $N \subset G$ is an invariant subgroup of G, then the left coset forms a group under the multiplication rule

$$g_1 N \cdot g_2 N \sim g_1 g_2 N \ . \tag{1.53}$$

Note that

$$g_1 N \cdot g_2 N = g_1 g_2 (g_2)^{-1} N g_2 N = g_1 g_2\, NN = g_1 g_2 N \ . \tag{1.54}$$

(h) If $K \subset G$ is the kernel of the homomorphism, $G \to G^*$,

$$g \in G \longrightarrow f(g) \in G^* \ , \tag{1.55}$$

$$k \in K \in G, \qquad f(k) = e^* \ , \tag{1.56}$$

K is an invariant subgroup of G (see the point (c)). The map from the left coset G/K to G^*,

$$G/K \to G^* \tag{1.57}$$

is an injection. If the map $G \to G^*$ is surjective (each element of G^* has at least one inverse image in G), then $G/K \to G^*$ is an isomorphism.

The proof goes as follows. Let a and a' be two elements of G, such that

$$f(a) = f(a') = a^* \ . \tag{1.58}$$

Then

$$a^{-1}a^{'} \longrightarrow f(a^{-1})f(a^{'}) = (f(a))^{-1}f(a^{'}) = (a^*)^{-1}a^* = e^* \ , \tag{1.59}$$

therefore

$$a^{-1}a^{'} \in K \ . \tag{1.60}$$

That is, a and $a^{'}$ belong to the same coset,

$$aK = aa^{-1}a^{'}K = a^{'}K \ . \tag{1.61}$$

Therefore the map $G/K \to G^*$ is injective. If $G \to G^*$ is also surjective, then the above shows that there is one-to-one map between the elements of the groups G/K and G^*: it is an isomorphism.

(i) Representation: Let V be an N dimensional vector space and $GL(V)$ be a group of linear transformations acting on V. That is:

$$x \in V \ ; \qquad R \in GL(V) \ , \qquad x \to Rx \ , \tag{1.62}$$

where R is a $N \times N$ regular matrix. A map from a group G to the group of matrices $GL(V)$,

$$g \in G \longrightarrow R(g) \in GL(V) \ , \tag{1.63}$$

such that

$$g_1 \circ g_2 \longrightarrow R(g_1 \circ g_2) = R(g_1)R(g_2) \ , \tag{1.64}$$

is known as a *representation*, and denoted as $R = \{R(g), V \,|\, g \in G\}$. Clearly, $R(g)$ themselves form a matrix group, and a representation is a homomorphism from G to $N \times N$ matrix group $R(g)$. It follows that

$$R(e) = \mathbb{1}_{N\times N} \ . \tag{1.65}$$

The Part III will be dedicated to the study of representation theory.

♣ Any group G has always the so-called trivial representation,

$$\forall g \in G \longrightarrow R(g) \equiv 1 \, . \tag{1.66}$$

♣ If there are two representations

$$R = \{R(g), V_{N\times N} \,|\, g \in G\} \qquad \text{and} \qquad \tilde{R} = \{\tilde{R}(g), \tilde{V}_{N\times N} \,|\, g \in G\} \tag{1.67}$$

of the same dimension N, and if there exists a fixed regular $N \times N$ matrix S such that

$$\tilde{R}(g) = S\, R(g)\, S^{-1} \; , \qquad \forall g \in G \; , \tag{1.68}$$

then the two representations are said to be *equivalent*.

♣ If a representation of a group G is described by unitary matrices R, then it is a *unitary representation*.

More about the representations (Part III) later.

(j) Given two elements a, b of any group G, the product

$$a\, b\, a^{-1}\, b^{-1} \tag{1.69}$$

is called the *commutator* of a and b. When the two elements *commute*, that is,

$$a\, b = b\, a \; , \tag{1.70}$$

their commutator is equal to the unit element, e. (Exercise: prove it).

An element of a group, $z \in G$, which commutes with all the elements of the group,

$$z \circ g = g \circ z \; , \qquad \forall g \in G \; , \tag{1.71}$$

is known as a *center element* of G. The set of all such elements form the subset called *center*, $\mathbf{C}$, of the group G. $\mathbf{C}$ is an Abelian invariant subgroup of G.

(k) In any group G, two elements $a, b \in G$ connected by a conjugation relation

$$\exists g \in G \; , \qquad b = g\, a\, g^{-1} \; , \tag{1.72}$$

are said to belong to the same *conjugacy class*. This is an example of the equivalence relation mentioned at (2.1), (ix).

The unit element forms its own conjugacy class.

2 Finite groups

Many of the characteristics of the group theory manifest themselves already in finite groups (groups with finite number of elements). Let us illustrate some of the concepts introduced above in simplest such groups of physical interest.

$g_1 \setminus g_2$	e	a
e	e	a
a	a	e

Table 1: Multiplication table for C_2

$g_1 \setminus g_2$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Table 2: Multiplication table for C_3

2.1 C_2

The simplest nontrivial group have just two elements (e, a), $a \neq e$. The consistency of the group multiplication rules tell us that the only possibility is the rule give in Table 1. Obviously, $a^{-1} = a$. Note that it would be inconsistent to assume

$$a \circ a = a \ . \tag{2.1}$$

It is also known as a cyclic group of order 2, $\mathbb{Z}_2$. Its one-dimensional representation is:

$$R(e) = 1 \ , \qquad R(a) = -1 \ . \tag{2.2}$$

2.2 C_3

There exists only one group of order three, C_3. Its three elements e, a, b are multiplied with each other according to Table 2. It is also called the cyclic group of order three, $\mathbb{Z}_3$. Its one-dimensional representation is

$$R(e) = 1 \ ; \qquad R(a) = e^{2\pi i/3} \ ; \qquad R(b) = e^{4\pi i/3} \ . \tag{2.3}$$

2.3 Cyclic groups C_n or $\mathbb{Z}_n$

The cyclic group of order n has n elements,

$$e, \, a, \, a^2, \ldots, \, a^{n-1} \ , \qquad a^n = e \ . \tag{2.4}$$

The cyclic groups are all Abelian. (Exercise: find the one-dimensional representation of $\mathbb{Z}_n$.)

$g_1 \setminus g_2$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Table 3: Multiplication table for D_2

$g_1 \setminus g_2$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

Table 4: Multiplication table of the cyclic group C_4 or $\mathbb{Z}_4$

2.4 Permutation group S_N

The group S_N of all permutations of N objects, $(1, 2, 3, \ldots, n)$ plays an important role in physics and mathematics. The generic element of S_N may be represented as

$$\begin{pmatrix} 1 & 2 & 3 & \ldots & N \\ a_1 & a_2 & a_3 & \ldots & a_N \end{pmatrix} \tag{2.5}$$

with an obvious notation, where $(a_1, a_2, \ldots, a_N)$ is some permutation of $(1, 2, \ldots, N)$. The unit element corresponds to

$$\begin{pmatrix} 1 & 2 & 3 & \ldots & N \\ 1 & 2 & 3 & \ldots & N \end{pmatrix} . \tag{2.6}$$

The product is defined by a successive application of two permutations, and the set of all such permutations form a finite, noncommutative group of order $N!$.

The irreducible representations (see below) of the group S_N can be summarized in Young tableaux, to be discussed later.

2.5 The group D_2

The simplest noncyclic group has order four. It is the *four group*, D_2, defined by the multiplication properties summarized in Table 3. The group D_2 arises as the symmetry group of transformations leaving invariant a rectangle, Fig. 1

Clearly, the elements (e, a) constitute a subgroup, identical to C_2, so do the pairs (e, b) or (e, c). But not the subset (e, a, b).

Note that the group D_2 and another group of order 4, C_4 or $\mathbb{Z}_4$ (see Table, 4) have indeed different multiplication rules among the four elements.

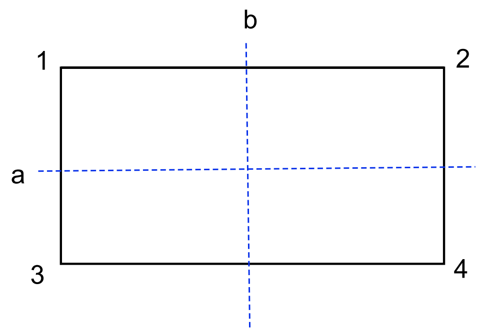


Figure 1: A rectangle left invariant under the reflections a, b, and angle-π rotation, c.

2.6 D_3 or S_3

Another important example is the *dihedral group* of order six, D_3. It can be defined as the group of transformations acting on a regular triangle and leaving it unchanged (reflections about the three axes passing through the three vertices, and angle $2\pi/3$ and $4\pi/3$ rotations around the center). See Fig. 2. The multiplication table is given in Table 5, where the names of the elements are given according to how the three vertices of the triangle are permuted by the operations of reflections and rotations. Not surprisingly, the dihedral group D_3 coincides with the order-three permutation group, S_3, mentioned in the example 4, Section 1.3.

The elements of D_3 or S_3 are divided in three conjugacy classes, [1] e itself; [2] three exchanges, (12), (23) and (31), and [3] two cyclic permutations, (123) and (321).

2.7 D_4 or D_8

The *dihedral group* of order eight, D_4 (or D_8), can be analogously constructed as the group of reflections and rotations acting on a square and leaving it invariant (Fig. 3). Note that, unlike the case of D_3, D_4 (D_8) is not equivalent to the permutation group S_4. The multiplication table for D_8 is given in Table 6 .

2.8 Generalizations

Clearly by considering more complicated two dimensional poligons, or three-dimensional crystals or molecules of different shapes and with different symmetries, one is led to large classes of finite groups. They will not be discussed here. Some of them are discussed in Landau and Lifshitz [9].

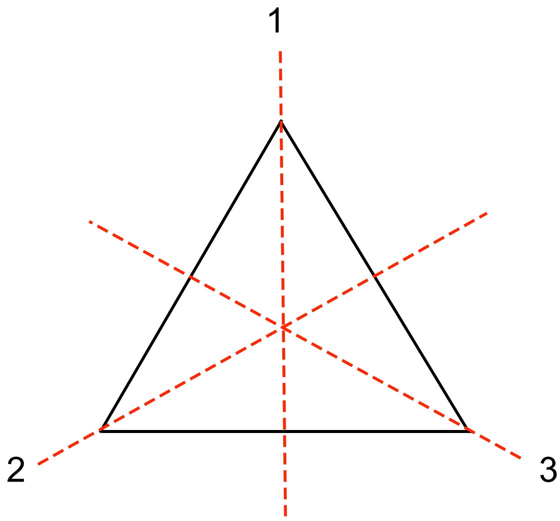


Figure 2: A regular triangle left invariant under the reflections (12), (23), (31), and angle $2\pi/3$ and $4\pi/3$ rotations (123) and (321).

$g_1 \setminus g_2$	e	(12)	(23)	(31)	(123)	(321)
e	e	(12)	(23)	(31)	(123)	(321)
(12)	(12)	e	(123)	(321)	(23)	(31)
(23)	(23)	(321)	e	(123)	(31)	(12)
(31)	(31)	(123)	(321)	e	(12)	(23)
(123)	(123)	(31)	(12)	(23)	(321)	e
(321)	(321)	(23)	(31)	(12)	e	(123)

Table 5: Multiplication table for D_3 or S_3

$g_1 \setminus g_2$	e	a	b	c	d	f	g	h
e	e	a	b	c	d	f	g	h
a	a	e	g	h	f	d	b	c
b	b	g	e	f	h	c	a	d
c	c	f	h	e	g	a	d	b
d	d	h	f	g	e	b	c	a
f	f	c	d	b	a	g	h	e
g	g	b	a	d	c	h	e	f
h	h	d	c	a	b	e	f	g

Table 6: Multiplication table for D_8 (or D_4)

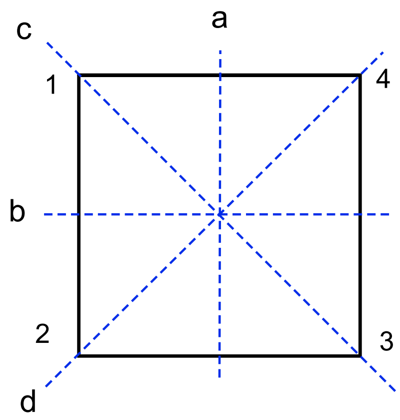


Figure 3: A square left invariant under the reflections a, b, c, d and angle $f = \pi/2$, $g = \pi$ and $h = 3\pi/2$ rotations.

Part II

Lie Groups and Lie algebras

The concepts of Lie groups and Lie algebras are introduced. The invariant integration over the group is defined. The local properties of a Lie group are described by the associated Lie algebra. Some global aspects of the Lie groups are discussed in terms of homotopy groups.

3 Lie groups

Consider a continuous group, G, whose elements are expressed by a set of continuous parameters $\{\alpha\} = \{\alpha_1, \alpha_2, \ldots\}$,

$$A \in G \; , \qquad A = A(\alpha) = A(\alpha_1, \alpha_2, \ldots, \alpha_r) \; . \tag{3.1}$$

The parameters are chosen such that

$$e = A(\mathbf{0}) = A(0, 0, \ldots, 0) \; . \tag{3.2}$$

We require the properties of:

(1) Closure:

$$A(\alpha)A(\beta) = A(\gamma) \; , \tag{3.3}$$

where γ is a differentiable function of α and β:

$$\gamma = f(\alpha, \beta) \; , \tag{3.4}$$

such that

$$f(0, \gamma) = f(\gamma, 0) = \gamma \; ; \tag{3.5}$$

(2) Inverse:

$$A(\alpha)^{-1} = A(\alpha') \; , \tag{3.6}$$

where α' is a differentiable function of α;

(3) Associativity:

$$A(\alpha)(A(\beta)A(\gamma)) = (A(\alpha)A(\beta))A(\gamma) \; , \tag{3.7}$$

that is,

$$f(\alpha, f(\beta, \gamma)) = f(f(\alpha, \beta)), \gamma) \; . \tag{3.8}$$

A continuous group with these properties are known as a Lie group.

Example 1

The group under the addition **R** of real numbers, form a Lie group.

Example 2

S (a circle)

$$\theta_1 \circ \theta_2 = \theta_1 + \theta_2 \ , \tag{3.9}$$

$$e = 0 \ , \qquad \theta^{-1} = -\theta \ , \tag{3.10}$$

is a Lie group.

Example 3

The groups, $SU(N)$, $SO(N)$, $USp(2N)$, etc. are all Lie groups.

3.1 Invariant integration

When the range of the parameters $\{\alpha\} = \{\alpha_1, \alpha_2, \ldots\}$ describing a given Lie group is finite, the group is said to be *compact*. The groups $O(N)$, $SO(N)$, $U(N)$, $SU(N)$ are all compact, whereas $SL(N,\mathbb{R})$, $SL(N,\mathbb{C})$, E_2, E_3 are non compact, as their parameters can become arbitrarily large.

The concept of invariant integration over the group can be illustrate by using the standard integral over the real variable, x, and regarding $x \to x + a$ as a group transformation in the additive group, $\mathbb{R}$ (see (1.16)). For an arbitrary function $f(x)$ the relation

$$\int_{-\infty}^{\infty} dx\, f(x+a) = \int_{-\infty}^{\infty} dx\, f(x) \tag{3.11}$$

holds, therefore the usual integration measure dx gives an *invariant measure* for the additive group $\mathbb{R}$. The total volume of integration is infinite in this case and the group is noncompact.

dx is however not an invariant measure for the multiplicative group $\mathbb{R}^+$ (see (1.17)), as

$$\int_0^{\infty} dx\, f(ax) \neq \int_0^{\infty} dx\, f(x) \ . \tag{3.12}$$

The invariant measure for $\mathbb{R}^+$ is given by $\frac{dx}{x}$, as

$$\int_0^{\infty} \frac{dx}{x} f(ax) = \int_0^{\infty} \frac{dx}{x} f(x) \ , \qquad \forall a \ . \tag{3.13}$$

In the case of the triangle matrix group, (1.18),

$$a = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \ , \qquad g = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \ , \tag{3.14}$$

$$a \circ g = \begin{pmatrix} x' & y' \\ 0 & z' \end{pmatrix} , \qquad x' = ax \; ; \quad y' = ay + bz \; , \quad z' = cz \; . \tag{3.15}$$

Therefore, by studying the Jacobian from (x, y, z) to $(x', y'z')$, the left invariant measure is seen to be given by

$$\frac{dx\,dy\,dz}{x^2 z} = \frac{dx'\,dy'\,dz'}{x'^{\,2} z'} \; . \tag{3.16}$$

The integration region is

$$\int_0^\infty dx \int_{-\infty}^\infty dy \int_0^\infty dz \; , \tag{3.17}$$

and the group is non compact.

The $SU(2)$ group leaves invariant (see (5.3))

$$z^\dagger z = |z_1|^2 + |z_2|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 \; . \tag{3.18}$$

Clearly a point on S^3 is transformed to another point on S^3: By introducing the four-dimensional spherical coordinates (at a fixed radius 1)

$$x_1 = \cos\theta_1, \quad x_2 = \cos\theta_2 \sin\theta_1, \quad x_3 = \cos\theta_3 \sin\theta_2 \sin\theta_1, \quad x_4 = \sin\theta_3 \sin\theta_2 \sin\theta_1, \tag{3.19}$$

the invariant measure is just the volume element on S^3:

$$dg = \frac{1}{2\pi^2} \sin^2\theta_1 \sin\theta_2 \, d\theta_1 d\theta_2 d\theta_3 \; , \qquad 0 \le \theta_1 \le \pi \; , \quad 0 \le \theta_2 \le \pi \; , \quad 0 \le \theta_3 \le 2\pi \; . \tag{3.20}$$

The total volume is normalized to 1: $\int dg = 1$.

In the case of $SO(3)$ group, the invariant Haar measure can be constructed in terms of the Euler angles, ϕ, θ, ψ, and invariant integratoin can be defined by

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{2\pi} \sin\theta \, d\phi \, d\theta \, d\psi \, f(\phi, \theta, \psi) \equiv \int_G dg \, f(g) \; , \tag{3.21}$$

$$dg = \frac{1}{8\pi^2} \sin\theta \, d\phi \, d\theta \, d\psi \; . \tag{3.22}$$

When the total volume of integration is finite, the group is *compact.* Otherwise, the group is *non compact.*

For a general compact Lie group G, the invariant measure, known as the left Haar measure, symbolically denoted as dg, satisfies

$$\int_G dg \, f(a \circ g) = \int_G dg \, f(g) \; ; \qquad \int_G dg = 1 \; , \tag{3.23}$$

where f is a function of the group element g and a is an element of G. They satisfy furthermore

(1) If b is a real number

$$\int_G dg\, b\, f(g) = b \int_G dg f(g)\ ; \tag{3.24}$$

(2) If f and h are two continuous functions then

$$\int_G dg\, (f(g) + h(g)) = \int_G dg\, f(g) + \int_G dg\, h(g)\ ; \tag{3.25}$$

(3) If $f \geq 0$ everywhere, and not identically zero, then

$$\int_G dg\, f(g) > 0\ ; \tag{3.26}$$

(4) If a is an element of G, then

$$\int_G dg\, f(g \circ a) = \int_G dg\, f(g)\ ; \qquad \int_G dg = 1\ , \tag{3.27}$$

(5)

$$\int_G dg\, f(g^{-1}) = \int_G dg\, f(g)\ . \tag{3.28}$$

Theorem: (without proof) For any compact Lie group there exists a unique invariant measure. Furthermore, the left invariant measure is automatically the right invariant measure [1]

For two functions of G, $f(g)$ and $h(g)$, a scalar product can be defined as

$$\langle h, g\rangle \equiv \int_G dg\, h^*(g) f(g)\ . \tag{3.29}$$

4 Lie algebras

Consider the group elements close to the unit element,

$$A(0) = e\ . \tag{4.1}$$

Thus in a representation

$$R = \{R(\alpha), V \mid A(\alpha) \in G\} \qquad (\&) \tag{4.2}$$

$$R(0) = \mathbb{1}\ . \tag{4.3}$$

[1] In the case of the group of triangle matrices, for instance, the left invariant measure (3.16) does not coincide with the right invariant measure.

By Taylor expanding $R(\alpha)$ near $\{\alpha\} = (0, 0, \ldots)$,

$$R(\alpha) = \mathbb{1} + i\,\alpha_a\,T^a + \ldots\ , \tag{4.4}$$

or

$$T^a \equiv -i\frac{\partial R(\alpha)}{\partial \alpha_a}|_{\alpha=0}\ , \qquad a = 1, 2, \ldots, r\ , \tag{4.5}$$

are known as the generators of the group G in the representation (&).

The set of generators of the group G, $\{T^a\}$ satisfy the following properties:

(1) $\{T^a\}$ form a basis of a vector space $\mathfrak{g}$ on the field of real numbers, i.e.,

$$T^a \in \mathfrak{g} \longrightarrow c_1 T^1 + c_2 T^2 \in \mathfrak{g}\ , \qquad c_a, c_b \in \mathbb{R}\ ; \tag{4.6}$$

Thus $\mathfrak{g}$ contains $\mathbf{0}$.

(2) The set $\{T^a\}$ is closed under commutations,

$$[T^a, T^b] \equiv T^a T^b - T^b T^a \in \mathfrak{g}\ . \tag{4.7}$$

Writing (2) in a standard form (by appropriately choosing the basis of $\{T^a\}$)

$$[T^a, T^b] = i f^{abc} T^c\ , \qquad f^{abc} = -f^{bac}\ , \tag{4.8}$$

$\{T^a\}$, forms the *Lie algebra* $\mathfrak{g} = Alg[G]$ of the group G. The constants f^{abc} characterize $\mathfrak{g}$: they are called the *structure constants.*

The way the generic group element is parametrized in terms of finite numbers $\{\alpha\}$ has a large arbitrariness. A convenient choice is the *exponential parametrization*, such that

$$R(\alpha) = e^{i\alpha_a T^a}\ . \tag{4.9}$$

This can be interpreted as a particular way of iterating the infinitesimal transformations to define a finite element

$$\lim_{k\to\infty} (\mathbb{1} + i\frac{\alpha_a T^a}{k})^k\ , \tag{4.10}$$

consistent with the definition of the infinitesimal generators, (4.5) .

The necessity of the commutation relations (4.8) follows from the group multiplication rules and the smoothness of the group elements in α. Consider the product of two elements $e^{i\alpha_a T^a} e^{i\beta_a T^a}$. It must be equal to another exponential representation with some other coefficients,

$$e^{i\alpha_a T^a} e^{i\beta_a T^a} = e^{i\delta_a T^a}\ . \tag{4.11}$$

Now assume that $\{\alpha_a\}$ and $\{\beta_a\}$ are all infinitesimal, so the product is close to the identity element: $\{\gamma_a\}$ must also be infinitesimal and related to $\{\alpha_a\}$ and $\{\beta_a\}$. Expanding the exponentials,

$$i\delta_a T^a = \log e^{i\alpha_a T^a} e^{i\beta_a T^a} = \log[\mathbb{1} + K] = K - \frac{1}{2}K^2 + \frac{1}{3}K^3 + \ldots, \tag{4.12}$$

where

$$K \equiv e^{i\alpha_a T^a} e^{i\beta_a T^a} - \mathbb{1} \; . \tag{4.13}$$

For small $\{\alpha_a\}$ and $\{\beta_a\}$ one has

$$\begin{aligned} K &\simeq (1 + i\alpha\cdot T + \frac{1}{2}(i\alpha T)^2 + \ldots)(1 + i\beta\cdot T + \frac{1}{2}(i\beta T)^2 + \ldots) - 1 \\ &= i\alpha\cdot T + i\beta\cdot T - (\alpha\cdot T)(\beta\cdot T) - \frac{1}{2}(\alpha T)^2 - \frac{1}{2}(\beta T)^2 + \ldots \end{aligned} \tag{4.14}$$

thus

$$i\delta_a T^a \simeq i\alpha\cdot T + i\beta\cdot T - \frac{1}{2}[\alpha\cdot T, \beta\cdot T] + \ldots \; . \tag{4.15}$$

Note that the terms $(\alpha T)^2$ and $(\beta T)^2$ cancel out in the sum (4.12). Thus it must be that

$$[\alpha_a T^a, \beta_b T^b] = -2i(\delta_c - \alpha_c - \beta_c)T^c \equiv i\gamma_c T^c \; , \tag{4.16}$$

$$\gamma_c = -2(\delta_c - \alpha_c - \beta_c) = \alpha_a \beta_b \, f^{abc} \; . \tag{4.17}$$

In other words the generators must satisfy

$$[T^a, T^b] = i f^{abc} T^c \; . \tag{4.18}$$

Another, simpler way to see the same thing is to consider the commutator of the two elements,

$$\begin{aligned} & g(\beta)^{-1} g(\alpha)^{-1} g(\beta) g(\alpha) \\ =\; & (1 - i\beta T + \ldots)(1 - i\alpha T + \ldots)(1 + i\beta T + \ldots)(1 + i\alpha T + \ldots) \\ =\; & 1 - \alpha_a \beta_b [T^b, T^a] + \ldots = 1 + i\gamma_c T^c + \ldots \;, \end{aligned} \tag{4.19}$$

which implies (4.8). Note that the terms $(\alpha T)^2$ and $(\beta T)^2$ again cancel out completely.

We shall be interested often in unitary representations (with unitary matrices $R(\alpha)$), in which case the generators are Hermitian matrices,

$$(T^a)^\dagger = T^a \; , \tag{4.20}$$

and the structure constants f^{abc} are real.

The Lie algebra (4.8) follows uniquely from the multiplication rules of the Lie group G, and characterizes its local properties (behavior of the group multiplication rules near the unit element) completely [2]. By the very definition of the representation, the Lie algebra and the structure constants f^{abc} are uniquely fixed [3] by the group G: f^{abc} are the same, independent of the particular representation (the dimension of the representation space, hence the matrix dimension). Indeed the concept of the Lie algebra, the generators and the structure constants could be defined independently of a particular representation, once the meaning of the group operations depending on continuous set of parameters is provided precisely and abstractly.

The Lie algebra relations are subject to a consistency condition

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0 \, , \tag{4.21}$$

known as the *Jacobi identity.*

Let us see some examples:

(i) $SU(N)$ group: the generators of the Lie algebra in the fundamental (the smallest nontrivial) representation are Hermitian, traceless $N \times N$ matrices

$$(T^a)^\dagger = T^a \, , \qquad \mathrm{Tr}\, T^a = 0 \, , \qquad a = 1, 2, \ldots N^2 - 1 \, . \tag{4.22}$$

(ii) The three generators of the $SU(2)$ group in the fundamental representation are

$$t^a = \frac{1}{2}\tau^a \, , \tag{4.23}$$

where

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{4.24}$$

are known as the Pauli matrices. The Lie algebra $su(2)$ is given by[4]

$$[t^a, t^b] = i\epsilon^{abc} t^c \, , \tag{4.25}$$

where $\epsilon^{123} = 1$, $\epsilon^{213} = -1$ and others are defined by cyclic permutations of the indices. ϵ^{abc} vanishes whenever two indices coincide. ϵ^{abc} is known as the totally antisymmetric invariant tensor of the $SU(2)$ group.

(iii) The generators of the special orthogonal group $SO(N)$ is the set of antisymmetric matrices

$$(T^a)^t = -T^a \, , \qquad a = 1, 2, \ldots, \frac{N(N-1)}{2} \, . \tag{4.26}$$

[2] Vice versa, the opposite is not always true: a Lie algebra does not always determine uniquely the corresponding Lie group: see later on the global aspects of Lie groups.

[3] More precisely, f^{abc} are determined uniquely once a choice of the independent generators is made. As in any vector space, the choice of the independent basis vectors has a large amount of arbitrariness.

[4] It is customary to use the lower case symbol for the associated Lie algebra of a given Lie group with the capital symbol.

The Lie algebra of the $SO(3)$ group (with three generators of rotations in $3D$) coincide with that of $SU(2)$, that is, $so(3) \sim su(2)$:

$$[T^a, T^b] = i\epsilon^{abc} T^c \ . \tag{4.27}$$

Explicitly,

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \ , \quad T^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \ , \quad T^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{4.28}$$

More about the relation between the two groups, $SU(2)$ and $SO(3)$ later. In quantum mechanics the algebra (4.28) or (4.25) appears as the commutation relations among the components of angular momentum (or spin) operators, and determine their quantization and composition rules completely.

4.1 Adjoint representation

Consider a Lie algebra $\mathfrak{g}$ with the number of the generators, $\#$. It follows from (4.8) that

$$[[T^a, T^b], T^d] = i f^{abc} [T^c, T^d] = -f^{abc} f^{cde} T^e \ . \tag{4.29}$$

Jacobi's identity tells then that

$$f^{abc} f^{cde} + f^{bdc} f^{cae} + f^{dac} f^{cbe} = 0 \ . \tag{4.30}$$

Now if we define

$$(\mathbb{T}^a)_{bc} \equiv i f^{bac} \ , \tag{4.31}$$

and consider them as matrices ($a, b, c = 1, 2, \ldots \#$), with bc considered as the row-column indices. It is a straightforward exercise to prove the matrix relations

$$[\mathbb{T}^a, \mathbb{T}^b] = i f^{abc} \mathbb{T}^c \ . \tag{4.32}$$

(The proof is left as an excercise). It follows that $\{\mathbb{T}^a\}$ form a particular representation of the algebra $\mathfrak{g}$, known as the *adjoint representation.*

4.2 Subalgebras and invariant subalgebras

Consider an algebra $X^a \in \mathfrak{g}$, and consider a subset $\mathfrak{h} \subset \mathfrak{g}$, and the generators belonging to it:

$$Y^{\dot{a}} \in \mathfrak{h} \subset \mathfrak{g} \ , \qquad X^a \in \mathfrak{g} \ . \tag{4.33}$$

If

$$[Y^{\dot{a}}, Y^{\dot{b}}] \in \mathfrak{h} \ , \tag{4.34}$$

then $\mathfrak{h}$ forms a *subalgebra* of $\mathfrak{g}$. Note that $\mathbf{0}$ and $\mathfrak{g}$ always form (trivial) subalgebras of any given algebra.

If furthermore

$$\forall \dot{a}, \quad \forall b, \qquad [Y^{\dot{a}}, X^b] = c_{\dot{d}} Y^{\dot{d}} \in \mathfrak{h} \tag{4.35}$$

then $Y^{\dot{a}}$ form an *invariant subalgebra* or *ideal* of $\mathfrak{g}$. $\mathbf{0}$ and $\mathfrak{g}$ are invariant subalgebras of $\mathfrak{g}$. If

$$[Y^{\dot{a}}, Y^{\dot{b}}] = 0, \qquad \forall Y^{\dot{a}} \in \mathfrak{h} \ , \tag{4.36}$$

then $Y^{\dot{a}}$ form an Abelian *invariant subalgebra* or Abelian *ideal*.

A Lie algebra $\mathfrak{g}$ without invariant subalgebras other than $\mathbf{0}$ or $\mathfrak{g}$, is *simple*. A Lie algebra without Abelian invariant subalgebras is *semi-simple*.

Note that a simple algebra is necessarily semi-simple, but not vice versa.

4.2.1 Invariant subalgebra generates invariant subgroup

Let $Y^{\dot{a}} \in Alg[H]$ where $Alg[H]$ is an invariant subalgebra and $X^a \in Alg[G]$, $H \subset G$, and

$$h = e^{i\alpha_{\dot{a}} Y^{\dot{a}}} \in H \ , \qquad \text{and} \qquad g = e^{i\beta_b X^b} \in G \ , \tag{4.37}$$

then

$$g^{-1} h g \in H \ . \tag{4.38}$$

The proof goes as follows.

$$g^{-1} h g = g^{-1} e^{i\alpha_{\dot{a}} Y^{\dot{a}}} g = e^{i\alpha_{\dot{a}} (g^{-1} Y^{\dot{a}} g)} \ . \tag{4.39}$$

Now

$$\begin{aligned}
g^{-1} Y^{\dot{a}} g &= (1 - i\beta X - \frac{1}{2!}\beta X \beta X + \ldots) Y^{\dot{a}} (1 + i\beta X - \frac{1}{2!}\beta X \beta X + \ldots) \\
&= Y^{\dot{a}} - i\beta_b [X^b, Y^{\dot{a}}] + \frac{(-i)^2}{2!}[\beta \cdot X, [\beta \cdot X, Y]] + \ldots \\
&+ \frac{(-i)^n}{n!}[\beta \cdot X, [\beta \cdot X, [\ldots [\beta \cdot X, Y]] \ldots] + \ldots \\
&= \gamma_{\dot{c}} Y^{\dot{c}} \in Alg[H]
\end{aligned} \tag{4.40}$$

by repeated use of (4.35).

4.2.2 Center of the algebra

The set of all elements of an algebra $\mathfrak{g}$ which commute with all elements of the algebra,

$$[T^{\dot{a}}, T^b] = 0 \ , \quad \forall T^b \in \mathfrak{g} \ , \tag{4.41}$$

form the center of the algebra $\mathcal{C}$

$$T^{\dot{a}} \in \mathcal{C} \subset \mathfrak{g} \ . \tag{4.42}$$

The center of the algebra generates the center of the group (but not necessarily, *vice versa*).

4.3 Killing form (metric tensor)

Define the metric

$$g^{ab} = g^{ba} \equiv f^{acd} f^{bdc} = f^{cad} f^{dbc} = -\mathrm{Tr}(\mathbb{T}^a \mathbb{T}^b) \ . \tag{4.43}$$

The *Cartan criterion* for an algebra to be semi-simple is that

$$\det |g^{ab}| \neq 0 \ . \tag{4.44}$$

The proof is easy: suppose that $Y^{\dot{a}} \in I \subset Alg[G]$ is an Abelian ideal, and $X^b \in Alg[G]$ is a generic element of the algebra. Then

$$g^{\dot{a}b} = f^{\dot{a}cd} f^{bdc} = f^{\dot{a}c\dot{d}} f^{bd\dot{c}} = f^{\dot{a}\dot{c}\dot{d}} f^{bd\dot{c}} = 0 \ , \qquad \forall b \tag{4.45}$$

hence

$$\det |g^{ab}| = 0 \ . \tag{4.46}$$

Let us study some examples:

(i) $so(3)$

$$[X^1, X^2] = iX^3 \ , \quad [X^2, X^3] = iX^1 \ , \quad [X^3, X^1] = iX^2 \ , \tag{4.47}$$

$$f^{abc} = \epsilon^{abc} \ , \tag{4.48}$$

so

$$g^{ab} = \epsilon^{acd} \epsilon^{bdc} = -2\delta^{ab} \ , \qquad \det g = -8 \ , \tag{4.49}$$

and $so(3)$ is semi-simple.

(ii) $so(2,1)$ algebra is

$$[X^1, X^2] = iX^3 \ , \quad [X^2, X^3] = -iX^1 \ , \quad [X^3, X^1] = iX^2 \ . \tag{4.50}$$

The structure constants are

$$f^{abc} = \mathcal{P}_{abc}\epsilon^{abc} = \begin{cases} -1, & (abc) = (231) \\ 1, & (abc) = (123) \\ 1, & (abc) = (312) \\ 0, & \text{otherwise} \end{cases} \tag{4.51}$$

$$g^{ab} = f^{acd} f^{bdc} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix} , \tag{4.52}$$

and

$$\det g = 8 \tag{4.53}$$

so $so(2,1)$ is semi-simple. Actually $so(2,1)$ and $so(3)$ are both simple.

(iii) E_2 is the group of translations

$$x \to x + a \ , \qquad y \to y + b \ , \qquad a, b \in \mathbb{R} \tag{4.54}$$

and rotations

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4.55}$$

and combinations theirof. The generators can be defined as

$$T^1 = -i\frac{\partial}{\partial x} \ , \qquad T^2 = -i\frac{\partial}{\partial y} \ , \tag{4.56}$$

for the two translations and

$$R = -i(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) \tag{4.57}$$

for the rotation. Indeed,

$$e^{iaT^1}\begin{pmatrix} x \\ y \end{pmatrix} \simeq (1 + iaT^1)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + a \\ y \end{pmatrix} , \tag{4.58}$$

$$e^{ibT^2}\begin{pmatrix} x \\ y \end{pmatrix} \simeq (1 + ibT^2)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y + b \end{pmatrix} , \tag{4.59}$$

$$e^{i\theta R}\begin{pmatrix} x \\ y \end{pmatrix} \simeq \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \theta y \\ -\theta x \end{pmatrix} . \tag{4.60}$$

The algebra of E_2 is then

$$[T^1, T^2] = 0 \ , \tag{4.61}$$

$$[T^1, R] = -\frac{\partial}{\partial x}(x\frac{\partial}{\partial y}) + x\frac{\partial}{\partial y}\frac{\partial}{\partial x} = -\frac{\partial}{\partial y} = -iT^2 \ , \tag{4.62}$$

$$[T^2, R] = iT^1 \ . \tag{4.63}$$

Thus

$$g^{ab} = f^{acd}f^{bdc} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad \det g = 0 \ . \tag{4.64}$$

Indeed, the translations $T^{1,2}$ form an Abelian ideal, as evident from the above algebra.

(iv) $so(4)$: It turns out (see later) the $SO(4)$ algebra is locally isomorphic to the direct product $su(2) \otimes su(2)$. $so(4)$ is thus semi-simple but not simple.

4.4 Casimir operators

Define

$$C \equiv g_{ab}T^aT^b \ , \qquad g_{ab} = (g^{-1})_{ab} \ , \quad g^{ab} = f^{ade}f^{bed} \ . \tag{4.65}$$

Then it can be shown that

$$[C, T^a] = 0 \ , \qquad \forall T^a \in \text{Alg[G]} \ . \tag{4.66}$$

The proof is straightforward, if somewhat lengthy. C takes a constant value inside a representation (Schur's lemma, see later), characterizing each representation. C is known as the quadratic Casimir operator.

Ex. 1: $so(3) \sim su(2)$. From the algebra it follows that the Casimir operator is given by

$$C = (T^1)^2 + (T^2)^2 + (T^3)^2 \ . \tag{4.67}$$

In the context of quantum mechanics these operators represent the square of the angular momentum, spin, or isospin operators. (See later).

Ex. 2: For $so(2,1)$ the Casimir operator is

$$C = (T^1)^2 + (T^2)^2 - (T^3)^2 \ . \tag{4.68}$$

5 Global and local aspects of Lie groups

As we have seen a Lie group G uniquely determines its Lie algebra $\mathfrak{g}$, but the opposite is not true in general. The Lie algebra describes the behavior of the group around the unit element. For instance, the Lie algebra of the $SU(2)$ and $SO(3)$ groups are isomorphic,

$$[t^a, t^b] = i\epsilon^{abc}t^c \ , \tag{5.1}$$

that is, $su(2) \sim so(3)$, but their global properties differ. It turns out that each element of $SO(3)$ has exactly two inverse images in $SU(2)$, the kernel of the unit element $\mathbb{1}$ in $SO(3)$ is $\{\mathbb{1}, -\mathbb{1}\}$. In the following a relevant concept is idea of group manifold, that is, to consider the set of all possible values the group parameters $\{\alpha_1, \alpha_2, \ldots\}$ as the coordinates of a manifold. In the case of $SU(2)$ group, the most general element satisfying

$$U^\dagger U = \mathbb{1} \ , \qquad \det U = 1 \ , \tag{5.2}$$

can be parametrized as

$$U = e^{i \frac{\tau^a}{2} \alpha^a} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \ , \qquad |a|^2 + |b|^2 = 1 \ . \tag{5.3}$$

It represents S^3.

5.1 Covering map

Such a relation is a special case of the *covering map* between two spaces. Consider two spaces M and N and a map

$$M \longrightarrow N \ , \qquad \forall x \in M \longrightarrow y = f(x) \in N \ . \tag{5.4}$$

If the neighborhood of each point $y \in N$, U_y, has the inverse image which consists of a disjoint, numerable set of open sets

$$f^{-1}(U_y) = V_{1y} \cup V_{2y} \cup V_{3y} \cup \ldots \ , \tag{5.5}$$

N is known as the base space; M is called the covering space of N. The number of the inverse images $x_1, x_2, \ldots \in M$ of each point $y \in N$ is the number of sheets of the map.

When the covering space M is simply connected,

$$\pi_1(M) = 1 \ , \tag{5.6}$$

(see the next section more about the first homotopy group π_1) it is called a **universal covering space** of N.

ex. 1: $\quad M = \mathbb{R}$, $N = S^1$ and

$$f(t) = e^{2\pi i t} \ , \qquad t \in \mathbb{R} \ . \tag{5.7}$$

The number of the sheets is infinite.

ex. 2: $M = S^1$, $N = S^1$ and

$$f(z) = z^n \ , \qquad |z| = 1 \quad \to \quad |f(z)| = 1 \, . \tag{5.8}$$

This is a n-sheeted cover. This can be reinterpreted as a map from $\mathbb{R}^2_{\{0\}} = \mathbb{C}^*$ to $\mathbb{C}^*$.
ex. 3: Consider the n-dimensional sphere $M = S^n$

$$x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1 \tag{5.9}$$

and the n-dimensional real projective space, $N = \mathbb{R}P^n$, defined by

$$(x_1, x_2, \ldots x_{n+1}) \sim \lambda(x_1, x_2, \ldots x_{n+1}) \ , \qquad \forall \lambda \in \mathbb{R}_{\{0\}} \ . \tag{5.10}$$

As the latter is equivalent to the points of S^n in which the equivalence relation $x \sim -x$ is imposed, the map $M \to N$ is a two-sheeted covering (or a double cover).

The case we mentioned already is

$$M = SU(2) \sim S^3 \ , \qquad N = SO(3) \sim \mathbb{R}P^3 \tag{5.11}$$

The group $SU(2)$ is thus a double cover of $SO(3)$. As $SU(2)$ is simply connected, it is a universal covering of $SO(3)$.
ex. 4: Another example in the context of group theory is

$$SU(2) \times SU(2) \longrightarrow SO(4) \tag{5.12}$$

which is a two-sheeted map (the kernel of the map is $(\mathbb{1}, \mathbb{1})$ and $(-\mathbb{1}, -\mathbb{1})$.) Again their algebras are isomorphic:

$$su(2) \times su(2) \sim so(4) \ . \tag{5.13}$$

ex. 5: $M = \mathbb{R}^n$, $N = T^n = \mathbb{R}^n/\mathbb{Z}^n$. $\mathbb{R}^n \to T^n$ is a infinite-sheeted covering. The case $n = 2$ it is a map from a plane to a torus.

5.2 Connected components

Another issue related to the global aspects of a group, as compared to the local properties (the algebra), is the connectedness of G as a manifold. In a manifold G, if two points a and b are connected by a continuous path, included entirely in G:

$$f(t) \in G \ , \qquad (\forall t, \quad 0 \leq t \leq 1), \tag{5.14}$$

such that

$$f(0) = a, \qquad f(1) = b \ , \tag{5.15}$$

then a and b are archwise connected. The number of the components connected is indicated as

$$\pi_0(G) \ . \tag{5.16}$$

For example, for the orthogonal groups,

$$O^t O = \mathbb{1} \ , \tag{5.17}$$

so

$$\det O = \pm 1 \ . \tag{5.18}$$

The special orthogonal group is defined as

$$O^t O = \mathbb{1} \ , \qquad \det O = 1 \ . \tag{5.19}$$

So the number of the connected components are indicated by π_0: thus

$$\pi_0(O(N)) = 2 \ , \qquad \pi_0(SO(N)) = 1 \ . \tag{5.20}$$

for the orthogonal and special orthogonal groups, respectively.

6 Homotopy groups

The global aspects of different Lie groups are further characterized by the general homotopy groups, the first of which is the fundamental group already mentioned. Here the fundamental group is introduced more systematically, and an introductory discussion on higher homotopy groups is given.

6.1 Fundamental group

Let us consider a manifold M which is archwise connected. Two paths in M,

$$\gamma_1(t) \ , \quad 0 \le t \le 1 \ ; \qquad \gamma_2(t) \ , \quad 1 \le t \le 2 \ ; \tag{6.1}$$

can be "multiplied", if

$$\gamma_1(1) = \gamma_2(1) \ . \tag{6.2}$$

The product $\gamma_2 \circ \gamma_1$ is defined as :

$$q(t) = \begin{cases} \gamma_1(t) & \text{if,} \quad 0 \le t \le 1 \ , \\ \gamma_2(t) & \text{if,} \quad 1 \le t \le 2 \ . \end{cases} \tag{6.3}$$

The inverse of $\gamma(t)$ can be defined as

$$\gamma^{-1}(t) = \gamma(1-t) \ , \qquad 0 \leq t \leq 1 \ , \tag{6.4}$$

or

$$\tilde{\gamma}^{-1}(t) \equiv \gamma(2-t) \ , \qquad 1 \leq t \leq 2 \ , \tag{6.5}$$

such that

$$\gamma^{-1} \circ \gamma = e \ , \qquad e(t) \equiv e = \gamma(0), \quad \forall t \ . \tag{6.6}$$

Two paths γ_1 and γ_2 are said to be equivalent, or homotopic, if

$$\gamma_1(t(\tau)) = \gamma_2(\tau) \ , \qquad \frac{\partial t}{\partial \tau} > 0 \ . \tag{6.7}$$

The set of all closed oriented paths in M starting and ending at the point x_0, is denoted as $\Omega(x_0, M)$. They can be classified in homotopically equivalence classes. The set of all oriented paths connecting from the points x_0 to x, is denoted as $\Omega(x_0, x, M)$. Note that two closed paths in $\Omega(x_0, M)$ can be multiplied. By substituting each of the paths by the class of equivalent paths, one can define the product of two classes of homotopic paths.

Theorem: The homotopic classes of orientable paths belonging to $\Omega(x_0, M)$ form a group under multiplication, in which an element corresponds to a class of equivalent paths. The inverse element corresponds to a class of inverse paths. The unit element corresponds to the class of paths equivalent to $e(t) \equiv x_0$. This group, indicated by $\pi_1(M, x_0)$, is known as *the fundamental group (or the first homotopy group)* of M. Actually it can be shown that the group structure does not depend on the initial and final point of the loops, $\pi_1(M, x_0) = \pi_1(M, x_1)$, therefore it is often indicated simply as

$$\pi_1(M) \ . \tag{6.8}$$

Examples:

(1) For any contractible space M (e.g., $\mathbb{R}^n$, a disk, D^n, $n = 1, 2, \ldots$),

$$\pi_1(M) = 1 \ . \tag{6.9}$$

(2) As for spheres,

$$\pi_1(S) = \mathbb{Z} \ ; \qquad \pi_1(S^n) = 1 \ , \quad n \geq 2 \ . \tag{6.10}$$

(3) The plane from which a point is removed, $\mathbb{R}^2/\{0\}$, is homotopically equivalent to a circle.

$$\pi_1(\mathbb{R}^2/\{0\}) = \mathbb{Z} \ . \tag{6.11}$$

(4) For a torus T,

$$\pi_1(T) = \mathbb{Z} \times \mathbb{Z} \ . \tag{6.12}$$

Each element of $\pi_1(T)$ is characterized by two integers (m, n), representing the winding numbers in one or the other nontrivial cycles over T.

(5) The three dimensional space from which a circle or a line is removed

$$\pi_1(\mathbb{R}^3/S) = \mathbb{Z} \ ; \qquad \pi_1(\mathbb{R}^3/R) \sim \pi_1(S) = \mathbb{Z} \ . \tag{6.13}$$

(6) The two dimensional plane from which two points a, b are removed is equivalent to two circles attached at a point, $S \vee S$. The fundamental group in this case is

$$\pi_1(S \vee S) = \mathbb{Z} * \mathbb{Z} \ , \tag{6.14}$$

where the free product (indicated by $*$) is generated by two elementary windings around the two circles. If α and β are generic, nontrivial elements of $\pi_1(S_1)$ $(m \neq 0)$ and of $\pi_1(S_2)$ $(n \neq 0)$, the elements of $\pi_1(S \vee S)$ are

$$\mathbf{1},\ \alpha,\ \beta,\ \alpha\beta\,,\beta\alpha,\ \alpha\beta\alpha\,,\beta\alpha\beta\,,\alpha\beta\alpha\beta\,,\ldots \tag{6.15}$$

Theorem:

If M is a Lie group, $\pi_1(M)$ is commutative.

6.2 Monodromy groups

Consider a map $M \to N$, with the base space N and the covering space M (see section 5.1). Each point $y \in N$ has a set of inverse images, $x_1, x_2, \ldots$, such that

$$f(x_i) = y \ , \qquad i = 1, 2, \ldots \tag{6.16}$$

Let us consider a closed path $\gamma(t)$ starting and ending at $y = y_0$ in N and consider the associated fundamental group $\pi_1(N, y_0)$. If the closed path $\gamma(t)$ is tiny, around y_0, each inverse image $\mu_i(t)$ will make a closed path around $x_i = f_i^{-1}(y_0)$. For finite closed loop $\gamma(t)$ in N, however, an inverse image x_i may not come back to itself but in general ends up at one of other inverse images, x_{σ_i}. In other words the inverse image of the closed path $\gamma(t)$ may not be a closed path, but a path connecting x_i to x_{σ_i}. The points make permutations

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} \xrightarrow{\gamma^{-1}} \begin{pmatrix} & \sigma(\gamma) & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} . \tag{6.17}$$

The $k \times k$ (k being the number of the sheets) matrix $\sigma(\gamma)$ is known as the monodromy matrix, and the rearrangement of the inverse images (6.17) as the monodromy transformation. Clearly, $\sigma(\gamma)$ depends only on the class of equivalent closed paths in N so is a function of $\pi_1(N)$. As

$$\sigma(\gamma_1 \circ \gamma_2) = \sigma(\gamma_1)\sigma(\gamma_2)\ , \tag{6.18}$$

σ's form a matrix group; it is a group homomorphism from the fundamental group $\pi_1(N)$. It is denoted as

$$\sigma(\pi_1(N)) \tag{6.19}$$

and is called the monodromy group.

Ex. 1 The map $\mathbb{R} \to S$,

$$x \to y = e^{2\pi i x}\ . \tag{6.20}$$

Clearly,

$$\sigma(m): \quad x \to x + m\ , \qquad m = 0, \pm 1, \pm 2, \ldots \tag{6.21}$$

$$\sigma(\pi_1(N)) = \mathbb{Z}\ , \tag{6.22}$$

the group of integers under addition.

Ex. 2 The map $S \to S$,

$$w = z^n\ , \qquad |z| = |w| = 1\ , \tag{6.23}$$

is a n-sheeted map from a circle to a circle. The inverse images of $1 \in S$ in the second circle (the base) are

$$e^{2\pi i k/n}\ , \qquad k = 0, 1, 2, \ldots, n-1\ . \tag{6.24}$$

$\pi_1(S) = \mathbb{Z}$ in the base space is mapped to a cyclic group $\mathbb{Z}_n$ generated by

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & \ldots & n-1 \\ 1 & 2 & 3 & \ldots & 0 \end{pmatrix}\ , \tag{6.25}$$

or

$$\sigma = \begin{pmatrix} 0 & & & & 0 & 1 \\ 1 & 0 & & \ldots & & 0 \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & \ddots & & \\ 0 & & & \vdots & & \\ & & & & 1 & 0 \end{pmatrix}\ , \tag{6.26}$$

acting on the n inverse images of 1. Thus

$$\sigma(\pi_1(S)) = \mathbb{Z}_n\ . \tag{6.27}$$

6.3 Projective spaces, $\mathbb{R}P^n$, $\mathbb{C}P^n$

The real projective space $\mathbb{R}P^n$ is defined by the points

$$(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \,, \qquad (x_1, \ldots, x_{n+1}) \sim c\,(x_1, \ldots, x_{n+1}) \,, \tag{6.28}$$

for arbitrary nonvanishing real number

$$c \in \mathbb{R}/\{0\} \,. \tag{6.29}$$

The elements of $\mathbb{R}P^1$ are the straight lines in a plane passing through the origin. The elements of $\mathbb{R}P^2$ are the straight lines in $\mathbb{R}^3$ passing through the origin. They are equivalent to the points on S^2 in which two antipodal points are identified, $S^2/\mathbb{Z}_2$. Thus

$$\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2 \,. \tag{6.30}$$

Similarly,

$$\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2 \,. \tag{6.31}$$

The points of the complex projective space $\mathbb{C}P^n$ are defined by

$$(z_1, z_2, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \,, \qquad (z_1, \ldots, z_{n+1}) \sim \lambda(z_1, \ldots, z_{n+1}) \,, \tag{6.32}$$

$$\lambda \in \mathbb{C}/\{0\} \,. \tag{6.33}$$

We need $n+1$ coordinate neighborhoods to cover the entire $\mathbb{C}P^n$ space. Introduce the j-th neighborhood U_j:

$$(z_1, z_2, \ldots, z_{n+1}) \in U_j \,, \qquad \text{if} \qquad z_j \neq 0 \,. \tag{6.34}$$

One can introduce the local coordinates there by $\zeta_i^{\{j\}} \equiv z_i/z_j$,

$$(z_1, z_2, \ldots, z_{n+1}) \sim (\zeta_1^{\{j\}}, \ldots, \zeta_{j-1}^{\{j\}}, 1, \zeta_{j+1}^{\{j\}}, \ldots \zeta_{n+1}^{\{j\}}) \in \mathbb{C}^n \,, \qquad \zeta_i \in \mathbb{Z} \,. \tag{6.35}$$

In an overlap region of the j-th and k-th neighborhoods, $U_j \cap U_k$ $z_j \neq 0$, $z_k \neq 0$, the local coordinates are related by

$$\zeta_i^{\{k\}} = \frac{z_j}{z_k}\,\zeta_i^{\{j\}} \,, \qquad \forall i \,. \tag{6.36}$$

These relations define the differential manifold $\mathbb{C}P^n$.

The simplest example of $\mathbb{C}P^n$ is $n = 1$:

$$\mathbb{C}P^1 \sim S^2 \,, \tag{6.37}$$

as is well known. The local coordinates (”North” and ”South”) correspond to two stereographic projections from the north or south poles onto the plane containing the equator.

$\mathbb{C}P^n$ is simply connected:

$$\pi_1(\mathbb{C}P^n) = 1 \ . \tag{6.38}$$

6.4 Higher homotopy groups

The i-th homotopy group $\pi_i(M)$ represents classes of homotopically equivalent maps from S^i to the space M. It can be defined as classes of homotopically inequivalent maps from a i-dimensional disk D^i to M

$$D^i \to M \ , \qquad D^i = \{(x^0, x^1, \dots x^i)\,, \quad (x^0)^2 + (x^1)^2 + \dots (x^i)^2 = 1\ , \quad x_0 \geq 0\} \tag{6.39}$$

such that the boundary circle (sphere) S^{i-1} (the equator)

$$S^{i-1} = \{(0, x^1, \dots x^i)\,, \quad (x^1)^2 + \dots (x^i)^2 = 1\} \tag{6.40}$$

is mapped to a fixed point $X_0 \in M$. Equivalently, since a disk D^i in which the boundary circle is considered to be a point is a sphere S^i, $\pi_i(M)$ can be defined to be the maps

$$S^i \to M \ , \qquad s_0 \in S^i \to X_0 \in M \ . \tag{6.41}$$

See Fig. 4.

The products in $\pi_i(M)$ can be defined as follows. Consider two maps α and β from a sphere S^i to M belonging to classes in $\pi_i(M)$, as in Fig. 4. Consider S^i as consisting of the northern hemisphere D^+ ($x^0 \geq 0$, see Eq. (6.39)) and southern hemisphere D^- ($x^0 \leq 0$) whose boundaries are the equator circle S^{i-1} ($x^0 = 0$), Eq. (6.40). By a map in which the points of the equator are transformed to a point, s_0, one defines a map ψ from S^i to $S^i \times S^i$ which are attached at a point s_0. One can define the product of α and β as a map from S^i to M:

$$\alpha\beta(x) \equiv \begin{cases} \alpha\,\psi\ , & x \in D^+ \subset S^i \\ \beta\,\psi\ , & x \in D^- \subset S^i\ , \end{cases} \tag{6.42}$$

as shown in Fig. 5. Clearly, $\alpha\beta(s_0) = X_0$.

It turns out that for $i > 1$ the homotopy group $\pi_i(M)$ is commutative:

$$\alpha\beta \sim \beta\alpha \ . \tag{6.43}$$

This can be proven by rotating S_i of the left of Fig. 5 continuously, around the straight axis passing through the center and the point s_0. At the end of angle π rotation, D_+ and D_- are interchanged.

The definition of an inverse element and the demonstration that the higher homotopy groups defined thus satisfy the group axioms, are given, for example, in [6]. A more detailed discussion of the higher homotopy groups goes beyond the scope of this lecture note. We

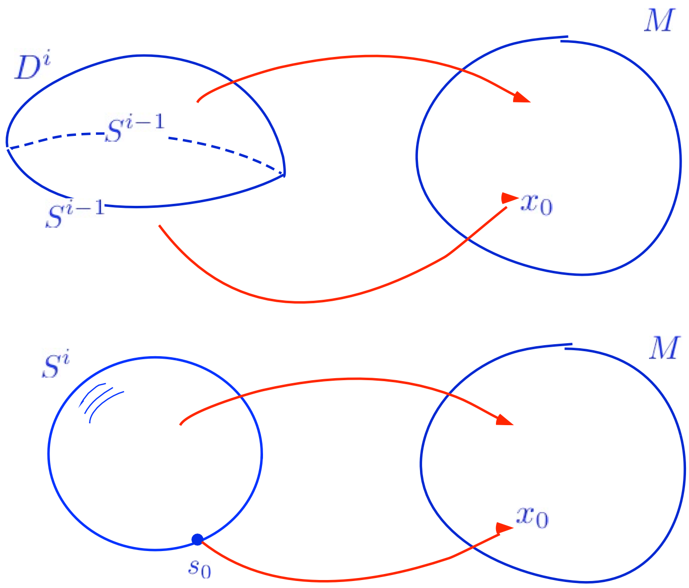


Figure 4:

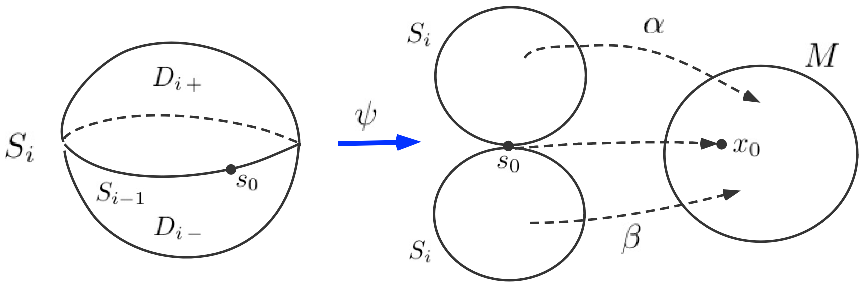


Figure 5:

limit ourselves to giving a few examples, some general results and several useful results in the cases $M = S^n$ or $M =$ a Lie group, below.

Example: $\pi_2(S^2) = \mathbb{Z}$

Nontrivial "standard" maps from S^2 to S^2 can be defined as follows. Let

$$x = (\theta, \phi) \ , \quad \text{or} \quad \hat{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \ , \qquad 0 \le \theta \le \pi \ , \quad 0 \le \phi \le 2\pi \ , \tag{6.44}$$

be the points on the first sphere; and the coordinates of the second sphere be

$$y = (\Theta, \Phi) \ , \quad \text{or} \qquad \hat{R} = (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta) \ . \tag{6.45}$$

Clearly the map $x \to y$:

$$\Theta = \theta \ , \qquad \Phi = n\,\phi \ , \qquad n = 0, \pm 1, \pm 2, \ldots \tag{6.46}$$

covers the second sphere n times (including the orientation), as the first sphere is scanned once: $\theta = 0 \to \pi$, $\phi = 0 \to 2\pi$.

Example: $\pi_3(S^2) = \mathbb{Z}$ (Hopf map)

The point of the base S^3 can be parametrized by

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1 \tag{6.47}$$

or

$$|z^0|^2 + |z^1|^2 = 1 \ , \qquad z^0 = x^1 + ix^2 \ , \quad z^1 = x^3 + ix^4 \ , \tag{6.48}$$

whereas S^2 has the coordinates (ξ_1, ξ_2, ξ_3),

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = 1 \ . \tag{6.49}$$

A nontrivial Hopf map is

$$\xi_1 = 2\,\mathrm{Re}\, z^0 (z^1)^* = 2(x^1x^3 + x^2x^4) \ ; \qquad \xi_2 = 2\,\mathrm{Im}\, z^0 (z^1)^* = 2(x^2x^3 - x^1x^4) \ ;$$

$$\xi_3 = |z^0|^2 - |z^1|^2 = (x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2 \ . \tag{6.50}$$

One has, indeed,

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = (|z^0|^2 + |z^1|^2)^2 = 1 \ . \tag{6.51}$$

Examples:

$$\pi_2(CP^n) = \mathbb{Z} \ , \qquad \pi_j(CP^n) = \pi_j(S^{2n+1}) \ , \quad j > 2 \ ; \tag{6.52}$$

$$\pi_{2n+1}(CP^n) = \mathbb{Z} \ . \tag{6.53}$$

6.4.1 Some general results

$\pi_j(M)$ is commutative for $j > 1$.

$\pi_i(M) = \mathbf{0}$ for any contractible M.

$\pi_1(M)$ is commutative if M is a Lie group.

$\pi_j(M \times N) = \pi_j(M) \times \pi_j(N)$, (for a direct product space) .

6.4.2 $M = S^n$

$$\pi_i(S^n) = \mathbf{0} \,, \qquad i < n \,. \tag{6.54}$$

$\pi_{n+k}(S^n)$ for some values of n, k are shown in Table 7 (from Monastyrsky, "Topology of Gauge Fields and Condensed Matter").

Table 7:

k \ n	2	3	4	5	6	7
1	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
3	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}\oplus\mathbb{Z}_{12}$	$\mathbb{Z}_{24}$	$\mathbb{Z}_{24}$	$\mathbb{Z}_{24}$
4	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$\mathbf{0}$
5	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbf{0}$
6	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2 \oplus \mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
7	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$	$\mathbb{Z}_{30}$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{120}$

Some $\Pi_{n+k}(S^n)$, with $n > k + 1$ is shown in Table 8 [6]

Table 8:

k	$\Pi_{n+k}(S^n)$		k	$\Pi_{n+k}(S^n)$
0	$\mathbb{Z}$		8	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
1	$\mathbb{Z}_2$		9	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$
2	$\mathbb{Z}_2$		10	$\mathbb{Z}_2$
3	$\mathbb{Z}_{24}$		11	$\mathbb{Z}_{504}$
4	$\mathbf{0}$		12	$\mathbf{0}$
5	$\mathbf{0}$		13	$\mathbb{Z}_3$
6	$\mathbb{Z}_2$		14	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
7	$\mathbb{Z}_{240}$		15	$\mathbb{Z}_{480} \oplus \mathbb{Z}_2$

6.4.3 $M =$ a Lie group

$\pi_k(M)$ for some Lie groups are shown in Table 9 (taken from "Current Algebra and Anomalies", S.B. Treiman, R. Jackiw, B. Zumino and E. Witten). Also, a random collection of interesting results is given below. These remarkable results show how the global properties of different Lie groups are characterized and described by the higher homotopy groups. These homotopy-group related results play exceedingly important roles in the understanding of *dynamics* of Abelian and nonAbelian gauge theories. Physics of solitons and related objects such as instantons are the central players for uncovering the nonperturbative dynamics of these theories, believed to describe the real world of fundamental interactions and condensed matters.

Table 9:)

k \ M	$U(N)$	$O(N)$	$Sp(N)$
	$N > \frac{k}{2}$	$N > k+1$	$N > \frac{k-2}{4}$
0	$\mathbf{0}$	$\mathbb{Z}_2$	$\mathbf{0}$
1	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbf{0}$
2	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
3	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
4	$\mathbf{0}$	$\mathbf{0}$	$\mathbb{Z}_2$
5	$\mathbb{Z}$	$\mathbf{0}$	$\mathbb{Z}_2$
6	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
7	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
8	$\mathbf{0}$	$\mathbb{Z}_2$	$\mathbf{0}$
period*	2	8	8

$$\pi_i(U(N)) = \pi_i(S^1) \oplus \pi_i(SU(N)) \ . \tag{6.55}$$

$$\pi_4(SU(3)) = \mathbf{0}, \qquad \pi_5(SU(3)) = \mathbb{Z} \ . \tag{6.56}$$

$$\pi_i(SO(3)) = \begin{cases} \mathbb{Z}_2 & i = 1 \ , \\ \mathbf{0} & i = 2 \ , \\ \mathbb{Z} & i = 3 \ . \end{cases} \tag{6.57}$$

$$\pi_i(U(1)) = \pi_i(SO(2)) = \begin{cases} \mathbb{Z} & \text{if} \quad i = 1 \ ; \\ \mathbf{0} & \text{if} \quad i > 1 \ . \end{cases} \tag{6.58}$$

$$\pi_i(SO(4)) = \pi_i(SO(3)) \oplus \pi_i(S^3) = \begin{cases} \mathbb{Z}_2 & i = 1 \ , \\ \mathbf{0} & i = 2 \ , \\ \mathbb{Z} \oplus \mathbb{Z} & i = 3 \ . \end{cases} \tag{6.59}$$

Part III

Representation theory

The representation theory is at the heart of the group theory. The specific characters of different groups and algebras are encoded in the properties of the representation matrices, and in various beautiful theorems governing them. The results of the representation theory severely restrict possible ways symmetries are realized in physics.

7 Definition

As already anticipated, the map from a group G to the matrices acting on an N dimensional vector space (called representation space),

$$G \longrightarrow R = \{M_{N\times N}, V\} \; ; \tag{7.1}$$

$$g \in G \longrightarrow M(g) \; , \tag{7.2}$$

such that the group multiplication rules are respected,

$$g_1 \circ g_2 \longrightarrow M(g_1 \circ g_2) = M(g_1)M(g_2) \; , \tag{7.3}$$

is called a *representation*

Example

In the case of the permutation group S_3, mentioned in Section 1.3, a representation is

$$M(e) = \mathbb{1}_{3\times 3} \; , \quad M(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; , \quad M(23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \; ,$$

$$M(31) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \; , \quad M(123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \; , \quad M(321) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \; . \tag{7.4}$$

The multiplication rules for S_3 are indeed respected, $M(12)M(23) = M(123)$, etc.

7.1 Equivalent representations

Consider two representations $R = \{M, V\}$ and $\tilde{R} = \{\tilde{M}, \tilde{V}\}$ of a group G, with the vector spaces V and $\tilde{V}$ having the same dimension. If there exists a regular matrix, S, such that

$$SM(g)S^{-1} = \tilde{M}(g), \qquad \forall g \in G \ , \tag{7.5}$$

then the two representations are said to be *equivalent*,

$$R \sim \tilde{R} \ . \tag{7.6}$$

7.2 Direct-product representations

Given two representations, $R = \{M, V\}$ and $\tilde{R} = \{\tilde{M}, \tilde{V}\}$ of a group G, their direct product,

$$R \otimes \tilde{R} = \{M \otimes \tilde{M}, V \oplus \tilde{V}\} \tag{7.7}$$

clearly is another representation. This is a very general statement: the dimensions of the two representations are arbitrary, and R and $\tilde{R}$ may be reducible or irreducible (see below). In general, however, when R and $\tilde{R}$ are two nontrivial irreducible representations, their direct product $R \otimes \tilde{R}$ will not be irreducible.

8 Reducible and irreducible representations

Let $R = \{M, V\}$ be a representation of a group G,

$$\forall g \in G \longrightarrow M(g) \ . \tag{8.1}$$

Let U be a subspace of V

$$U \subset V \ . \tag{8.2}$$

If

$$\forall g \in G \ , \qquad M(g)U \subset U \ , \tag{8.3}$$

and

$$U \neq \emptyset, \qquad U \neq V \ , \tag{8.4}$$

that is, *if there is a nontrivial invariant subspace*, then the representation is *reducible*. If the condition (8.3) implies

$$U = \emptyset \qquad \text{or} \qquad U \equiv V \ , \tag{8.5}$$

then R is *irreducible*.

If

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_n \tag{8.6}$$

such that

$$M(g) \sim \begin{pmatrix} M_1(g) & & & \\ & M_2(g) & & \\ & & \ddots & \\ & & & M_n(g) \end{pmatrix} , \qquad \forall g \tag{8.7}$$

such that each $\{M_i, V_i\}$ is irreducible, then $\{M, V\}$ is said to be completely reducible. In other words, a completely reducible representation is a representation which is equivalent to a direct product of representations,

$$V \sim V_1 \oplus V_2 \oplus V_3 \ldots \ , \tag{8.8}$$

$$\{M, V\} \sim \{M_1(g), V_1\} \otimes \{M_2(g), V_2\} \otimes \{M_3(g), V_3\} \otimes \ldots \ . \tag{8.9}$$

Example

An example of a reducible but not completely reducible representation is given by the group of triangular matrices, (1.18):

$$T = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} , \qquad a, b, c \in \mathbf{R}, \quad a > 0, \quad c > 0 \right\} , \tag{8.10}$$

Clearly the subspace $\begin{pmatrix} * \\ 0 \end{pmatrix}$ is an invariant subspace, but T cannot be brought to a block-diagonal form.

9 Unitary representations

Suppose

$$M^\dagger = M^{-1} , \qquad M^\dagger M = \mathbb{1} , \qquad \forall g \in G . \tag{9.1}$$

If in V a scalar product between two vectors $z, w \in V$ is defined

$$\langle w, z \rangle = w^\dagger \cdot z , \tag{9.2}$$

such that

$$\langle w, z \rangle \xrightarrow{g} \langle Mw, Mz \rangle = \langle w, z \rangle , \tag{9.3}$$

then $R = \{M, V\}$ is a *unitary representation.* For a unitary representation matrix $M(g)$

$$(M(g))^\dagger = M(g^{-1}) . \tag{9.4}$$

Here are some theorems.

Theorem: Every representation of a finite group is equivalent to a unitary representation.

The proof is not difficult, but interesting. See [3] .

Theorem: Any representation of a compact group is equivalent to a unitary representation.

Let $\langle x|y\rangle$ be an arbitrary Hermitian, semipositive definite scalar product in V:

$$\langle x|y\rangle \ ; \qquad \langle x|x\rangle \geq 0 \ ; \qquad \langle x|y\rangle^* = \langle y|x\rangle \ , \qquad x, y \in V \ . \tag{9.5}$$

Now if a new scalar product is defined by

$$(x, y) = \int_G dg \langle M(g)x|M(g)y\rangle \tag{9.6}$$

Then by using the fact that for a compact group the integration measure is unique and left and right invariant, it follows that for $\forall g_0 \in G$,

$$(M(g_0)x, M(g_0)y) = \int_G dg\langle M(gg_0)x|M(gg_0)y\rangle = \int_G dg\langle M(g)x|M(g)y\rangle = (x, y) \ : \tag{9.7}$$

that is $M(g_0)$ is a unitary representation with respect to the scalar product, (x, y).

Theorem: A unitary representation of any group is completely reducible.

Let (M, V) be a unitary representation of a group G. If it is irreducible, it is of course completely reducible. If not, then there is an invariant subspace $U \subset V$, $U \neq 0, \neq V$, such that

$$M(g)U = U \ . \tag{9.8}$$

Let $\hat{U} \subset V$ be the complement of U,

$$\hat{U} = \{z; \ \langle z, w\rangle = 0 \ , \quad \forall w \in U\} \ , \tag{9.9}$$

or simply,

$$\langle \hat{U}, U\rangle = 0 \ . \tag{9.10}$$

Then $\hat{U}$ is also an invariant subspace of G, as

$$\langle M(g)\hat{U}, U\rangle = \langle \hat{U}, M(g)^\dagger U\rangle = \langle \hat{U}, M(g^{-1})U\rangle = \langle \hat{U}, U\rangle = 0 \ . \tag{9.11}$$

If $\{M, \hat{U}\}$ is irreducible, the proof is done. Otherwise, the procedure can be repeated, until the form (8.7) is reached.

9.1 Real and complex representations

Consider a representation

$$G \longrightarrow R = \{M, V\} \;, \tag{9.12}$$

or

$$g \in G \longrightarrow M(g) \in R \;, \qquad M(g_1 \circ g_2) = M(g_1)M(g_2) \;. \tag{9.13}$$

If $M(g)$ are complex matrices, then

$$g \in G \longrightarrow M^*(g) \tag{9.14}$$

also form a representation. If there exists a fixed matrix S such that

$$M(g) = SM(g)^* S^{-1}, \qquad \forall g \in G \;, \tag{9.15}$$

then the representatiothe ns $R = \{M, V\}$ and $R^* = \{M^*, V^*\}$ are equivalent, and the representation R is said to be *real*. Otherwise, a representation is *complex*.

In the case of a Lie group,

$$M = e^{iT^a \alpha^a} \tag{9.16}$$

the condition for the reality of a representation is that the generators in that representation satisfy

$$T^a = -ST^{a\,*} S^{-1} \;, \qquad \forall a \tag{9.17}$$

with some fixed S. See Subsection 12.4 for examples in the $SU(2)$ group.

Actually there is a finer distinction between a *real representation* and a *pseudoreal representation*. From the reality condition and Hermiticity of T^a, it follows that

$$(T^a)^T = -S\, T^a\, S^{-1} \;. \tag{9.18}$$

By taking the transpose

$$T^a = -(S^{-1})^T\, (T^a)^T\, S^T \;. \tag{9.19}$$

Insert (9.18) now in (9.19) to get

$$T^a = (S^{-1})^T\, S\, T^a\, S^{-1}\; S^T \;, \tag{9.20}$$

or

$$S^{-1}\; S^T\; T^a = T^a\, S^{-1}\; S^T \;, \tag{9.21}$$

namely, $S^{-1}\; S^T$ commutes with all T^a's. By Schur's lemma, see below, it follows that

$$S^{-1}\; S^T = \lambda\, \mathbb{1} \;, \qquad \therefore \quad S^T = \lambda\, S \;. \tag{9.22}$$

But as $(S^T)^T = S$, one gets

$$\lambda^2 = 1\,, \qquad \therefore \quad S^T = \pm S \ : \tag{9.23}$$

the matrix S is either symmetric or antisymmetric. When S is symmetric, the representation is called *real*; when it is antisymmetric, the representation is *pseudoreal*. Thus the fundamental representation of $SU(2)$ group is pseudoreal, as $S = \tau^2$ is antisymmetric.

10 Schur's Lemma

One of the powerful criteria for a representation to be irreducible is Schur's Lemma.

10.1 Schur's Lemma I

Let $\{M, V\}$ and $\{N, W\}$ be two irreducible representations of the group G, and let a linear map $A : V \to W$

$$x \in V \longrightarrow y = Ax \in W \ , \tag{10.1}$$

be such that

$$N(g)A = AM(g) \ , \qquad \forall g \in G \ . \tag{10.2}$$

Then either

$$A \equiv \mathbf{0} \ , \tag{10.3}$$

or $A : V \to W$ is a one-to-one map (i.e., $\{M, V\} \sim \{N, W\}$).

The proof goes as follows. Let K be the kernel of the map A:

$$K = \{x \in V, \qquad Ax = 0\} \ . \tag{10.4}$$

It forms an invariant subspace, as

$$AMx = NAx = 0 \ , \qquad \therefore \quad Mx \in K \ . \tag{10.5}$$

Since by assumption $\{M, V\}$ is an irreducible representation, either $K = V$ or $K = 0$. If $K = V$ then $A = \mathbf{0}$ which is one of the possibilities.

If $A \neq \mathbf{0}$, then $K = 0$, which means that

$$A\,x = 0 \to x = 0 \ . \tag{10.6}$$

That is if $Ax = Ay$, $A(x - y) = 0$ so $x = y$. Thus each $y \in W$ has a unique inverse in V. It remains to show that that each point in V has an image in W, that is

$$AV = W \ . \tag{10.7}$$

To show it, we note that

$$N(Ax) = AMx \ , \qquad \forall x \in V \ , \quad \forall g \in G \ , \tag{10.8}$$

thus AV is an invariant subspace of W. As $\{N, W\}$ is irreducible, it means that either $AV = 0$ or $AV = W$. As $A \neq 0$, it follows that $AV = W$, which proves that the map $V \to W$ is one-to-one.

10.2 Schur's Lemma II

A second theorem, also known as Schur's lemma, is valid for a complex irreducible representation, $\{M, V\}$. If $\{M_N, V\}$ is a complex irreducible representation of a group G, and if there exists a $N \times N$ matrix A such that

$$M_N(g)A = AM_N(g); \, , \qquad \forall g, \tag{10.9}$$

then

$$A = a \, \mathbb{1}_N \ , \qquad a \in \mathbb{C} \ . \tag{10.10}$$

Let a be an eigenvalue of A, so that

$$\det(A - a\mathbb{1}) = 0 \ . \tag{10.11}$$

$B = A - a\mathbb{1}$ commutes with all $M(g)$, $\forall g$, so by the first Schur lemma, either $B = 0$ or B is a regular one-to-one map from V to V. But $\det B = 0$, so $B = 0$, that is, $A = a \, \mathbb{1}_N$ (10.10).

Note: Actually the condition (10.10) is not only necessary but also sufficient for the representation to be irreducible.

10.2.1 A theorem

Any irreducible complex representation of an Abelian group is one-dimensional. Proof: for an Abelian group

$$M(g)M(h) = M(h)M(g) \ , \qquad \forall g, h \in G \ , \tag{10.12}$$

so by Schur's lemma,

$$M(g) = a(g) \, \mathbb{1} \ , \qquad a(g) \in \mathbb{C} \tag{10.13}$$

that is, such a representation is one-dimensional.

e.g. For the $SO(2)$ group, the complex irreducible representations are

$$\chi_m(\phi) = e^{im\phi} \ , \qquad m = 0, \pm 1, \pm 2, \ldots \ , \tag{10.14}$$

i.e., an integer m specifies an irreducible representation of $SO(2)$.

10.3 Orthogonality Theorem

Let $R = \{M, V\}$ and $S = \{N, W\}$ be two unitary irreducible representations of a group G, of order m and n respectively (M and N are, respectively, $m \times m$ and $n \times n$ matrices). The theorem states that

$$\int_G dg\, M_{ij}(g)(N_{\ell k}(g))^* = \begin{cases} 0\,, & R \not\sim S\,, \\ \frac{1}{m}\,\delta_{i\ell}\,\delta_{jk}\,, & R = S\,. \end{cases} \tag{10.15}$$

Let B be an arbitrary $m \times n$ matrix and define

$$A \equiv \int_G dg\, M(g) B N(g^{-1})\ . \tag{10.16}$$

Now for any element $g_0 \in G$,

$$M(g_0)\, A = A\, N(g_0)\ , \qquad \forall g_0 \tag{10.17}$$

holds (the proof is left for the reader as an exercise). Therefore, by Schur's lemma, either $A \equiv 0$ (if $R \not\sim S$) or $R \sim S$.

If $R \not\sim S$, by choosing the matrix B

$$B_{jk} = 1\ , \quad \text{for some } (j,k)\ , \qquad B_{j'k'} = 0, \quad \forall j' \neq j\ , \forall k' \neq k\ , \tag{10.18}$$

one gets (remember $N(g^{-1}) = N^\dagger(g)$ for a unitary representation)

$$A_{i\ell} = \int_G dg\, M_{ij}(g)(N^\dagger_{k\ell}(g)) = \int_G dg\, M_{ij}(g)(N_{\ell k}(g))^* = 0\ . \tag{10.19}$$

If $R = S$, $M(g) = N(g)$, we have $A = c\,\mathbb{1}$ by Schur's lemma II. The $(i\ell)$ element of (10.16) now gives

$$\int_G dg\, M_{ij}(g) B_{jk} (N_{\ell k}(g))^* = \delta_{i\ell}\, c\ . \tag{10.20}$$

To determine c, choose B as before, and take the trace of the above, which gives

$$\mathrm{Tr}\, A = m\, c = \int_G dg\, \mathrm{Tr}(M(g) B M(g^{-1})) = \int_G dg\, \mathrm{Tr} B = \mathrm{Tr} B = \delta_{jk}\ , \tag{10.21}$$

thus

$$\int_G dg\, M_{ij}(g)(M_{\ell k}(g))^* = \frac{1}{m}\delta_{jk}\delta_{i\ell} \tag{10.22}$$

11 Character

The *character* of a representation $R = \{M, V\}$ of a group G is defined as

$$\chi_R(g) \equiv \mathrm{Tr}\, M(g) \ . \tag{11.1}$$

Due to the properties of the trace the character satisfies:

(1)

$$\chi_R(g) = \chi_R(h^{-1}gh) \ ; \tag{11.2}$$

thus the character depends only on the conjugacy class of the elements: such a function is known as a class function.

(2) If $R = \{M, V\} \sim \tilde{R} = \{\tilde{M}, \tilde{V}\}$ then

$$\chi_R(g) = \chi_{\tilde{R}}(g) \tag{11.3}$$

As the following theorems well illustrate, in a compact group various properties about the representations, such as the irreducibility of a representation, the way a reducible representation decomposes into a direct sum of irreducible representations, etc. are all concisely summarized by the character of the representations.

11.1 Orthogonality theorem

Consider a compact group G and consider two irreducible representations $R = \{M, V\}$ and $\tilde{R} = \{\tilde{M}, \tilde{V}\}$ of G. The characters of the two representations are given by

$$\chi_R = \mathrm{Tr}\, M(g) \ , \qquad \chi_{\tilde{R}} = \mathrm{Tr}\, \tilde{M}(g) \ . \tag{11.4}$$

As they are functions of the group, their scalar product can be defined by (3.29):

$$\langle \chi_{\tilde{R}}, \chi_R \rangle \equiv \int_G dg\, (\mathrm{Tr}\, \tilde{M}(g))^* \mathrm{Tr}\, M(g) \ . \tag{11.5}$$

The theorem states that

$$\langle \chi_{\tilde{R}}, \chi_R \rangle = \begin{cases} 0 & \text{if} \quad R \not\sim \tilde{R} \ , \\ 1 & \text{if} \quad R \sim \tilde{R} \ . \end{cases} \tag{11.6}$$

The theorem follows at once from the orthogonality theorem of Subsection 10.3. The proof is left to the reader.

11.2 Character and irreducibility of the representation

For a compact group G a necessary and sufficient condition for a representation R to be irreducible is simply:

$$\langle \chi_R, \chi_R \rangle = 1 \ . \tag{11.7}$$

That this is necessary has been already proven in (11.6).

In order to show that it is also sufficient, let a generic unitary representation $R = \{M, V\}$ be a direct sum of irreducible representations

$$R = \oplus N_i \, R_i \ , \tag{11.8}$$

where nonnegative integers $N_i \in \mathbb{Z}_+$ is the number of times a particular irreducible representation appears in the reduction, (8.8), (8.9). Note that this is always possible as a unitary representation is completely reducible, as has been proven in Section 9. Consider the norm of the character χ_R of R, (11.8),

$$\langle \chi_R, \chi_R \rangle = \sum_i N_i^2 \ , \tag{11.9}$$

where the orthonormality of irreducible representations R_i, already proven, has been used. The only way (11.9) is compatible with (11.7) is that

$$N_i = 1 \ , \qquad N_j = 0 \ , \quad j \neq i \ , \tag{11.10}$$

for an index i, that is, the representation R is irreducible.

11.3 Criterion for the equivalence of two representations

A necessary and sufficient condition for two representations R and $\tilde{R}$ to be equivalent, is that

$$\chi_R = \chi_{\tilde{R}} \, . \tag{11.11}$$

That this is necessary is a straightforward consequence of the definition of the character and of the equivalence between two representations.

That this is also sufficient can be seen as follows. From (11.11) it follows that

$$\langle \chi_{\tilde{R}}, \chi_R \rangle = \langle \chi_R, \chi_R \rangle = 1 \ , \qquad \langle \chi_{\tilde{R}}, \chi_{R'} \rangle = \langle \chi_R, \chi_{R'} \rangle = 0 \ , \tag{11.12}$$

for $\forall R' \not\sim R$. Assuming that $\tilde{R}$ has the generic form

$$\tilde{R} = \oplus N_i \, R_i \ , \tag{11.13}$$

(11.12) gives

$$N_R = 1 \; , \qquad N_{R'} = 0 \; . \tag{11.14}$$

Thus

$$\tilde{R} \sim R \; . \tag{11.15}$$

11.4 Completeness

Consider a generic unitary representation of the form, (11.8). It follows from a property of the trace on a matrix of block diagonal form, (8.7), that

$$\chi_R(g) = \sum_i N_i \, \chi_{R_i}(g) \; . \tag{11.16}$$

The orthonormality relation (11.6) gives then

$$\langle \chi_R, \chi_{R_i} \rangle = \int_G dg \, \chi_R(g)^* \chi_{R_i}(g) = N_i \; . \tag{11.17}$$

By resubstituting this into (11.16), one gets

$$\chi_R(g) = \int_G dg' \, \chi_R(g') \sum_i \chi_{R_i}(g')^* \chi_{R_i}(g) \; , \tag{11.18}$$

for any g. Consistency then requires the completeness relation

$$\sum_i \chi_{R_i}(g') \chi_{R_i}(g)^* = \delta(g' - g) \; , \tag{11.19}$$

where the delta function in the group parameter space is appropriately defined, such that

$$\delta(g' - g) = 0 \; , \quad g' \neq g \; , \qquad \int_G dg' \, \delta(g' - g) = 1 \; . \tag{11.20}$$

For instance, for the S^3 parametrization of the $SU(2)$ group, (3.19), (3.20), the delta function in the group is given by

$$\delta(g' - g) \equiv \frac{2\pi^2}{\sin^2 \theta_1 \sin \theta_2} \delta(\theta_1' - \theta_1) \delta(\theta_2' - \theta_2) \delta(\theta_3' - \theta_3) \; . \tag{11.21}$$

For an Euler-angle representation of $SO(3)$ group elements, (3.21), one has

$$\delta(g' - g) \equiv \frac{8\pi^2}{\sin \theta} \delta(\theta' - \theta) \delta(\phi' - \phi) \delta(\psi' - \psi) \; . \tag{11.22}$$

All the results of the Representation theory, Section 7 $\sim$ Section 11, concerning compact groups, apply to finite groups, where the invariant group integration is replaced by the sum over the group elements, g_ℓ,

$$\int_G dg \to \sum_\ell \ . \tag{11.23}$$

For some characteristic features of the finite groups, see [3].

Part IV

Familiar Lie groups in physics and their representations

We discuss now the groups frequently used in physics, taking advantage of the artilleries so far developed. The $SU(2)$ group, which plays the central role in the theory of angular momentum and of isospin in quantum mechanics, as well as in the gauge theory of the fundamental interactions, is studied in detail. The $su(2)$ algebra furthermore constitutes the building block for all other semi-simple Lie algebras, as will be seen in later sections. Other useful groups treated here, not as in details as for $SU(2)$, are $SU(3)$, $SU(N)$, $SO(4)$, E_2, Lorentz and Poincaré groups.

12 $SU(2)$ Group

The $SU(2)$ group is generated by the generators J_1, J_2, J_3 as

$$g = e^{i\alpha^a J_a} \ , \tag{12.1}$$

The algebra $su(2)$ is

$$[J_i, J_j] = i\epsilon^{ijk} J_k \ , \tag{12.2}$$

or

$$[J_1, J_2] = iJ_3 \ ; \qquad [J_2, J_3] = iJ_1 \ ; \qquad [J_3, J_1] = iJ_2 \ . \tag{12.3}$$

The Casimir operator is given by

$$\mathbf{J}^2 \equiv J_1^2 + J_2^2 + J_3^2 \tag{12.4}$$

which commutes with all the generators,

$$[\mathbf{J}^2, J_i] = 0 \ , \qquad i = 1, 2, 3, \tag{12.5}$$

as can be explicitly verified easily. $\mathbf{J}^2$ commutes with any representation matrix

$$[\mathbf{J}^2, M] = 0 \ , \qquad M = e^{iJ^a\alpha^a} \tag{12.6}$$

hence by Schur's lemma is a constant within an irreducible representation,

$$\mathbf{J}^2 = c\,\mathbb{1} \ . \tag{12.7}$$

The constant c characterizes each irreducible representation of $SU(2)$.

In the context of quantum mechanics, (12.2) is nothing but the commutation relations among the three components of the angular momentum operators [5] (or of the isospin operators) and the constant c corresponds to the value [6] of the square of the angular momentum (isospin) operators in that representation.

Below we are going to construct all the irreducible representations of the $SU(2)$ group explicitly.

Isospin

Before starting the analysis, let us remark that the $SU(2)$ group manifests itself in nature in the form of "isospin" symmetry of the nuclear forces (in fact, of the strong interactions). Historically it appeared first in the form of phenomenological observation of

(i) Charge symmetry, namely

$$V_{pp} = V_{nn} \ , \tag{12.8}$$

i.e., that the nuclear forces between two protons are the same as those between two neutrons; then

(ii) Charge indendence,

$$V_{pp} = V_{nn} = V_{pn} \ , \tag{12.9}$$

i.e., that the forces between a proton and a neutron is also the same as those between the same particles, and finally as

(iii) Isospin symmetry, i.e., the nuclear forces are invariant under a continuous transformations

$$\begin{pmatrix} p \\ n \end{pmatrix} \to U \begin{pmatrix} p \\ n \end{pmatrix} \tag{12.10}$$

where U is an $SU(2)$ matrix.

The idea which came out of these development is that the proton and neutron are two different quantum isospin "states" of the same particle, called the nucleon [7]. Indeed $m_p \simeq m_n \simeq 940\,\mathrm{MeV}/c^2$. See more about the isospin-invariant form of the nucleon interactions later.

[5] More precisely, of the angular momentum operators written in the unit of $\hbar$, the Planck constant.

[6] More precisely, the eigenvalue.

[7] If the isospin symmetry were exact, the proton and neutron would have to be considered as just two quantum states of a single particle, the nucleon. Actually, the small mass difference ($m_p \simeq 938.27$, $m_n \simeq 939.57$), and the electromagnetic interactions ($e_P = 1, e_n = 0$) break explicitly the $SU(2)$ symmetry, and this makes the distinction of the two particles unambiguous.

12.1 The fundamental and adjoint representations

The elementary (fundamental) representation of $SU(2)$ group is the action of 2×2 special unitary matrices U acting on a complex two component vectors,

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to U \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} . \tag{12.11}$$

In this representation the generators take the form of the Pauli matrices, (4.23),

$$t^a = \frac{1}{2}\tau^a ,$$

where

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{12.12}$$

known as the Pauli matrices. A useful identity for writing a generic finite element of $SU(2)$ in terms of three real parameters $\mathbf{b} = (b_1, b_2, b_3)$ is

$$U = e^{i\frac{\tau}{2}\cdot\mathbf{b}} = \cos\frac{b}{2} + i\frac{\tau\cdot\mathbf{b}}{b}\sin\frac{b}{2} = \begin{pmatrix} \cos\frac{b}{2} + i\frac{b_3}{b}\sin\frac{b}{2} & i\frac{b_1 - ib_2}{b}\sin\frac{b}{2} \\ i\frac{b_1+ib_2}{b}\sin\frac{b}{2} & \cos\frac{b}{2} - i\frac{b_3}{b}\sin\frac{b}{2} \end{pmatrix} , \qquad b = |\mathbf{b}| . \tag{12.13}$$

The adjoint representation has the generators

$$(\mathbb{T}^a)_{bc} \equiv if^{bac} = i\epsilon^{bac} , \qquad a, b, c = 1, 2, 3. \tag{12.14}$$

The representation space is three dimensional. Using the example of isospin, the three kinds of pions,

$$\pi^+ = \frac{\pi^1 - i\pi_2}{\sqrt{2}} , \qquad \pi^- = \frac{\pi^1 + i\pi_2}{\sqrt{2}} , \qquad \pi^0 = \pi^3 , \tag{12.15}$$

can be regarded as the three basis states (vectors) of the adjoint (triplet) representation, transforming as

$$\begin{pmatrix} \pi^1 \\ \pi^2 \\ \pi^3 \end{pmatrix} \longrightarrow \begin{pmatrix} \pi'^1 \\ \pi'^2 \\ \pi'^3 \end{pmatrix} = \mathcal{U} \begin{pmatrix} \pi^1 \\ \pi^2 \\ \pi^3 \end{pmatrix} , \qquad \mathcal{U} = e^{i\alpha^a\mathbb{T}^a} . \tag{12.16}$$

A very useful way to express this transformation law is to write the pion state as

$$\frac{\pi^a\tau^a}{2} , \tag{12.17}$$

by using a 2×2 matrices of the fundamental representation, and to consider the transformation law

$$\frac{\pi^a\tau^a}{2} \longrightarrow \frac{\pi'^a\tau^a}{2} = U\,\frac{\pi^a\tau^a}{2}\,U^{-1}\,, \qquad U = e^{i\alpha^a\tau^a/2}\,. \tag{12.18}$$

It is left as an exercise to prove that (12.16) and (12.18) indeed yield the identical transformation law for π^a's.

As a little application one may now write an isospin-invariant pion-nucleon interaction vertices as

$$V_Y = g_Y\,(\bar{p},\bar{n})\frac{\pi^a\tau^a}{2}\begin{pmatrix} p \\ n \end{pmatrix}\,, \tag{12.19}$$

(known as the Yukawa interaction) which is manifestly invariant under any $SU(2)$ transformation in view of (12.10) and (12.18). Writing explicitly,

$$V_Y = g_Y\,\Big[(\bar{p}p - \bar{n}n)\pi^0 + \sqrt{2}\,\bar{p}n\pi^+ + \sqrt{2}\,\bar{n}p\pi^-\Big]\,. \tag{12.20}$$

12.2 Construction of the irreducible representations

A useful tool is that of the raising and lowering operators [8]

$$J_+ \equiv J_1 + iJ_2\,, \qquad J_- \equiv J_1 - iJ_2\,, \tag{12.21}$$

in terms of which the $su(2)$ algebra can be written as

$$[J_3, J_+] = J_+\,, \qquad [J_3, J_-] = -J_-\,, \qquad [J_+, J_-] = 2J_3\,. \tag{12.22}$$

Also,

$$[\mathbf{J}^2, J_\pm] = [\mathbf{J}^2, J_3] = 0\,. \tag{12.23}$$

Also very useful below are the two expressions of $\mathbf{J}^2$, in terms of $J_\pm$:

$$\begin{aligned} \mathbf{J}^2 &= J_+J_- + J_3^2 - J_3 \\ &= J_-J_+ + J_3^2 + J_3 \end{aligned} \tag{12.24}$$

which can be proven easily.

The starting point of the analysis is that in a given representation, characterized by the Casimir

$$\mathbf{J}^2 = c\,\mathbb{1}\,, \tag{12.25}$$

where the unit matrix has the dimension depending on the particular representation, there

[8] The details of these constructions can be found in many quantum mechanics textbooks, see e.g., [10].

will be various eigenvalues of the generator J_3, m:[9]

$$J_3|c,m\rangle = m|c,m\rangle\ , \qquad \mathbf{J}^2|c,m\rangle = c\,|c,m\rangle\ , \qquad \langle c,m|c,m\rangle = 1\ . \tag{12.27}$$

The possible values taken by c and m in a given representation will be determined below. As we are interested in a finite representation, i.e., with a representation space of finite dimension, there will be the maximum value of m,

$$\mathrm{Max}\{m\}_c = j\ , \tag{12.28}$$

(def. of j) within the representation. Thus j depends on c (see below). The "highest" state $|c,m\rangle$ then satisfies [10]

$$J_3|c,j\rangle = j|c,j\rangle\ , \tag{12.29}$$

Now consider various states which can be obtained by applying J_+ or J_- to a state $|c,m\rangle$ in the representation under consideration. One finds

$$J_3(J_+\,|c,m\rangle) = (J_+J_3 + J_+)|c,m\rangle = (m+1)(J_+\,|c,m\rangle)\ ; \tag{12.30}$$

$$J_3(J_-\,|c,m\rangle) = (J_-J_3 - J_-)|c,m\rangle = (m-1)(J_-\,|c,m\rangle)\ ; \tag{12.31}$$

where use was made of the algebras (12.22). This means that $J_+\,|c,m\rangle$ and $J_-\,|c,m\rangle$ are eigenstates of J_3, with eigenvalue $m \pm 1$, unless J_+ or J_- annihilates it. Does this happen? For instance, consider the highest state, $|c,j\rangle$. By hypothesis there are no other states with higher value of J_3 in this representation, so it must be that

$$J_+|c,j\rangle = 0\ , \tag{12.32}$$

for otherwise it would mean that there is another state with the eigenvalue, $j+1$: a contradiction.

Let us apply the second of (12.24) as

$$c = \langle c,j|\mathbf{J}^2|c,j\rangle = \langle c,j|J_-J_+ + J_3^2 + J_3|c,j\rangle = j^2 + j\ , \tag{12.33}$$

[9] A more familiar notation from the matrix theory would be to write

$$J_3\,\mathbf{v}^{(m)} = m\,\mathbf{v}^{(m)}\ , \qquad (\mathbf{v}^{(m)})^\dagger\mathbf{v}^{(m)} = 1\ , \tag{12.26}$$

with $\mathbf{v}^{(m)}$ a column vector, and m is the associated eigenvalue. We adopt the notation which is more commonly used in quantum mechanics. Indeed the content of the present section covers good part of the theory of angular momentum found in any standard textbook of quantum mechanics. See e.g. [10].

[10] Here we use the term "state" in the sense of the common eigenvectors of the matrices $\mathbf{J}^2$ and J_3. Even though in the application in quantum mechanics, these objects acquire automatically the physical meaning of angular momentum "quantum states", this fact does not concern us here: no knowledge of quantum mechanics is required to follow this section.

thus

$$c = j(j+1) \ . \tag{12.34}$$

Namely we have proven that the Casimir operator $\mathbf{J}^2 = J_1^2 + J_2^2 + J_3^2$ in the representation in which the highest J_3 eigenvalue is j takes the value $j(j+1)$.

Applying J_- on $|c,j\rangle$ would in general not give zero, but another eigenvector,

$$J_-|c,j\rangle \propto |c,j-1\rangle \ . \tag{12.35}$$

By applying repeatedly J_-, one finds a tower of states,

$$|c,j\rangle, \quad |c,j-1\rangle, \quad |c,j-2\rangle, \ldots \tag{12.36}$$

with J_3 eigenvalues, $j, j-1, j-2, \ldots$, respectively. Note that as J_- commutes with $\mathbf{J}^2$, the eigenvalue with respect to $\mathbf{J}^2$ is unchanged: they are all eigenvectors of $\mathbf{J}^2$ with eigenvalue, $c = j(j+1)$.

How many such states (12.36) are there? As our representation is finite, there must be the lowest vector, "the minimum state" $|c,j-n\rangle$ such that

$$J_- \, |c,j-n\rangle = 0 \ , \tag{12.37}$$

for otherwise there would be a vector with the eigenvalue of J_3 even smaller, which is a contradiction. (12.37) however implies that n and j are related. Indeed by using this time the first of (12.24) one finds

$$c = j(j+1) = \langle c,j-n|\mathbf{J}^2|c,j-n\rangle = \langle c,j-n|J_+J_- + J_3^2 - J_3|c,j-n\rangle = (j-n)^2 - (j-n) \ , \tag{12.38}$$

where (12.37) has been used. Solving this equation one finds that

$$j = \frac{n}{2} \ . \tag{12.39}$$

As n is a nonnegative integer, this means that the possible values of j in various representations are limited to

$$j = 0\,, \quad \frac{1}{2}\,, \quad 1\,, \quad \frac{3}{2}\,, \quad 2\,, \quad \frac{5}{2}\,, \quad \ldots \ : \tag{12.40}$$

either a nonnegative integer or a half-integer.[11]

Now that we found the relation (12.34) between the Casimir $\mathbf{J}^2 = c$ and the highest $j = \mathrm{Max}(J_3)$ belonging to that representation, it is more convenient to use the notation

[11] In quantum mechanics this corresponds to the universal quantization rule of the angular momentum, of the spin, or of the isospin, rigorously obeyed in Nature. In that context, another fundamental result is that the *orbital* angular momenta are quantized by integer values only: this reflects [10] the property of the space we live in: $\pi_1(\mathbb{R}^3) = 1$.

$j = \mathrm{Max} J_3$	0	1/2	1	3/2	2	...	j	...
Casimir $\mathbf{J}^2$	0	3/4	2	15/4	6	...	$j(j+1)$	...
Dimension	1	2	3	4	5	...	$2j+1$	...

Table 10: Irreducible representations of $SU(2)$ group, the Casimir operator and the dimension of the representations

to indicate each eigenstate of $(\mathbf{J}^2, J_3)$ as $|j,\, m\rangle$ rather than as $|c,\, m\rangle$. The base vectors belonging to the j-representation are thus

$$|j,\, m\rangle\ , \qquad m = j, j-1, \ldots, -j\ . \tag{12.41}$$

Applying now the formulas (12.24) again, one finds

$$\begin{aligned} j(j+1) &= \langle j,\, m|\mathbf{J}^2|j,\, m\rangle = \langle j,\, m|J_+J_- + J_3^2 - J_3|j,\, m\rangle \\ &= |\langle j,\, m-1|J_-|j,\, m\rangle|^2 + m^2 - m\ , \end{aligned} \tag{12.42}$$

thus

$$\langle j,\, m-1|J_-|j,\, m\rangle = \sqrt{(j+m)(j-m+1)}\ , \tag{12.43}$$

and similarly [12]

$$\langle j,\, m+1|J_+|j,\, m\rangle = \sqrt{(j-m)(j+m+1)}\ . \tag{12.44}$$

Recalling $J_\pm = J_1 \pm iJ_2$, these relations, together with

$$\langle j,\, m^{'}|J_3|j,\, m\rangle = \delta_{m'm}\, m \tag{12.45}$$

determine all the matrix elements of the generators J_1, J_2, J_3 in any given representation j.

For $j = \frac{1}{2}$, these reproduce the familiar expressions

$$t^a = \frac{1}{2}\tau^a\ , \qquad \tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\ , \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ , \tag{12.46}$$

in terms of the Pauli matrices. For $j = 1$, one finds

$$t^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\ , \quad t^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}\ , \quad t^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\ . \tag{12.47}$$

[12] A careful reader will have noted that the phase factor was taken to be 1 in taking the square root in (12.43) and (12.44). Such a choice fixes conventionally the relative phases of the states $|j, m\rangle$ within the same multiplet. See more about this later, in connection to the Clebsch-Gordan coefficients.

12.3 Direct product representations and their decomposition in the direct sum of irreducible representation

Without going into details, the direct products of two $SU(2)$ representations decompose as a direct sum of irreducible representations, as [13]

$$\underline{j_1} \otimes \underline{j_2} = \underline{j_1 + j_2} \oplus \underline{j_1 + j_2 - 1} \oplus \underline{j_1 + j_2 - 2} \oplus \ldots \underline{|j_1 - j_2|} \ . \tag{12.48}$$

For example, the product of two fundamental representations decompose as

$$\underline{1/2} \otimes \underline{1/2} = \underline{1} \oplus \underline{0} \ , \tag{12.49}$$

the $2 \times 2 = 4$ states of the direct product decompose as the sum of a triplet and a singlet. By using the notation

$$|\uparrow\rangle = |\tfrac{1}{2}, \tfrac{1}{2}\rangle \ , \quad |\downarrow\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \ , \tag{12.50}$$

representing $|p\rangle$, $|n\rangle$ of the isospin doublet (or the spin up or down states of a spin 1/2 particle), the decomposition rule (12.49) can be written explicitly,

$$|1,1\rangle = |\uparrow\rangle|\uparrow\rangle \ , \quad |1,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle) \ , \quad |1,-1\rangle = |\downarrow\rangle|\downarrow\rangle \ , \tag{12.51}$$

for the triplet, and

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle) \ . \tag{12.52}$$

for the singlet. The various coefficients appearing in (12.51), (12.52), are examples of the Clebsh-Gordan coefficients [10]. Each state in the direct-product basis on the left hand side of (12.48) can be expressed in terms of the states appearing on the right hand side (i.e., states with various total J), as

$$|j_1, m_1; j_2, m_2\rangle = \sum_{J,M} |j_1, j_2, J, M\rangle\langle j_1, j_2, J, M|j_1, m_1; j_2, m_2\rangle \ . \tag{12.53}$$

The inverse relation reads

$$|j_1, j_2, J, M\rangle = \sum_{m_1,m_2} |j_1, m_1; j_2, m_2\rangle\langle j_1, m_1; j_2, m_2|j_1, j_2, J, M\rangle \ . \tag{12.54}$$

or dropping somewhat redundant $j_{1,2}$

$$|J, M\rangle = \sum_{m_1,m_2} |j_1, m_1; j_2, m_2\rangle\langle j_1, m_1; j_2, m_2|J, M\rangle \ . \tag{12.55}$$

[13] In the context of quantum mechanics this is known as the composition-decomposition rule of angular momenta. See [10] for details.

The expansion coefficients

$$\langle j_1, m_1; j_2, m_2 | j_1, j_2, J, M \rangle = \langle j_1, j_2, J, M | j_1, m_1; j_2, m_2 \rangle \tag{12.56}$$

are known as Clebsch-Gordan (or CG-) coefficients.

As one can easily glimpse from an explicit construction of the composition-decomposition, (12.48), see [10], it is necessary to fix certain phase convention among the states $|j_1, m_1\rangle$, $|j_2, m_2\rangle$ and $|J, M\rangle$ appering in (12.48). Such a convention must be such that all of the CG coefficients be unambiguously and exhaustively determined. A well-known phase convention (usually used) is known as the Condon-Shortley convention. It consists of the following three rules [14]:

(i) The highest states in the two bases are identified with coefficient 1.

(ii) All matrix elements of $J_{1\,-}$, $J_{2\,-}$, J_- are real and semi-positive definite [15].

(iii) The matrix elements

$$\langle j_1, j_2, J, M | J_{1\,z} | j_1, j_2, J \pm 1, M \rangle \tag{12.57}$$

are all real and semi-positive definite.

In Mathematica, the command to get a CG coefficient is

$$\text{ClebschGordan}[\{j_1, m_1\}, \{j_2, m_2\}, \{J, M\}] \ . \tag{12.58}$$

12.4 Irreducible representations of $su(2)$ are (pseudo-)real

An interesting (and important) aspect of the $SU(2)$ group is that its irreducible representations are all real (or pseudo real, see below). As any $SU(N)$ generators are Hermitian, the reality of the representation

$$M(g) \sim M(g)^* \tag{12.59}$$

requires

$$-(T^a)^* = S\, T^a\, S^{-1} \ , \qquad \forall a \ . \tag{12.60}$$

for some fixed matrix S. In the case of $SU(2)$, the generators in the fundamental representation are given by the Pauli matrices,

$$T^1 = \frac{\tau^1}{2} \ , \quad T^2 = \frac{\tau^2}{2} \ , \quad T^3 = \frac{\tau^3}{2} \ . \tag{12.61}$$

By choosing

$$S = \tau^2 \ , \qquad S^{-1} = \tau^2 \ , \tag{12.62}$$

[14] The demonstration is found in A.R. Edmonds, "Angular Momentum in Quantum Mechanics".

[15] This was used in (12.43), (12.44).

and by using the properties of the Pauli matrices, it is easy to prove (left as an exercise) that

$$-\tau^2(\tau^a)^*\tau^2 = \tau^a \;, \qquad a = 1,2,3 \;. \tag{12.63}$$

Thus the representation $\underline{2}^*$ and $\underline{2}$ are equivalent,

$$\underline{2}^* \sim \underline{2} \;, \tag{12.64}$$

and consequently, other irreducible representations which can be constructed from the decompositions of the direct products of $\underline{2}$ representations, are all real.

Actually there is a finer distinction between a *real representation* and a *pseudoreal representation*. From the reality condition it follows that

$$(T^a)^T = -S\,T^a\,S^{-1} \;. \tag{12.65}$$

By taking the transpose

$$T^a = -(S^{-1})^T\,(T^a)^T\,S^T \;. \tag{12.66}$$

Insert (9.18) now in (9.19) to get

$$T^a = (S^{-1})^T\,S\,T^a\,S^{-1}\;S^T \;, \tag{12.67}$$

or

$$S^{-1}\;S^T\,T^a = T^a\,S^{-1}\;S^T \;, \tag{12.68}$$

namely, $S^{-1}\;S^T$ commutes with all T^a's. By Schur's lemma, it follows that

$$S^{-1}\;S^T = \lambda\,\mathbb{1}\,, \qquad \therefore \quad S^T = \lambda\,S\,. \tag{12.69}$$

But as $(S^T)^T = S$, one gets

$$\lambda^2 = 1\,, \qquad \therefore \quad S^T = \pm S \;: \tag{12.70}$$

the matrix S is either symmetric or antisymmetric. When S is symmetric, the representation is called *real*; when it is antisymmetric, the representation is *pseudoreal*. Thus the fundamental representation of $SU(2)$ group is pseudoreal, as $S = \tau^2$ is antisymmetric.

abc	123	147	156	246	257	345	367	458	678
f^{abc}	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$

Table 11:

13 $SU(3)$ Group, Quark Model and Young tableaux

13.1 $su(3)$ algebra and weight vectors

The algebra of the $SU(3)$ group are given by the following eight generators

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{13.1}$$

$$T^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad T^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{13.2}$$

$$T^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{13.3}$$

known as Gell-Mann's matrices. The structure constants f^{abc} in

$$[T^a, T^b] = i f^{abc} T^c \tag{13.4}$$

turn out to be completely antisymmetric under exchanges of the indices, a, b, c. The Killing form is given simply by $g^{ab} = -\frac{3}{2}\delta^{ab}$. The nonvanishing structure constants are listed in Table 11.

Let us define

$$T_\pm = T^1 \pm iT^2 , \qquad U_\pm = T^6 \pm iT^7 , \qquad V_\pm = T^4 \mp iT^5 , \tag{13.5}$$

$$U^3 = (\sqrt{3}T^8 - T^3)/2 \qquad V^3 = (\sqrt{3}T^8 + T^3)/2 \tag{13.6}$$

In terms of these raising and lowering operators the $su(3)$ algebra can be written as

$$[T^3, T_\pm] = \pm T_\pm , \qquad [T_+, T_-] = 2\,T^3 , \tag{13.7}$$

$$[U^3, U_\pm] = \pm U_\pm , \qquad [U_+, U_-] = 2\,U^3 , \tag{13.8}$$

$$[V^3, V_\pm] = \pm V_\pm , \qquad [V_+, V_-] = 2\,V^3 , \tag{13.9}$$

and

$$[T^8, T_\pm] = 0 \ , \qquad [T^3, U_\pm] = \mp\frac{1}{2}U_\pm \ , \qquad [T^3, V_\pm] = \pm\frac{1}{2}V_\pm \ , \tag{13.10}$$

$$[T^8, U_\pm] = \pm\frac{\sqrt{3}}{2}U_\pm \ , \qquad [T^8, V_\pm] = \pm\frac{\sqrt{3}}{2}V_\pm \ . \tag{13.11}$$

There are two diagonal generators, T^3 and T^8. As they commute, there are vectors which are simultaneous eigenvectors of T^3 and T^8. In the representation where T^3 and T^8 are diagonal, such vectors (of the minimum number of components) are simply:

$$|q_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ , \quad |q_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \ , \quad |q_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \ , \tag{13.12}$$

which form together the basis of

$$\underline{3} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \ . \tag{13.13}$$

Indeed,

$$T^3|q_1\rangle = \frac{1}{2}|q_1\rangle \ , \quad T^3|q_2\rangle = -\frac{1}{2}|q_2\rangle \ , \quad T^3|q_3\rangle = 0 \ , \tag{13.14}$$

$$T^8|q_1\rangle = \frac{1}{2\sqrt{3}}|q_1\rangle \ , \quad T^8|q_2\rangle = \frac{1}{2\sqrt{3}}|q_2\rangle \ , \quad T^8|q_3\rangle = -\frac{1}{\sqrt{3}}|q_3\rangle \ . \tag{13.15}$$

A very useful way to interpret these formulas is to represent the vectors $|q_1\rangle$, $|q_2\rangle$, and $|q_3\rangle$, by a kind of vectors in a real *two-dimensional* space by using their eigenvalues with respect to T^3, T^8 as their components, namely, as

$$|q_1\rangle \to \mu_1 = (\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}), \qquad |q_2\rangle \to \mu_2 = (-\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}), \qquad |q_3\rangle \to \mu_3 = (0, -\tfrac{1}{\sqrt{3}}). \tag{13.16}$$

The vectors expressed this way are known as the *weight vectors*. Eq. (13.16) are the weight vectors of $\underline{3}$. They form the vertices of a regular triangle in the "weight diagram", see Fig. 6.

Analogously, the weight vectors of $\underline{3}^*$ (the representation given by the complex conjugate triplet vectors (q_1^*, q_2^*, q_3^*)) are simply

$$|q_1^*\rangle \to -\mu_1 = (-\tfrac{1}{2}, -\tfrac{1}{2\sqrt{3}}), \qquad |q_2^*\rangle \to -\mu_2 = (\tfrac{1}{2}, -\tfrac{1}{2\sqrt{3}}), \qquad |q_3^*\rangle \to -\mu_3 = (0, \tfrac{1}{\sqrt{3}}). \tag{13.17}$$

(Exercise: prove that the weight vectors of a complex conjugate representation $\underline{r}^*$ are simply minus of the weight vectors of $\underline{r}$.) The weight diagrams for $\underline{3}^*$ and $\underline{8}$ are also shown in Fg. 6.

As will be discussed in Section 18, the weight vectors μ play an important role in the formal development of the Lie algebras. In order to avoid confusion, it must be kept in

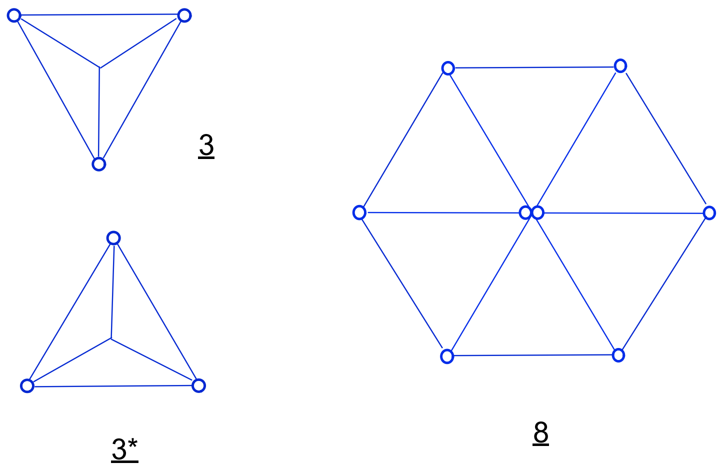


Figure 6: The weight vectors of $\underline{3}$, $\underline{3}^*$ and $\underline{8}$ of the $SU(3)$ group. The weight vectors are the lines connecting the center to the vertices. In $\underline{8}$ there are two null weight vectors.

mind however that they just *label* - but they are not themselves - the basis states (or vectors) in the vector space of a given representation.

The irreducible representations of the $SU(3)$ group, the decomposition of the direct product of two irreps r_1 and r_2, etc. are best worked out by using the Young tableaux discussed in the following Subsection. Some of the examples are

$$\underline{3} \otimes \underline{3} = \underline{6} \oplus \underline{3}^* \ , \qquad \underline{3} \otimes \underline{3}^* = \underline{1} \oplus \underline{8} \ ; \qquad \underline{3} \otimes \underline{3} \otimes \underline{3} = \underline{1} \oplus \underline{8} \oplus \underline{8} \oplus \underline{10} \ , \tag{13.18}$$

$$\underline{8} \otimes \underline{8} = \underline{27} \oplus \underline{10} \oplus \underline{10}^* \oplus \underline{8} \oplus \underline{8} \oplus \underline{1} \ , \qquad \underline{10} \otimes \underline{8} = \underline{35} \oplus \underline{27} \oplus \underline{10} \oplus \underline{8} \ . \tag{13.19}$$

13.2 Young tableaux

Consider the set of products of N objects,[16] each of which can take one of p ”values”, a_1, a_2, ... a_p,

$$\psi = \psi_{a_1}(1)\psi_{a_2}(2)\ldots\psi_{a_1}(N) \ . \tag{13.20}$$

We ask what are the possible symmetry types of these ψ under the permutation of the objects. Indeed, the answer, given below in terms of the Young tableaux, amounts to the irreducible representations of the permutation group S_N considered earlier.

(1) For $N = 1$ there is only one type, ψ_a, $a = 1, 2, \ldots, p$. This is represented by a box

(13.21)

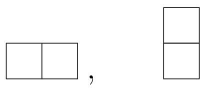
(2) For $N = 2$ one can construct symmetric combinations (there are $p(p+1)/2$ of them)

$$\psi^S = (\psi_{a_1}(1)\psi_{a_2}(2) + \psi_{a_2}(1)\psi_{a_1}(2))/\sqrt{2} \ , \tag{13.22}$$

or antisymmetric ones (there are $p(p-1)/2$ of them)

$$\psi^A = (\psi_{a_1}(1)\psi_{a_2}(2) - \psi_{a_2}(1)\psi_{a_1}(2))/\sqrt{2} \ . \tag{13.23}$$

They are represented by the Young tableaux

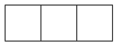


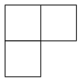
(13.24)

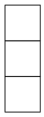
respectively.

[16] Each of $\psi_{a_i}(J)$ can be regarded simply as the color of the Jth box, the orientation of the J-th vector, or the wave function of the J-th particle, etc. These details are unimportant for the considerations here.

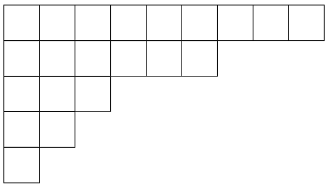
(3) For $N = 3$ there are in general three symmetry types, represented by

 totally symmetric

 mixed symmetry

 totally antisymmetric (13.25)

(4) The most general Young tableau looks like



(13.26)

with the number of the boxes in the rows, N_1, N_2, ... , N_r, such that

$$N_i \geq N_{i-1} \ , \tag{13.27}$$

$$N_1 + N_2 + \ldots N_r = N \ . \tag{13.28}$$

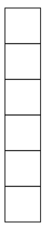
(5) Given the maximum number of values, p, each box can take, it is clear that

$$r \leq p \ , \tag{13.29}$$

as more than $N = p$ objects cannot by anisymmetrized.

(6) For $N = p$, there is exactly one combination which is totally antisymmetric,

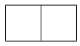
$$\epsilon^{a_1, a_2, \ldots a_p} \psi_{a_1}(1) \psi_{a_2}(2) \ldots \psi_{a_p}(p) = \tag{13.30}$$



(13.31)

(with height p) which is necessarily a singlet.

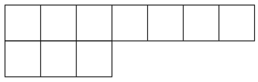
(7) For $SU(N)$ group, each Young tableau corresponds to an irreducible representation. This is not so for, e.g., $SO(N)$ group. For $SO(3)$ group, for instance, the Young tableau



(13.32)

which are the symmetric product of two vectors, represents a direct sum of a rank two tensor $\underline{5}$ and a scalar $\underline{1}$.

(8) Applying these tools to the $SU(2)$ group, a box represents the fundamental representation, $\underline{2}$. A general irreducible representation of $SU(2)$ is represented by a Young tableau of the type



, (13.33)

where the number of the horizontal boxes are n_1 and n_2. As the height 2 columns represent singlets, (13.33) representa the irreducible representation j ($\mathbf{J}^2 = j(j+1)$), where

$$j = \frac{n_1 - n_2}{2} \ . \tag{13.34}$$

(9) For $SU(3)$, a box represents $\underline{3}$, a height 3 column represents a singlet.

(10) The multiplicity (the dimension) of an irreducible representation corresponding to a Young tableau can be read off from the tableau. Denoting the *differences* between the length of the successive rows,

$$p_1 = N_1 - N_2; \qquad p_2 = N_2 - N_3; \qquad \ldots \ , \tag{13.35}$$

the multiplicity (the dimension) of the $SU(3)$ representations is given by

$$N(p_1, p_2) = \frac{1}{2}(p_1 + 1)(p_1 + p_2 + 2)(p_2 + 1) \ ; \tag{13.36}$$

that of the irreps of $SU(4)$ is

$$N(p_1, p_2, p_3) = \frac{1}{2!\,3!}(p_1+1)(p_1+p_2+2)(p_1+p_2+p_3+3)(p_2+1)(p_2+p_3+2)(p_3+1) \ ; \tag{13.37}$$

etc.

13.3 From the Quark model to the contemporary theory of fundamental interactions

In the quark model (Gell-Mann, Zweig, Neeman) the hadrons are described as $M = \bar{q}q$ (mesons) or $B = qqq$ (baryons) bound states. The lowest pseudoscalar mesons and baryons are assigned to $\underline{8}$. The quarks and the light pseudoscalar mesons are arranged as in Fig. 7. The lightest baryons and the first excited baryons (resonances) are assigned to an $\underline{8}$ and to an $\underline{10}$. See Fig. 8. One of the first important results in understanding the systematics

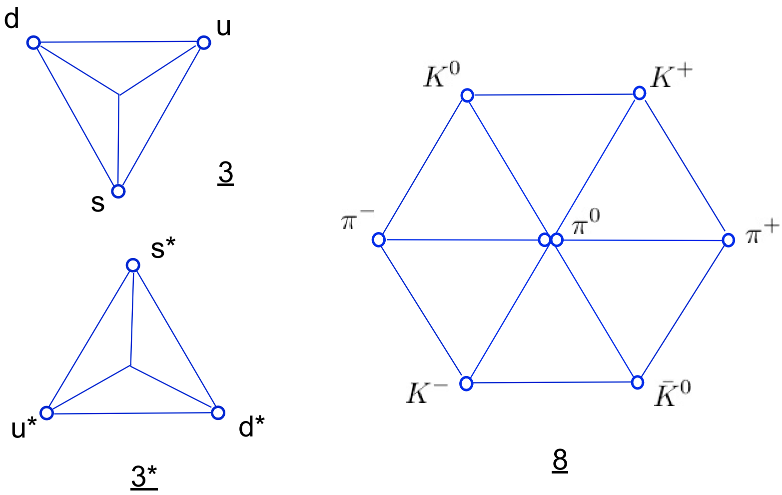


Figure 7: The quarks are in the fundamental $\underline{3}$ and in the antifundamental $\underline{3}^*$. The known lightest pseudoscalar mesons are in an $\underline{8}$.

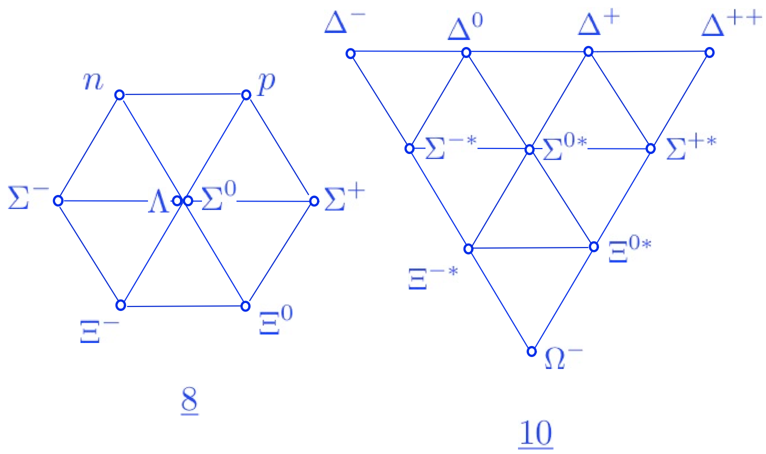


Figure 8: The known baryons and excited baryons (resonances) are in the octet $\underline{8}$ and in a decouplet $\underline{1}0$ representations.

in the quark model was the Nishijima-Gell-Mann-Okubo relation

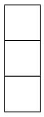
$$Q = I_3 + \frac{Y}{2} = I_3 + \frac{B+S}{2} \ , \tag{13.38}$$

where $I_3 = T^3$ is the third component of the isospin, Y is the hypercharge, related to T^8 by

$$T^8 = \frac{\sqrt{3}}{2} Y \ , \qquad Y = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \ . \tag{13.39}$$

B is the baryon number, S is the strangeness.

Historically the discovery of the baryon Ω^- and the experimental verification of the quark model has brought an important puzzle. Ω^- has the "flavor" content $|sss\rangle$, the spin $\frac{3}{2}$ and the orbital angular momentum $L = 0$ (being the ground state): each of these factors of the wave function is totally symmetric under the exchanges of a pair of quarks. As the quarks are fermions, such a state could not exist in Nature according to the principles of Quantum Mechanics (Fermi-Dirac statistics). This deep puzzle has led to the introduction (a hypothesis) of an extra quantum number, the color (which come in three varieties) – Han and Nambu –, with the assumption that the baryon states are completely antisymmetric with respect to $SU(3)_{color}$,

 , (13.40)

i.e., they are singlets of $SU(3)_{color}$.

Another brilliant and fundamental step was to realize that the very extra degrees of freedom (the color) introduced to solve the statistics puzzle, could be responsible for the *dynamics* of quarks, i.e., could explain the forces binding them together to form (color singlet) baryons and mesons. In 1973, Fritzsch, Leutwyler and Gell-Mann proposed the theory of Quantum Chromodymanics (QCD), the color $SU(3)$ gauge theory [17] .

Immediately, the property of *asymptotic freedom* of QCD was discovered by Politzer, Gross and Wilczek, by studying the renormalization group properties of the theory. In simple terms, it states that the effective interaction strength becomes weak at shorter distances. And this explained Feynman's "parton" structure of hadrons: existence of some pointlike constituents inside hadrons, experimentally observed in deep-inelastic (i.e., with large energy and large momentum transfer) scattering processes.

On the other hand such a property could imply that at large distances the force binding quarks inside hadron becomes stronger, leading to the idea of *quark confinement*. Only color-singlet hadrons (mesons $\sim |\bar{q}q\rangle$ and baryons $\sim |qqq\rangle$) can exist as finite mass observable states. For instance, the inter-quark force may be described by a linearly rising potential, as certain features of hadron spectrum suggest. If this is the case, to free a quark from inside a hadron would require an infinite amount of energy, so that the quarks are permanently confined inside the hadrons [18]. Only recently some new types of color-singlet hadrons, of the type, $\sim |\bar{q}\bar{q}qq\rangle$, called tetraquark states, have been discovered in the experiments, Belle, and at LHC.

This brings us to this day: in spite of great efforts dedicated to this problem, and enormous success (both experimental and theoretical) which provided us with countless checks of the standard model of the fundamental interactions $\{SU_L(2) \times U(1)\}_{WS} \times SU(3)_{QCD}$ and which finally led to the discovery of the Higgs particle (2012), a proper understanding of quark confinement is still an open problem.

[17] By 1972 the electroweak interactions were known to be described by a nonAbelian (Yang-Mills) type gauge theory, $SU(2) \times U(1)$ Weinberg-Salam theory, which unified the Quantum Electrodynamics and the weak interactions. It remained to explain the nature of the strong nuclear forces.

[18] This should be compared with an atom: to liberate a valence electron requires a finite energy, known as the *ionization potential*, of the order of eV. To free a nucleon from a nucleus one needs a much bigger energy, of the order of MeV, roughly $10^5 - 10^6$ times larger than a typical ionization potential, but still finite.

14 $SO(4)$ group and its representations

The orthogonal group $SO(4)$ is the group of rotations in four dimensional Euclidean space. The basic object is the four component vector, $\underline{4}$,

$$a_i \sim \underline{4} \; , \tag{14.1}$$

rank-two tensors,

$$a_i b_j \sim \underline{4} \otimes \underline{4} \; , \tag{14.2}$$

and so on. The latter can be decomposed into symmetric and antisymmetric parts,

$$a_i b_j = \frac{a_i b_j + a_j b_i}{2} + \frac{a_i b_j - a_j b_i}{2} \; , \tag{14.3}$$

the symmetric part can further be decomposed into the sum of the singlet and the traceless second rank tensor

$$\frac{a_i b_j + a_j b_i}{2} = \frac{\delta_{ij}}{4} a_k b_k + \left(\frac{a_i b_j + a_j b_i}{2} - \frac{\delta_{ij}}{4} a_k b_k \right) \; . \tag{14.4}$$

In the elementary representation $\underline{4}$, the generators of $SO(4)$ are rotations in six planes

$$(\Sigma^{ij})_{k\ell} = -i \left(\delta^i_k \delta^j_\ell - \delta^j_k \delta^i_\ell \right) \; . \tag{14.5}$$

For instance,

$$\Sigma^{12} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \qquad e^{i\theta\Sigma^{12}} = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{14.6}$$

$$\Sigma^{13} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \qquad e^{i\phi\Sigma^{13}} = \begin{pmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \sin\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{14.7}$$

etc. The algebra of $SO(4)$ can be found easily:

$$[\Sigma^{ij}, \Sigma^{k\ell}] = -i\{\Sigma^{i\ell}\delta^{jk} - \Sigma^{j\ell}\delta^{ik} - \Sigma^{ik}\delta^{j\ell} + \Sigma^{jk}\delta^{i\ell}\} \; . \tag{14.8}$$

$$\Sigma^{ij} = -\Sigma^{ji} \; , \qquad \Sigma^T = -\Sigma^T \; , \tag{14.9}$$

These appear fairly straightforward generalizations of the group $SO(3)$.

The surprise is that the algebra $so(4)$ actually splits into two commuting factors $su(2) \times$

$su(2)$. This can be seen by defining

$$S^1 = \frac{1}{2}(\Sigma^{23} + \Sigma^{41}) \; ; \qquad S^2 = \frac{1}{2}(\Sigma^{31} + \Sigma^{42}) \; ; \qquad S^3 = \frac{1}{2}(\Sigma^{12} + \Sigma^{43}) \; ; \tag{14.10}$$

$$\hat{S}^1 = \frac{1}{2}(\Sigma^{23} - \Sigma^{41}) \; ; \qquad \hat{S}^2 = \frac{1}{2}(\Sigma^{31} - \Sigma^{42}) \; ; \qquad \hat{S}^3 = \frac{1}{2}(\Sigma^{12} - \Sigma^{43}) \; . \tag{14.11}$$

It can be readily checked that

$$[S^i, S^j] = i\epsilon^{ijk}\, S^k \; ; \qquad [\hat{S}^i, \hat{S}^j] = i\epsilon^{ijk}\, \hat{S}^k \; ; \qquad [S^i, \hat{S}^j] = 0 \; , \tag{14.12}$$

showing

$$so(4) \sim su(2) \times su(2) \; . \tag{14.13}$$

This means that the smallest representation of $SO(4)$ is not the elementary vector representation $\underline{4}$, but the spinor representations of two types,

$$(\underline{2}, \underline{1}) \qquad \text{and} \qquad (\underline{1}, \underline{2}) \; . \tag{14.14}$$

whereas the vector $\underline{4}$ transforms as

$$\underline{4} = (\underline{2}, \underline{2}) \; . \tag{14.15}$$

The general irreducible representations of $SO(4)$ are classified as

$$(j_1, j_2) \; , \qquad j_1, j_2 = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \dots \; . \tag{14.16}$$

The Casimir operators of the two $su(2)$ algebras,

$$\mathbf{S}^2 \; , \qquad \hat{\mathbf{S}}^2 \; , \tag{14.17}$$

have eigenvalues (see Sec. 12.2)

$$j_1(j_1 + 1) \; , \qquad j_2(j_2 + 1) \; . \tag{14.18}$$

Eq. (14.13) shows that $SO(4)$ and $SU(2) \times SU(2)$ have isomorphic algebras. As groups however they are different. As in the case of $SO(3)$ and $SU(2)$ groups, each element of $SO(4)$ group has two inverse images in $SU(2) \times SU(2)$: in particular the unit element of $SO(4)$ has two inverse images,

$$\mathbb{1}_4 \longleftarrow (\mathbb{1}_2, \mathbb{1}_2) \; , (-\mathbb{1}_2, -\mathbb{1}_2) \; , \tag{14.19}$$

thus their relation is

$$SO(4) \sim SU(2) \times SU(2)/\mathbb{Z}_2 \; . \tag{14.20}$$

As $SU(2)$ is simply connected, it follows that

$$\pi_1(SO(4)) = \mathbb{Z}_2 \ . \tag{14.21}$$

$SU(2) \times SU(2)$ is the universal covering group of $SO(4)$.

The generators of $so(4)$ algebra in the spinor representations can be constructed explicitly as follows. Define two-by-two matrices

$$\sigma_\mu \equiv (i\mathbb{1}, \sigma^i) \ , \qquad \bar{\sigma}_\mu \equiv (-i\mathbb{1}, \sigma^i) \qquad \mu = 4, 1, 2, 3 \ . \tag{14.22}$$

and

$$\sigma_{\mu\nu} \equiv \frac{\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu}{4i} \ , \tag{14.23}$$

$$\bar{\sigma}_{\mu\nu} \equiv \frac{\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu}{4i} \ , \tag{14.24}$$

The $so(4)$ generators in the spinorial representations are given by

$$\Sigma_{\mu\nu} = \begin{pmatrix} \sigma_{\mu\nu} & \mathbf{0} \\ \mathbf{0} & \bar{\sigma}_{\mu\nu} \end{pmatrix} \ . \tag{14.25}$$

It is easy to compute

$$S_1 = \frac{1}{2}(\Sigma_{23} + \Sigma_{41}) = \begin{pmatrix} \frac{1}{2}\sigma_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \ ; \quad S_2 = \begin{pmatrix} \frac{1}{2}\sigma_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \ ; \quad S_3 = \begin{pmatrix} \frac{1}{2}\sigma_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \ ; \tag{14.26}$$

which are the generators of the first $SU(2)$ and

$$\hat{S}_1 = \frac{1}{2}(\Sigma_{23} - \Sigma_{41}) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\sigma_1 \end{pmatrix} \ ; \quad \hat{S}_2 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\sigma_2 \end{pmatrix} \ ; \quad \hat{S}_3 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\sigma_3 \end{pmatrix} \ ; \tag{14.27}$$

which are the generators of the second $SU(2)$ factor. They trivially commute with each other. The chirality operator

$$\gamma_5 \equiv \begin{pmatrix} \sigma_4\sigma_1\sigma_2\sigma_3 & \mathbf{0} \\ \mathbf{0} & \bar{\sigma}_4\bar{\sigma}_1\bar{\sigma}_2\bar{\sigma}_3 \end{pmatrix} = \begin{pmatrix} -\mathbb{1} & \mathbf{0} \\ \mathbf{0} & \mathbb{1} \end{pmatrix} \tag{14.28}$$

obviously commutes with all the generators of $su(2) \times su(2)$. It is a Casimir operator; it takes the value ∓ 1 in the spinor representation $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, respectively.

15 Euclidean groups, E_2

The Euclidean group E_n is a group of rotations and translations in n-dimensional Euclidean space, $\mathbf{R}^n$. Their actions on a vector $x \in \mathbf{R}^n$ are defined by

$$x \to Rx + a \ , \tag{15.1}$$

where $R_{n\times n} \in SO(n)$ and a is a constant vector. They leave the metric δ_{ij} and

$$ds = dx^i \delta_{ij} dx^j \tag{15.2}$$

invariant.

Let us study the simplest nontrivial Euclidean group, E_2, in some details. It acts on a two-vector as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1 \cos\theta - x_2 \sin\theta + b_1 \\ x_1 \sin\theta + x_2 \cos\theta + b_2 \end{pmatrix} \ : \tag{15.3}$$

a rotation followed by a translation. Let $g(\mathbf{b}, \theta) \in E_2$ be a generic element. Clearly the group product rule is

$$g(\mathbf{b}_2, \theta_2) g(\mathbf{b}_1, \theta_1) = g(\mathbf{b}_3, \theta_3) \ , \tag{15.4}$$

where

$$\theta_3 = \theta_1 + \theta_2 \ ; \qquad \mathbf{b}_3 = R(\theta_2)\mathbf{b}_1 + \mathbf{b}_2 \ . \tag{15.5}$$

and

$$R(\theta) \equiv \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \ . \tag{15.6}$$

The infinitesimal generators corresponding to (15.3) are

$$P^1 = -i\frac{\partial}{\partial x_1} \ ; \qquad P^2 = -i\frac{\partial}{\partial x_2} \ ; \qquad J = -i(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}) \ . \tag{15.7}$$

Acting on the basis vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{15.8}$$

the above (15.7) correctly generate the infinitesimal transformations,

$$\delta_{\mathbf{P}}\mathbf{x} = i\mathbf{b}\cdot\mathbf{P}\,\mathbf{x} = \mathbf{b} \ , \qquad \delta_\theta \mathbf{x} = iJ\theta\,\mathbf{x} = \begin{pmatrix} \theta x_2 \\ -\theta x_1 \end{pmatrix} \ . \tag{15.9}$$

The algebra of E_2 follows from (15.7) straightforwardly:

$$[P^i, P^j] = 0 \ ; \qquad [J, P^i] = i\epsilon^{ij} P^j \ , \qquad \epsilon^{12} = -\epsilon^{21} = 1 \ . \tag{15.10}$$

The finite transformations are given by

$$T(\mathbf{b}) = e^{-i\mathbf{b}\cdot\mathbf{P}} \; ; \qquad R(\theta) = e^{-i\theta J} \; . \tag{15.11}$$

The algebra (15.10) shows that $P^{1,2}$ form an Abelian invariant subalgebra (ideal). Thus the algebra of E_2 is neither simple nor semi-simple.

The defining rules of E_2, (15.4), (15.5), show that a generic element of E_2 can be written in a factorized form

$$g(\mathbf{b}, \theta) = T(\mathbf{b})R(\theta) = g(\mathbf{b}, 0)g(\mathbf{0}, \theta) \; . \tag{15.12}$$

The important property of E_2 is that $T(\mathbf{b})$ forms an invariant subgroup of E_2. The group is non semi-simple. Another characteristics of the group E_2 is that it is non compact, as the translation parameters b_i can take arbitrarily large values. These characteristics restrain indeed the properties of representations as will be seen below.

Theorem: $T(\mathbf{b})$ forms a normal subgroup $N = T^2$. In fact,

$$e^{-i\theta J} P_k \, e^{i\theta J} = P_m \, (R(\theta))_k^m \; , \tag{15.13}$$

(the proof of which is left to the reader), so

$$e^{-i\theta J} (\mathbf{P} \cdot \mathbf{b}) \, e^{i\theta J} = P_m R_k^m b^k = \mathbf{P} \cdot \mathbf{b}' \; , \qquad \mathbf{b}' = (R(\theta)\mathbf{b}) \; . \tag{15.14}$$

Therefore

$$e^{-i\theta J} T(\mathbf{b}) \, e^{i\theta J} = T(R(\theta)\mathbf{b}) \, . \tag{15.15}$$

At this point, it is easy to prove it:

$$\begin{aligned} & g(\mathbf{b}, \theta)T(\mathbf{a})g(\mathbf{b}, \theta)^{-1} = T(\mathbf{b}) \; \left[R(\theta)T(\mathbf{a})R(\theta)^{-1}\right] T(\mathbf{b})^{-1} \\ = \; & T(\mathbf{b}) \, T(R(\theta)\mathbf{a}) \, T(-\mathbf{b}) = T(R(\theta)\mathbf{a}) \in N \; . \end{aligned} \tag{15.16}$$

Theorem: The coset $G/N = E_2/T^2 \sim SO(2)$.

Let us recall that the right coset associated with an invariant subgroup $N \subset G$ is defined by

$$N g_1 N g_2 = N g_1 N (g_1)^{-1} g_1 g_2 = N N g_1 g_2 = N g_1 g_2 \; . \tag{15.17}$$

From $T^2 = N$, $g = g(\mathbf{b}, \theta) = T(\mathbf{b})R(\theta)$, one has

$$\begin{aligned} & T(\mathbf{a})(T(\mathbf{b}_1)R(\theta_1))T(\mathbf{a}')(T(\mathbf{b}_2)R(\theta_2)) \sim TR(\theta_1)TR(\theta_2) \\ \sim \; & TR(\theta_1)TR(\theta_1)^{-1}R(\theta_1)R(\theta_2) \sim TR(\theta_1 + \theta_2) \; . \end{aligned} \tag{15.18}$$

Thus $E_2/T^2 \sim SO(2)$.

15.1 Unitary representations of E_2: the coset construction

By using the coset group it is possible to construct a unitary representation of E_2. Define the unitary "matrix" $U(\mathbf{b},\theta)$ such that

$$g(\mathbf{b},\theta) \to U(\mathbf{b},\theta) = e^{im\theta} \;, \qquad \forall \mathbf{b} \;, \qquad m \in \mathbb{Z} \;. \tag{15.19}$$

Clearly these satisfy the group multiplication rules (15.4), (15.5), and thus $U(\mathbf{b},\theta)$ is a finite, unitary representation of E_2: more precisely, it is a finite, irreducible representation of its coset $E_2/T^2 \sim SO(2)$.

More general unitary representations can be constructed as follows. Define

$$P_\pm = P_1 \pm iP_2 \;; \qquad [J, P_\pm] = \pm P_\pm \,. \tag{15.20}$$

We note also that

$$\mathbf{P}^2 = P_1^2 + P_2^2 = P_+P_- = P_-P_+ \;; \tag{15.21}$$

$$[\mathbf{P}^2, J] = 0 \;; \qquad [\mathbf{P}^2, P_\pm] = 0 \;. \tag{15.22}$$

Namely $\mathbf{P}^2$ is a Casimir operator, with eigenvalues $p^2 \geq 0$. As $\mathbf{P}^2$ and J commute, their simultaneous eigenvectors $|p,m\rangle$

$$\mathbf{P}^2|p,m\rangle = p^2|p,m\rangle \qquad J|p,m\rangle = m|p,m\rangle, \qquad -\infty < p < \infty \;, \quad m = 0, \pm1, \pm2, \dots \;, \tag{15.23}$$

may be used as the basis vectors of a representation. Our task is to find the (infinite-dimensional) matrix representation of generic element $g(\mathbf{b},\theta) = T(\mathbf{b})R(\theta)$.

We start from observing

$$P_\pm|p,m\rangle = c\,|p,m\pm1\rangle \;. \tag{15.24}$$

c can be found by

$$|c|^2 = \langle p,m|P_\mp P_\pm|p,m\rangle = \langle p,m|\mathbf{P}^2|p,m\rangle = p^2 \;, \tag{15.25}$$

so

$$c = -ip \;, \tag{15.26}$$

where the phase factor is chosen for later convenience. That is,

$$|p,m\pm1\rangle = \pm\frac{i}{p}P_\pm|p,m\rangle \;. \tag{15.27}$$

(i) $p^2 = 0$,

$$P_\pm|0,m\rangle = 0 \;. \tag{15.28}$$

Other elements act as

$$J|0,m\rangle = m\,|0,m\rangle\ , \quad R(\theta)\,|0,m\rangle = e^{-im\theta}|0,m\rangle\ ; \quad T(\mathbf{b})|0,m\rangle = |0,m\rangle\ , \tag{15.29}$$

so

$$g(\mathbf{b},\theta) \Rightarrow U_m(\mathbf{b},\theta) = e^{-im\theta}\ . \tag{15.30}$$

(ii) $p^2 > 0$

The matrix elements of the generators are

$$\langle p,m'|J|p,m\rangle = m\,\delta_{m',m}\ ; \tag{15.31}$$

$$\langle p,m'|P_\pm|p,m\rangle = \mp ip\,\delta_{m',m\pm1}\ . \tag{15.32}$$

Finite representation matrix is given by

$$g(\mathbf{b},\theta) \to D^p(\mathbf{b},\theta)_{m'm} = e^{i(m'-m)\phi}J_{m-m'}(p\,b)e^{-im\theta}\ , \tag{15.33}$$

where $\cos\phi = \frac{\mathbf{p}\cdot\mathbf{b}}{p\,b}$ and $J_k(z)$ is the Bessel function (of the first kind) of order k (see below, Eq. (15.44)).

The rest of the section is dedicated to the proof of (15.33).

By definition

$$D^p(\mathbf{b},\theta)_{m'm} = \langle p,m'|g(\mathbf{b},\theta)|p,m\rangle = \langle p,m'|T(\mathbf{b})R(\theta))|p,m\rangle\ . \tag{15.34}$$

Now

$$R(\theta))|p,m\rangle = e^{-i\theta J}|p,m\rangle = e^{-i\theta m}|p,m\rangle\ , \tag{15.35}$$

so that

$$D^p(\mathbf{b},\theta)_{m'm} = e^{-i\theta m}\langle p,m'|T(\mathbf{b})|p,m\rangle\ . \tag{15.36}$$

Let ϕ be the angle $\mathbf{b}$ makes with an arbitrarily fixed $\hat{x}$ axis:

$$\mathbf{b} = R(\phi)\mathbf{b}_0\ , \quad \mathbf{b}_0 = \begin{pmatrix} b \\ 0 \end{pmatrix}\ , \tag{15.37}$$

then

$$T(\mathbf{b}) = T(R(\phi)\mathbf{b}_0) = e^{-i\phi J}T(\mathbf{b}_0)e^{i\phi J} \tag{15.38}$$

so that

$$\langle p,m'|T(\mathbf{b})|p,m\rangle = e^{-i(m'-m)\phi}\langle p,m'|T(\mathbf{b}_0)|p,m\rangle = e^{-i(m'-m)\phi}\langle p,m'|e^{-iP_1 b}|p,m\rangle$$
$$= \quad e^{-i(m'-m)\phi}\langle p,m'|e^{-\frac{ib}{2}(P_+ + P_-)}|p,m\rangle = e^{-i(m'-m)\phi}\sum_{k,\ell}\frac{(\frac{-ib}{2})^{k+\ell}}{k!\ell!}\langle p,m'|P_+^k P_-^\ell|p,m\rangle \ . \tag{15.39}$$

But

$$P_-|p,m\rangle = ip|p,m-1\rangle \ ; \qquad P_+|p,m\rangle = -ip|p,m+1\rangle \ ; \tag{15.40}$$

so

$$\langle p,m'|T(\mathbf{b}_0)|p,m\rangle = \sum_{k,\ell}\frac{(\frac{-ib}{2})^{k+\ell}}{k!\ell!}(ip)^\ell(-ip)^k\langle p,m'|p,m+k-\ell\rangle$$
$$= \quad \sum_{k,\ell}\frac{(-)^k(\frac{pb}{2})^{k+\ell}}{k!\ell!}\delta_{m'-m,k-\ell} \ . \tag{15.41}$$

Now in the summation over k and ℓ the condition $k,\ell \geq 0$ and the Kronecker delta must both be taken into account. The solution can be summarized as:

(i) $m \geq m'$.

The summation ranges for k,ℓ are

$$k = 0,1,2,\ldots, \qquad \ell = k+m-m' \geq 0 \ , \tag{15.42}$$

thus

$$\langle p,m'|T(\mathbf{b}_0)|p,m\rangle = \left(\frac{pb}{2}\right)^{m-m'}\sum_{k=0}^{\infty}\frac{(-)^k(\frac{pb}{2})^{2k}}{k!(k+m-m')!}$$
$$= \quad J_{m-m'}(pb) \ , \tag{15.43}$$

where $J_\nu(z)$ is the Bessel function of the first kind,

$$J_\nu(z) \equiv \left(\frac{z}{2}\right)^\nu\sum_{n=0}^{\infty}\frac{(-)^n(\frac{z}{2})^{2n}}{n!\,\Gamma(\nu+n+1)} \ . \tag{15.44}$$

$J_\nu(z)$ is the solution of Bessel's equation

$$\frac{d^2 Z_\nu}{dz^2} + \frac{1}{z}\frac{dZ_\nu}{dz} + (1-\frac{\nu^2}{z^2})Z_\nu = 0 \ , \tag{15.45}$$

regular at the origin. Thus

$$g(\mathbf{b},\theta) \to D^p(\mathbf{b},\theta)_{m'm} = e^{i(m'-m)\phi} J_{m-m'}(p\,b) e^{-im\theta} \,, \tag{15.46}$$

(ii) $m' \geq m$

In this case the summation range is

$$\ell = 0, 1, 2, \ldots, \qquad k = \ell + m' - m \geq 0 \,. \tag{15.47}$$

so that

$$\begin{aligned} \langle p, m'|T(\mathbf{b}_0)|p, m\rangle &= (-)^{m'-m} \left(\frac{p\,b}{2}\right)^{m'-m} \sum_{\ell=0}^{\infty} \frac{(-)^\ell (\frac{p\,b}{2})^{2\ell}}{\ell!(\ell+m'-m)!} \\ &= (-)^{m'-m} J_{m'-m}(p\,b) = J_{m-m'}(p\,b) \end{aligned} \tag{15.48}$$

where use was made of the relation

$$J_{-n}(z) = (-)^n J_n(z) \,. \tag{15.49}$$

The proof of this relation is left to the reader.

15.2 Unitary representations of E_2: induced representation

Let us consider another possible unitary representations of E_2 by the *induced representation* method. For $p^2 = 0$, i.e., $p = 0$, the states of the basis are $|0, m\rangle$ considered already. The generators act

$$P_\pm|0, m\rangle = 0 \,, \qquad J|0, m\rangle = m|0, m\rangle \,. \tag{15.50}$$

Finite elements act as

$$R(\theta)|0, m\rangle = e^{-im\theta}|0, m\rangle \,, \qquad T(\mathbf{b})|0, m\rangle = |0, m\rangle \,. \tag{15.51}$$

From now on we take $p^2 > 0$. Consider a reference momentum

$$\mathbf{p} = \mathbf{p}_0 = (p, 0) \,. \tag{15.52}$$

Thus

$$P_1|\mathbf{p}_0\rangle = p|\mathbf{p}_0\rangle \,; \qquad P_2|\mathbf{p}_0\rangle = 0 \,, \qquad P^2|\mathbf{p}_0\rangle = p^2|\mathbf{p}_0\rangle \,. \tag{15.53}$$

$R(\theta)$ acts nontrivially on it:

$$\begin{aligned} P_k(R(\theta)|\mathbf{p}_0\rangle) &= R(\theta)R(\theta)^{-1}P_k R(\theta)|\mathbf{p}_0\rangle \\ &= R(\theta)\left(R(-\theta)\right)_{\ell k} P_\ell|\mathbf{p}_0\rangle = p_k\, R(\theta)|\mathbf{p}_0\rangle \end{aligned} \tag{15.54}$$

where

$$p_k = (p_0)_\ell \left(R(-\theta)\right)_{\ell k} = \left(R(\theta)\right)_{k\ell} (p_0)_\ell \tag{15.55}$$

is just $\mathbf{p}_0$ rotated by angle θ. (15.54) shows that the state $R(\theta)|\mathbf{p}_0\rangle$ is an eigenstate of $\mathbf{P}$ with eigenvalue, $\mathbf{p}$,

$$|\mathbf{p}\rangle \equiv R(\theta)|\mathbf{p}_0\rangle \ . \tag{15.56}$$

E_2 acts on it as

$$T(\mathbf{b})|\mathbf{p}\rangle = e^{-i\mathbf{b}\cdot\mathbf{p}}\,|\mathbf{p}\rangle \ ; \tag{15.57}$$

$$R(\phi)|\mathbf{p}\rangle = R(\phi)R(\theta)|\mathbf{p}_0\rangle = |\mathbf{p}'\rangle \ . \tag{15.58}$$

In other words ensemble of $|\mathbf{p}\rangle$ with $\mathbf{p}$ in all possible directions and magnitudes form the basis of a new representation space of E_2 [19]. It is not difficult to recover the representation (15.33) from the above by appropriate change of the basis [2].

16 The Lorentz group

16.1 Definition; algebra of the proper Lorentz group

The Lorentz group is defined by

$$x^\mu \to \Lambda^\mu_\nu x^\nu \tag{16.1}$$

such that the four-vector squared

$$x^\mu x_\mu = x^\mu x^\nu g_{\mu\nu} = (x^0)^2 - \mathbf{x}^2 \tag{16.2}$$

is invariant; equivalently the Minkowski metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{16.3}$$

is left invariant:

$$g_{\mu\nu} \to g_{\mu\nu}\Lambda^\mu_\lambda \Lambda^\nu_\sigma = g_{\lambda\sigma} \ . \tag{16.4}$$

It follows from the above that

$$(\det\Lambda)^2 = 1 \ , \qquad \therefore \quad \det\Lambda = \pm 1 \ . \tag{16.5}$$

Let us choose

$$\det\Lambda = \Lambda^0_\mu \Lambda^1_\nu \Lambda^2_\rho \Lambda^3_\sigma \epsilon^{\mu\nu\rho\sigma} = 1 \ , \tag{16.6}$$

[19] This is analogous to the use of the plane wave basis in solving $3D$ Schrödinger equations as compared to the angular-momentum-spherical-wave basis, which is similar to the representation of Subsection 15.1.

where $\epsilon^{\mu\nu\rho\sigma}$ is a totally antisymmetric tensor with

$$\epsilon^{0123} = 1 \ . \tag{16.7}$$

From (16.4) it follows that

$$(\Lambda^0_0)^2 - \sum_i (\Lambda^i_0)^2 = 1 \ , \qquad \therefore \ \ \Lambda^0_0 \geq 1 \ , \quad \text{or} \quad \Lambda^0_0 \leq -1 \ . \tag{16.8}$$

The transformations with the properties (16.6) and

$$\Lambda^0_0 \geq 0 \ , \tag{16.9}$$

form proper, ortho-chronous Lorentz group, sometime indicated as $\tilde{L}_+$.

Eq. (16.6) can be written in a more general form,

$$\epsilon^{\mu\nu\rho\sigma}\Lambda^\alpha_\mu \Lambda^\beta_\nu \Lambda^\gamma_\rho \Lambda^\delta_\sigma \epsilon^{\mu\nu\rho\sigma} = \epsilon^{\alpha\beta\gamma\delta} \ . \tag{16.10}$$

This and (16.4) show that $g_{\mu\nu}$ and $\epsilon^{\mu\nu\rho\sigma}$ are invariant tensor of the Lorentz group.

Some explicit forms are:

$$\Lambda^{(rotation)_{12}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{16.11}$$

for rotations in (12) plane, and

$$\Lambda^{(boost)} = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \tag{16.12}$$

for a boost in the z direction, where

$$\beta = \frac{v}{c} \ ; \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \ . \tag{16.13}$$

Or by introducing

$$\beta = \tanh\xi \ , \qquad \gamma = \frac{1}{\sqrt{1-\tanh^2\xi}} = \cosh\xi \ , \qquad -\infty < \xi < \infty \ , \tag{16.14}$$

$$\Lambda^{(boost)} = \begin{pmatrix} \cosh\xi & 0 & 0 & \sinh\xi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\xi & 0 & 0 & \cosh\xi \end{pmatrix} , \qquad (16.15)$$

which looks more similar to a rotation, i.e., a rotation with a pure imaginary angle.

The six generators of the Lorentz group, in the vector representation, are simply [20]

$$(J_{\lambda\sigma})_\mu^\nu = i(\delta_\lambda^\nu\, g_{\sigma\mu} - \delta_\sigma^\nu\, g_{\lambda\mu}) \ , \qquad (16.16)$$

cfr. (14.5) for the $SO(4)$ generators. The group $\tilde{L}_+$ is also known as $SO(3,1)$. The $so(3,1)$ algebra can be worked out straightforwardly:

$$[J_{\mu\nu}, J_{\lambda\sigma}] = -i\{J_{\lambda\nu}g_{\mu\sigma} - J_{\lambda\mu}g_{\nu\sigma} - J_{\sigma\nu}g_{\mu\lambda} + J_{\sigma\mu}g_{\nu\lambda}\} \ . \qquad (16.17)$$

Let us denote the rotation and boost generators as [21]

$$J_k \equiv \frac{1}{2}\epsilon^{k\ell m} J_{\ell m} \ , \qquad K_m \equiv J_{m0} \ . \qquad (16.18)$$

In terms of these, the $so(3,1)$ algebra reads

$$[J_m, J_n] = i\epsilon_{mn\ell}J_\ell \ , \qquad [K_m, J_n] = i\epsilon_{mn\ell}K_\ell \ , \qquad [K_m, K_n] = -i\epsilon_{mn\ell}J_\ell \ . \qquad (16.19)$$

Now we define [22]

$$M_m \equiv \frac{J_m + iK_m}{2} \ ; \qquad N_m \equiv \frac{J_m - iK_m}{2} \ ; \qquad (16.20)$$

in terms of these the algebra decouples:

$$[M_m, M_n] = i\epsilon_{mn\ell}M_\ell \ , \qquad [N_m, N_n] = i\epsilon_{mn\ell}N_\ell \ , \qquad [M_m, N_n] = 0 \ , \qquad (16.21)$$

showing

$$so(3,1) \sim su(2) \times su(2) \ . \qquad (16.22)$$

[20] A remark on the notation: here the label for the six generators are in lower indices, whereas the matrix indices are given as mixed suffix-index, as is appropriate with the Minkowski metric.

[21] We use the Latin letters $k, \ell, m, \ldots$ for the space indices, $1, 2, 3$, whereas the Greek letters $\mu, \nu, \lambda, \ldots$ for spacetime indices, $0, 1, 2, 3$.

[22] This is very similar to what was done for $SO(4)$; note however the crucial factor i.

16.2 Finite representations of $\tilde{L}_+$

The isomorphism $so(3,1) \sim su(2) \times su(2)$ allows us to introduce finite irreducible representations, labeled by the Casimirs

$$(j_1, j_2), \qquad j_1, j_2 = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, \tag{16.23}$$

and the results found for $SU(2)$ group before. The two Casimirs are

$$\mathbf{M}^2 = j_1(j_1+1) \ , \qquad \mathbf{N}^2 = j_2(j_2+1) \, ; \tag{16.24}$$

the basis vectors are

$$|j_1, m_1\rangle |j_2, m_2\rangle \ , \quad M_3 = m_1 = j_1, j_1 - 1, \ldots, -j_1 \ , \quad N_3 = m_2 = j_2, j_2 - 1, \ldots, -j_2 \ . \tag{16.25}$$

Some of the small representations are

$$(0,0), \quad (\frac{1}{2}, 0), \quad (0, \frac{1}{2}), \quad (1,0), \quad (0,1), \quad (\frac{1}{2}, \frac{1}{2}), \qquad \ldots . \tag{16.26}$$

The scalar fields, Weyl fermions of plus and minus chiralities (see more about these below), the electric and magnetic field, and vectors fields, respectively, are example of physical fields transforming according to these representations.

The explicit form of the (j_1, j_2) representation of $\tilde{L}_+$ can be found readily by using the results from the $SU(2)$ theory, (12.43)-(12.45). By inverting (16.20),

$$J_3 = M_3 + N_3 \ , \qquad J_\pm = M_\pm + N_\pm \ ; \tag{16.27}$$

$$K_3 = i(N_3 - M_3) \ , \qquad K_\pm = i(N_\pm - M_\pm) \ ; \tag{16.28}$$

and writing

$$|j_1, m_1\rangle |j_2, m_2\rangle \Rightarrow |m_1, m_2\rangle \tag{16.29}$$

one finds

$$J_3 |m_1, m_2\rangle = (m_1 + m_2) |m_1, m_2\rangle \ ; \tag{16.30}$$

$$\begin{aligned} J_+ |m_1, m_2\rangle &= \sqrt{(j_1 - m_1)(j_1 + m_1 + 1)} |m_1 + 1, m_2\rangle \\ &+ \sqrt{(j_2 - m_2)(j_2 + m_2 + 1)} |m_1, m_2 + 1\rangle \ ; \end{aligned} \tag{16.31}$$

$$\begin{aligned} J_- |m_1, m_2\rangle &= \sqrt{(j_1 + m_1)(j_1 - m_1 + 1)} |m_1 - 1, m_2\rangle \\ &+ \sqrt{(j_2 + m_2)(j_2 - m_2 + 1)} |m_1, m_2 - 1\rangle \ ; \end{aligned} \tag{16.32}$$

$$K_3 |m_1, m_2\rangle = i(m_2 - m_1) |m_1, m_2\rangle \ ; \tag{16.33}$$

$$
\begin{aligned}
K_+|m_1, m_2\rangle &= i\sqrt{(j_2-m_2)(j_2+m_2+1)}|m_1, m_2+1\rangle \\
&- i\sqrt{(j_1-m_1)(j_1+m_1+1)}|m_1+1, m_2\rangle \; ; \qquad (16.34)
\end{aligned}
$$

$$
\begin{aligned}
K_-|m_1, m_2\rangle &= i\sqrt{(j_2+m_2)(j_2-m_2+1)}|m_1, m_2-1\rangle \\
&- i\sqrt{(j_1+m_1)(j_1-m_1+1)}|m_1-1, m_2\rangle \; ; \qquad (16.35)
\end{aligned}
$$

These give complete $(2j_1+1)(2j_2+1)\times(2j_1+1)(2j_2+1)$ matrix elements of the generators J_i, K_i ($\sim M_{\mu\nu}$).

Let us make several observations.

(i) Due to the factor i in (16.28) the generators are non Hermitian: the representation is non unitary. This reflects the noncompact nature of the Lorentz group (the Boosts!).

(ii) In a compact group, each finite representation is equivalent to a unitary representation.

(iii) Vice versa, an irreducible representation in a noncompact group is either finite but non unitary, or unitary but infinite-dimensional.

(iv) Can one make a good use of these finite but nonunitary representations (j_1, j_2) in physics? The answer is: **YES.** In quantum mechanics or especially, in quantum field theories, a central role in the theory is played by the operators. Thus field operators such as

$$\phi(\text{scalar}) \sim (0, 0) \; , \qquad (16.36)$$

$$\psi_L(\text{Weyl fermion}) \sim (\frac{1}{2}, 0) \; , \qquad (16.37)$$

$$\psi_R(\text{Weyl fermion}) \sim (0, \frac{1}{2}) \; , \qquad (16.38)$$

$$A_\mu(\text{vector}) \sim (\frac{1}{2}, \frac{1}{2}) \; , \qquad (16.39)$$

$$
\begin{aligned}
F_{\mu\nu}(\text{fieldtensor}) &\sim (\frac{1}{2}, \frac{1}{2}) \otimes (\frac{1}{2}, \frac{1}{2}) \\
&\sim (1, 0) \oplus (0, 1) \; , \qquad (16.40)
\end{aligned}
$$

and so on, play the role of the basic building blocks with which the standard model of fundamental interactions (based on the gauge group $SU(3)\times SU_L(2)\times U_Y(1)$) or the Grand Unified Theories (such as based on $SU(5)$ or $SO(10)$ groups) is constructed.

16.3 Spinor representations and Chiralities

Another powerful observation in the analysis of the Lorentz group is the isomorphism $so(3,1) \sim sl(2, \mathbf{C})$. The group $SL(2, \mathbf{C})$ is a special linear group of regular 2×2 complex matrices, with

$$\det M = 1 \ . \tag{16.41}$$

Clearly the number of independent degrees of freedom in M

$$4 \times 2 - 2 = 6 \ , \tag{16.42}$$

matches with that of the Lorentz group, 3 rotations and 3 boosts. The actual correspondence can be constructed as follows, by first defining four 2×2 Hermitian matrices,

$$\sigma^\mu \equiv (-\mathbb{1}, \sigma^i) \ , \qquad \bar{\sigma}^\mu \equiv (-\mathbb{1}, -\sigma^i) \ , \tag{16.43}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ ; \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ ; \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{16.44}$$

are the Pauli matrices. Now given a generic four vector p_μ one defines a 2×2 matrix

$$P = p^\mu \sigma_\mu = \begin{pmatrix} -p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & -p^0 + p^3 \end{pmatrix} \ . \tag{16.45}$$

As

$$\mathrm{Tr}\,(\bar{\sigma}^\mu \sigma^\nu) = 2\, g^{\mu\nu} \ , \tag{16.46}$$

the above relation can be inverted:

$$p^\mu = \frac{1}{2} \mathrm{Tr}(\bar{\sigma}^\mu P) \ . \tag{16.47}$$

Now consider a transformation of P,

$$P \Rightarrow M P M^\dagger = P' \ , \tag{16.48}$$

where $M \in SL(2, \mathbf{C})$. Clearly,

$$\det P \Rightarrow \det(M P M^\dagger) = \det P' \ , \tag{16.49}$$

as $\det M = 1$. But as

$$\det P = (p^0)^2 - \mathbf{p}^2 = p^\mu p_\mu \tag{16.50}$$

(16.48) clearly represents a Lorentz transformation on a four vector.

The relation between (16.49) and the Lorentz transformation can be made more explicit: actually it is easy to verify that $M = e^{\frac{i}{2}\phi \cdot \sigma}$ and $M = e^{\frac{1}{2}\omega \cdot \sigma}$ describe, respectively, the three rotations and three boosts, respectively.

As a group, $SL(2,\mathbf{C})$ is simply connected: it represents a universal (double-)covering of $SO(3,1)$.

A spinor of left or right chirality can be defined as quantities transforming as $\psi \Rightarrow M\,\psi$ or $\bar\psi \Rightarrow \bar\psi M^\dagger$. By introducing the $su(2)\times su(2)$ indices $\alpha, \dot\alpha = 1,2$,

$$\psi_\alpha \Rightarrow M_\alpha^\beta \psi_\beta \; ; \qquad \bar\psi_{\dot\alpha} \Rightarrow (M^*)_{\dot\alpha}^{\dot\beta} \bar\psi_{\dot\beta} \; . \tag{16.51}$$

It is also useful to raise or lower the spinor indices by antisymmetric $su(2)\times su(2)$ tensors

$$\epsilon^{12} = -\epsilon^{21} = -\epsilon_{12} = \epsilon_{21} = 1 \; , \tag{16.52}$$

as

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \; , \qquad \bar\psi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\alpha} \; , \tag{16.53}$$

then these spinors transform as

$$\psi^\alpha \Rightarrow (M^{-1})_\beta^\alpha \, \psi^\beta \; ; \qquad \bar\psi^{\dot\alpha} \Rightarrow ((M*)^{-1})_{\dot\beta}^{\dot\alpha} \, \bar\psi^{\dot\beta} \tag{16.54}$$

(the proof of which is left to the reader). The 2×2 matrix P can then be seen to transform as

$$P_{\alpha\dot\beta} = p_\mu(\sigma^\mu_{\alpha\dot\beta}) \sim \psi_\alpha \bar\psi_{\dot\beta} \sim (\tfrac{1}{2}, \tfrac{1}{2}) \; , \tag{16.55}$$

whereas $\psi \sim (\frac{1}{2}, 0)$, $\bar\psi \sim (0, \frac{1}{2})$.

Other useful combinations are

$$\psi^\alpha \psi_\alpha \; , \qquad \bar\psi_{\dot\alpha}\bar\psi^{\dot\alpha} \; , \qquad i\psi^\alpha (\sigma^\mu)_{\alpha\dot\beta}\partial_\mu \bar\psi^{\dot\alpha} \tag{16.56}$$

which are all invariant under $su(2)\times su(2)$ or $sl(2,\mathbf{C})$: they are Lorentz invariant. They can be used to write a mass term or the kinetic terms for the fermions.

The 4×4 Dirac matrices are

$$\gamma^\mu = \begin{pmatrix} \mathbf{0} & \sigma^\mu \\ \bar\sigma^\mu & \mathbf{0} \end{pmatrix} \tag{16.57}$$

they act on four-component Dirac spinor (e.g., an electron)

$$\psi_D = \begin{pmatrix} \chi_\alpha \\ \bar\psi^{\dot\alpha} \end{pmatrix} \; , \qquad \bar\psi_D \equiv \psi_D^\dagger \gamma^0 = (-\psi^\alpha \; - \bar\chi_{\dot\alpha}) \; . \tag{16.58}$$

The free Dirac equation has the form

$$(i\gamma^\mu\partial_\mu - m)\psi_D = 0 \; , \qquad \begin{pmatrix} -m & i\sigma^\mu\partial_\mu \\ i\bar\sigma^\mu\partial_\mu & -m \end{pmatrix} \begin{pmatrix} \chi_\alpha \\ \bar\psi^{\dot\alpha} \end{pmatrix} = 0 \; , \tag{16.59}$$

Massless fermion of definite chirality is describer by Weyl's equation,

$$i\sigma^\mu \partial_\mu \bar{\psi}^{\dot{\alpha}} = 0 \ , \tag{16.60}$$

or

$$E\bar{\psi} = i\frac{\partial}{\partial x_0}\bar{\psi} = -\sigma \cdot \mathbf{p}\,\bar{\psi} = 0 \ . \tag{16.61}$$

So

$$E > 0 \Rightarrow \sigma \cdot \mathbf{p} < 0 \tag{16.62}$$

and $\bar{\psi}$ can be interpreted as a wave function of negative helicity.

The generators of the Lorentz group in the spinor representations are given by (cfr. Eq. (16.16))

$$J^{\mu\nu} = \frac{i}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \begin{pmatrix} i\frac{\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu}{4} & 0 \\ 0 & i\frac{\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu}{4} \end{pmatrix} \ . \tag{16.63}$$

Define the chirality

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \ . \tag{16.64}$$

with eigenvalues $\pm$ for chirality $\pm$ states.

The spinor generators (16.63) show that the particles (states) of positive and negative chiralities $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ transform separately and independently, under the Lorentz group. Indeed,

$$[\gamma_5, J^{\mu\nu}] = 0 \ : \tag{16.65}$$

the chirality is conserved.

17 Poincaré group

The set of transformations of the type,

$$g(b, \Lambda): \quad x^\mu \Rightarrow (x^\mu)' = \Lambda^\mu_\nu x^\nu + b^\mu \ , \tag{17.1}$$

i.e., a Lorentz transformation followed by translation, which leaves invariant the Minkowski geodesic

$$ds = dx^\mu g_{\mu\nu} dx^\nu \ , \tag{17.2}$$

form the Poincaré group, $\tilde{P}$. The group product rule is

$$g(b', \Lambda')g(b, \Lambda) = g(\Lambda' b + b', \Lambda'\Lambda) \ , \tag{17.3}$$

as can be easily verified. By construction a generic element of the Poincaré group can be written in a factorized form,

$$g(b, \Lambda) = T(b)\,\Lambda\ , \tag{17.4}$$

with an obvious notation, so that

$$g(0, \Lambda) = \Lambda, \qquad g(b, \mathbb{1}) = T(b)\ . \tag{17.5}$$

Let us introduce the generator of the translation as P_μ, i.e.,

$$T(b) = e^{-ib^\mu P_\mu}\ , \tag{17.6}$$

The algebra of the Poincaré group is

$$[P_\mu, P_\nu] = 0\ ; \qquad [P_\mu, J_{\lambda,\sigma}] = i(P_\lambda g_{\mu\sigma} - P_\sigma g_{\mu\Lambda})\ ; \tag{17.7}$$

$$[J_{\mu\nu}, J_{\lambda\sigma}] = i(J_{\lambda\nu} g_{\mu\sigma} - J_{\lambda\mu} g_{\nu\sigma} - J_{\sigma\nu} g_{\mu\lambda} + J_{\sigma\mu} g_{\nu\lambda})\ \ , \tag{17.8}$$

where the last ones are just the Lorenz group subalgebra. The Lorentz transformations and the translations form two subalgebras; the translations form Abelian ideal (invariant subalgebra). The Poincaré algebra is non semi-simple and non compact.

Theorem: The translation subgroup $T(b)$ forms an Abelian, invariant subgroup; more precisely

$$\Lambda T(b) \Lambda^{-1} = T(\Lambda b)\ . \tag{17.9}$$

The proof is straightforward: in fact, use of (17.3), (17.5) gives

$$g(0, \Lambda) g(b, \mathbb{1}) g(0, \Lambda^{-1}) = g(0, \Lambda) g(b, \Lambda^{-1}) = g(\Lambda b, \mathbf{1}) = T(\Lambda b)\ . \tag{17.10}$$

In terms of the generators $\{P^\mu, J_{\lambda\sigma}\} \sim \{P^0, P_m, J_m, K_m\}$ the algebra becomes:

$$[P^0, J_n] = 0\ ; \qquad [P_m, J_n] = i\epsilon_{mnk} P_k\ ; \tag{17.11}$$

$$[P_m, K_n] = i\delta_{mn} P^0\ ; \qquad [P^0, K_n] = iP_n\ ; \tag{17.12}$$

$$[J_m, J_n] = i\epsilon_{mn\ell} J_\ell\ , \qquad [K_m, J_n] = i\epsilon_{mn\ell} K_\ell\ , \qquad [K_m, K_n] = -i\epsilon_{mn\ell} J_\ell\ . \tag{17.13}$$

Out of these generators, two Casimir operators can be formed:

$$C_1 \equiv P^\mu P_\mu\ , \qquad C_2 \equiv W^\lambda W_\lambda\ , \tag{17.14}$$

where

$$W^\lambda = \epsilon^{\lambda\mu\nu\sigma} J_{\mu\nu} P_\sigma / 2\ , \tag{17.15}$$

is the Pauli-Lubanski vector. It satisfies several relations

$$W^\lambda P_\lambda = 0 \; ; \qquad [W^\lambda, P^\mu] = 0 \; ; \tag{17.16}$$

$$[W^\lambda, J^{\mu\nu}] = i(W^\mu g^{\lambda\nu} - W^\nu g^{\mu\lambda}) \; ; \qquad [W^\lambda, W^\sigma] = i\epsilon^{\lambda\sigma\mu\nu} W_\mu P_\nu \; . \tag{17.17}$$

It can be easily shown that $C_{1,2}$ commute with all generators, so they can be used to label the possible irreducible representations.

Note that in the subspace with $P^\mu = p^\mu$ (i.e., eigenstates of the four momentum),

$$W^\lambda = \epsilon^{\lambda\mu\nu\sigma} J_{\mu\nu} p_\sigma / 2 \; , \tag{17.18}$$

so that, for instance, in states with momenta $p^\mu = (M, \mathbf{0})$,

$$W^0 = 0 \; , \qquad W^i = \frac{M}{2} \epsilon^{ijk} J_{jk} \tag{17.19}$$

which are just the angular momentum operators: the little (stability) group of ($p^\mu = (M, \mathbf{0})$.

Theorem 1

The irreducible representations of $\tilde{P}$ are classified by the eingenvalues of $C_{1,2}$.

Theorem 2

In the subspace with $P^\mu = p^\mu$ the independent components of W^μ generate a subgroup of $G(p^\mu) \subset \tilde{P}$. This fact immediately follows from the last of Eq. (17.17). It is the little group of p^μ.

Theorem 3

The representation of $\tilde{P}$ can be constructed from the irreducible representations of $G(p^\mu)$ by repeated applications of Lorentz transformations. They are known as the induced representations. Some examples of physical interest are discussed below.

17.1 Zero momentum states $p^\mu = 0$; $C_{1,2} = 0$

In this case

$$G(p^\mu = 0) = \tilde{L}_+ \; , \tag{17.20}$$

and the problem reduces to that of the irreducible representations of the Lorentz group, already solved in the previous section.

17.2 Timelike momenta $C_1 > 0$; $p^\mu p_\mu = M^2 > 0$

Let us take as a reference momentum,

$$p^\mu = (M, \mathbf{0}) \ , \tag{17.21}$$

or consider the center-of-mass system. The Pauli-Lubanski vector reduces to, in this subspace

$$W^i = M\, J^i \in so(3) \ : \tag{17.22}$$

the rotation generators. The irreps of this little group are known: they are completely classified by the value of the spin, j, with the base vectors

$$|\mathbf{0}, j, m\rangle \ , \tag{17.23}$$

such that

$$P^\mu |\mathbf{0}, j, m\rangle = p_0^\mu |\mathbf{0}, j, m\rangle \ , \qquad p_0^\mu = (M, \mathbf{0}) \ ; \tag{17.24}$$

$$\mathbf{J}^2 |\mathbf{0}, j, m\rangle = j(j+1)|\mathbf{0}, j\rangle \ ; \qquad J_3 |\mathbf{0}, j, m\rangle = m|\mathbf{0}, j, m\rangle \ . \tag{17.25}$$

The actions of $J_{1,2}$ have been worked out before. All other vectors can be constructing by acting the rotations and boosts on these vectors.

For instance,

$$|p\hat{z}, j, m\rangle = L_3(\xi)|\mathbf{0}, j, m\rangle \ , \qquad p = M \sinh \xi \tag{17.26}$$

where L_3 is a boost in the $\hat{z}$ direction, (16.15). A general vector is obtained by rotating it:

$$|\mathbf{p}, j, m\rangle = R(\alpha, \beta, 0)|p\hat{z}, j, m\rangle = H(p)|\mathbf{0}, j, m\rangle \ , \tag{17.27}$$

$$H(p) \equiv R(\alpha, \beta, 0) L_3(\xi) \ . \tag{17.28}$$

Theorem: The vectors $|\mathbf{p}, j, m\rangle$ thus constructed provides the basis vectors of all possible irreducible unitary representations of $\tilde{P}$.

The generators act on them as:

$$T(b)|\mathbf{p}, j, m\rangle = e^{-ib^\mu p_\mu} |\mathbf{p}, j, m\rangle \ ; \tag{17.29}$$

$$\Lambda |\mathbf{p}, j, m\rangle = D^j_{m'm}(R(\Lambda, \mathbf{p}))\, |\mathbf{p}', j, m'\rangle \ , \tag{17.30}$$

where [2]

$$p' = \Lambda p \ ; \qquad R(\Lambda, \mathbf{p}) = H^{-1}(p')\Lambda H(p) \ . \tag{17.31}$$

17.3 Lightlike momenta $C_1 = P^\mu P_\mu = 0$, $M^2 = 0$

The states with a generic momentum p can be obtained from a reference momentum p_ℓ,

$$p_\ell^\mu = (\omega_0, 0, 0, \omega_0) \tag{17.32}$$

by appropriate Lorentz transformation. Let us first find out the irreducible representations of the little group of p_ℓ.

The Pauli-Lubanski vectors take, in this subspace the form

$$W^0 = -W^3 = \omega_0 J_{12} = \omega_0 J_3 \ ; \tag{17.33}$$

$$W^1 = \omega_0(J_{23} + J_{20}) = \omega_0(-J_1 + K_2) \ ; \tag{17.34}$$

$$W^2 = \omega_0(J_{31} - J_{10}) = \omega_0(-J_2 - K_1) \ ; \tag{17.35}$$

from which follows the second Casimir,

$$C_2 = W^\mu W_\mu = -(W_1^2 + W_2^2) \tag{17.36}$$

and

$$[W^1, W^2] = 0 \ ; \qquad [W^2, J_3] = iW^1 \ ; \qquad [W^1, J_3] = -iW^2 \ . \tag{17.37}$$

Remarkably, these are isomorphic to the algebra of E_2 studied in Section 15, if a formal identification $W^{1,2} \Rightarrow T^{1,2}$ is made.

As in the $SO(2)$ coset representation of E_2 (for $p = 0$ there) the representation of the little group is one-dimensional,

$$P^\mu |\mathbf{p}_\ell, \lambda\rangle = p_\ell^\mu |\mathbf{p}_\ell, \lambda\rangle \ ; \tag{17.38}$$

$$J_3 |\mathbf{p}_\ell, \lambda\rangle = \lambda |\mathbf{p}_\ell, \lambda\rangle \ ; \qquad W_i |\mathbf{p}_\ell, \lambda\rangle = 0 \ . \tag{17.39}$$

where

$$\lambda = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots \lambda halfint \tag{17.40}$$

correspond to the helicity $\mathbf{p}\cdot\mathbf{s}/p$ of the state.

Theorem:

The vectors

$$|p, \lambda\rangle = H(p)|p_\ell, \lambda\rangle \ , \qquad H(p) = R(\alpha, \beta, 0) L_3(\xi) \tag{17.41}$$

or

$$|p, \lambda\rangle = R(\alpha, \beta, 0)|p\hat{z}, \lambda\rangle \ ; \qquad p = p^0 = \omega_0 e^\xi \ , \tag{17.42}$$

form the basis of unitary irreducible representations of $\tilde{P}$.

The finite elements act as

$$T(b)|p,\lambda\rangle = e^{-ib^\mu p_\mu}|p,\lambda\rangle \ ; \qquad \Lambda|p,\lambda\rangle = e^{-i\lambda\theta(\Lambda,p)}|\Lambda p,\lambda\rangle \ , \tag{17.43}$$

where the angle $\theta(\Lambda,p)$ is defined by (see [2])

$$e^{-i\lambda\theta(\Lambda,p)} = \langle p_\ell,\lambda|H(\Lambda p)^{-1}\Lambda H(p)|p_\ell,\lambda\rangle \ . \tag{17.44}$$

Note that the helicity λ is invariant.

Remark:

The proper representations of $SO(2)$ group are characterized by integer winding numbers,

$$\lambda = m = 0, 1, 2, \ldots \tag{17.45}$$

(see Eq. (10.14)). However we do know that the rotation group $SO(3)$ (hence its subgroup $SO(2)$) allows for improper (*spinor*) representations. They are the irreps of its covering group $SU(2)$. From physics point of view, on the other hand, we know that in Nature there exist particles carrying half integer spins (intrinsic angular momentum), e.g., electrons, neutrinos, nucleons, quarks, etc. Therefore it is appropriate to allow for half-integer helicity states in the representations of the Poincaré group, as our description of Nature.

Part V

Roots, weights and Dynkin diagrams

The roots and weight systems for the Lie algebras will be discussed here. They provide us with a deep understanding of the structure of the different Lie algebras and their relations. These systems are so restrictive that in terms of a few inputs (the simple roots) the entire algebra can be reconstructed, and furthermore such a construction leads to the complete classification of semi-simple Lie algebras of compact groups.

18 Root and weight vectors of a semi-simple algebra

A semi-simple Lie algebra can be cast in the following form (Cartan-Weyl basis):

$$\begin{aligned}
&[H_i, H_j] = 0; , \qquad i, j = 1, 2, \ldots m \ , \\
&[H_i, E_\alpha] = \alpha_i \, E_\alpha \ , \\
&[E_\alpha, E_\beta] = N_{\alpha\beta} \, E_{\alpha+\beta} \ , \qquad (\alpha + \beta \neq 0 \ , \quad \alpha + \beta \in \Phi) \ , \\
&[E_\alpha, E_{-\alpha}] = \alpha_i H_i = \alpha \cdot \mathbf{H} \ .
\end{aligned} \tag{18.1}$$

The commuting generators H_i together form a subalgebra known as the Cartan subalgebra. Its dimension m is the *rank* of the algebra (and of the group it generates). The vectors α,

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \tag{18.2}$$

are known as *the root vectors.* The set of all root vectors of the algebra is denoted as Φ. The normalization of the generators can be conveniently fixed by

$$Tr \ H_i H_j = k \ \delta_{ij} \ . \tag{18.3}$$

The raising and lowering operators $J_\pm$ of $su(2)$ algebra (see Eq. (12.21), Eq. (12.22) are example of $E_\alpha, E_{-\alpha}$. Given two root vectors α and β their inner product is defined as

$$\alpha \cdot \beta = \sum_i \alpha_i \beta_i \ , \qquad \alpha^2 = \sum_i \alpha_i \alpha_i \ , \tag{18.4}$$

etc.

We define a triplet of generators associated to a root vector α, as [23]

$$J_1 = \frac{1}{\sqrt{2\alpha^2}}(E_\alpha + E_{-\alpha}) \; , \qquad J_2 = \frac{-i}{\sqrt{2\alpha^2}}(E_\alpha - E_{-\alpha}) \; , \tag{18.5}$$

$$J_3 = \alpha^* \cdot \mathbf{H} \; , \qquad \alpha^* = \frac{\alpha}{\alpha \cdot \alpha} \; . \tag{18.6}$$

It can be easily checked that J_i's satisfy an $su(2)$ algebra:

$$[J_i, J_j] = i\epsilon^{ijk} J_k \; . \tag{18.7}$$

It is a remarkable property of any semi-simple Lie algebra that it contains various $su(2)$ subalgebras, each associated with a triplet of generators, E_α, $E_{-\alpha}$ and $\alpha \cdot \mathbf{H}$. From this simple fact, there follow three fundamental theorems of group theory. Define also a weight vector $|\mu, R\rangle$

$$H_i|\mu, R\rangle = \mu_i|\mu, R\rangle \tag{18.8}$$

in any reresentation R. As seen already in the examples in $SU(3)$ theory, *the weight vectors* are vectors whose components are the simultaneous eigenvalues with respect to the commuting generators, H_i.[24]

(1) Theorem 1: if α and β are two roots, then

$$\frac{2\,\alpha \cdot \beta}{\alpha \cdot \alpha} \in \mathbb{Z} \; ; \tag{18.9}$$

(2) Theorem 2: if α and β are two roots, then

$$\beta - \frac{2\,\alpha \cdot \beta}{\alpha \cdot \alpha}\,\alpha \tag{18.10}$$

is a root. This is known as Weyl's reflection;

(3) Theorem 3: for any weight vector μ in any representation, and for any root vector α,

$$\frac{2\,\alpha \cdot \mu}{\alpha \cdot \alpha} \in \mathbb{Z} \tag{18.11}$$

holds.

As the root vectors are nothing but the weight vectors in the adjoint representation, The-

[23]As this $su(2)$ subalgebra is associated with a particular root vector α, it would be more adequate to indicate these generators as J_1^α, J_2^α, etc., but these suffixes will be omitted for simplicity of writing. No confusion should arise.

[24]In quantum mechanics, an analogous theorem states that the states can be chosen such that they are simultaneous eigenstats of such operators, implying that these operators represent mutually compatible observables, i.e., they can have definite values simultaneously. Vice versa, the operators which do not commute represent dynamical variables which suffer from Heisenberg's uncertainty relations.

orem 1 can be regarded as a special case of Theorem 3. These theorems will be proven below.

18.1 Illustrations with the $SU(3)$ group

The concepts of the roots and weights have already been introduced in the discussion of the $SU(3)$ group. It can be seen easily that the $SU(3)$ algebras (13.7)-(13.11) have already the form of the Cartan basis. The three $su(2)$ subalgebras associated with the isospin $I_\pm = T_1 \pm iT_2, T^3$, with the V spin, $V_\pm = T_4 \pm iT_5, V^3$ and the U spin, $U_\pm = T_6 \pm iT_7, U^3$ can be identified with (18.5), (18.6), associated with the three pairs of nonzero root vectors, $\pm\alpha_i$,

$$\alpha_1 = (1, 0) \; ; \qquad \alpha_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2}) \; ; \qquad \alpha_3 = (\frac{1}{2}, -\frac{\sqrt{3}}{2}) \; . \tag{18.12}$$

The precise relations are

$$E_{\pm\alpha_1} = \frac{1}{\sqrt{2}}(T_1 \pm iT_2) \; ; \tag{18.13}$$

$$E_{\pm\alpha_2} = \frac{1}{\sqrt{2}}(T_4 \pm iT_5) = \frac{1}{\sqrt{2}} V_\pm \; ; \tag{18.14}$$

$$E_{\pm\alpha_3} = \frac{1}{\sqrt{2}}(T_6 \pm iT_7) = \frac{1}{\sqrt{2}} U_\pm \; ; \tag{18.15}$$

$$H_1 = T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \; ; \qquad H_2 = T_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \; . \tag{18.16}$$

The weight vectors of the fundamental representations have been given in (13.16):

$$\mu_1 = (\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}), \qquad \mu_2 = (-\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}), \qquad \mu_3 = (0, -\tfrac{1}{\sqrt{3}}). \tag{18.17}$$

(The weight vectors of a complex conjugate representation $\underline{r}^*$ are simply minus of the weight vectors of $\underline{r}$). See Fig. 6.

The adjoint representation $\underline{8}$ arises from the decomposition $\underline{3} \times \underline{3}^* = \underline{8} + \underline{1}$, the root vectors are related to the weight vectors of the fundamentals, and made to correspond to the $|qq^*\rangle$ states:

$$\begin{array}{rcl} \text{root (weight) vector} & & \text{vector in } \underline{8}\ , \\ \alpha_1 = \mu_1 - \mu_2 = (1,0) = |E_{\alpha_1}\rangle & \longleftrightarrow & |q_1\rangle|q_2^*\rangle\ ; \\ \alpha_2 = \mu_1 - \mu_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2}) = |E_{\alpha_2}\rangle & \longleftrightarrow & |q_1\rangle|q_3^*\rangle\ ; \\ \alpha_3 = \mu_3 - \mu_2 = (\frac{1}{2}, -\frac{\sqrt{3}}{2}) = |E_{\alpha_3}\rangle & \longleftrightarrow & |q_3\rangle|q_2^*\rangle\ ; \\ -\alpha_1 = \mu_2 - \mu_1 = (-1,0) = |E_{-\alpha_1}\rangle & \longleftrightarrow & |q_2\rangle|q_1^*\rangle\ ; \\ -\alpha_2 = \mu_3 - \mu_1 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2}) = |E_{-\alpha_2}\rangle & \longleftrightarrow & |q_3\rangle|q_1^*\rangle\ ; \\ -\alpha_3 = \mu_2 - \mu_3 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}) = |E_{-\alpha_3}\rangle & \longleftrightarrow & |q_2\rangle|q_3^*\rangle\ ; \end{array} \qquad (18.18)$$

furthermore, there are two null root vectors

$$\alpha_{0\,1} = (0,0) = |E_{01}\rangle \longleftrightarrow \frac{|q_1\rangle|q_1^*\rangle - |q_2\rangle|q_2^*\rangle}{\sqrt{2}} \qquad (18.19)$$

$$\alpha_{0\,2} = (0,0) = |E_{02}\rangle \longleftrightarrow \frac{|q_1\rangle|q_1^*\rangle + |q_2\rangle|q_2^*\rangle - 2|q_3\rangle|q_3^*\rangle}{\sqrt{6}}\ , \qquad (18.20)$$

eight in all.

N.B. Another $|qq^*\rangle$ state

$$\frac{|q_1\rangle|q_1^*\rangle + |q_2\rangle|q_2^*\rangle + |q_3\rangle|q_3^*\rangle}{\sqrt{3}} \qquad (18.21)$$

is invariant: it corresponds to $\underline{1}$ of $SU(3)$.

19 $SU(2)$ substructures of a general semi-simple Lie algebra

The diagonal generators commute with each other:

$$[H_i, H_j] = 0\ : \qquad \mathrm{Tr}(H_i H_j) = k_D \delta_{ij}\ , \qquad i,j, = 1,2,\ldots,m\ , \qquad H_i^\dagger = H_i\ . \qquad (19.1)$$

They form the Cartan subalgebra. The simultaneous eigenvectors with respect to them,

$$H_i|\mu, R\rangle = \mu_i|\mu, R\rangle\ , \qquad i = 1,2,\ldots,m\ , \qquad (19.2)$$

where

$$\mu = (\mu_1, \mu_2, \ldots, \mu_m)\ , \qquad (19.3)$$

defines a *weight vector* in R representation. ($m=$ the rank of the algebra). Let us introduce a scalar product between a root vector and a weight vector:

$$(\alpha \cdot \mu) = \alpha_i \, \mu_i \; . \tag{19.4}$$

The algebra (see (18.1))

$$[H_i, E_\alpha] = \alpha_i E_\alpha \; , \qquad i = 1, 2, \ldots, m \; , \tag{19.5}$$

defines a root vector $\alpha = (\alpha_1, \alpha_2, \ldots)$. Now by taking the Hermitian conjugate of the above,

$$[H_i, E_\alpha^\dagger] = -\alpha_i E_\alpha^\dagger \tag{19.6}$$

one sees that

$$E_{-\alpha} = E_\alpha^\dagger, \qquad cfr. \quad [H_i, E_{-\alpha}] = -\alpha_i E_{-\alpha} \tag{19.7}$$

$E_{\pm\alpha}$ acts as the raising and lowering operator (cfr. $J_\pm$ of $su(2)$). Indeed,

$$\begin{aligned} H_i E_{\pm\alpha} |\mu\rangle &= ([H_i, E_{\pm\alpha}] + E_{\pm\alpha} H_i)|\mu\rangle \\ &= (\pm\alpha_i + \mu_i)(E_{\pm\alpha}|\mu\rangle) \; , \end{aligned} \tag{19.8}$$

that is,

$$E_{\pm\alpha}|\mu\rangle \propto |\mu \pm \alpha\rangle \; , \tag{19.9}$$

unless

$$E_{\pm\alpha}|\mu\rangle = 0 \; . \tag{19.10}$$

The other commutators are

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha+\beta} E_{\alpha+\beta} & \text{if,} \qquad \alpha + \beta \in \Phi \; ; \\ 0 & \text{otherwise.} \end{cases} \tag{19.11}$$

And finally

$$[E_\alpha, E_{-\alpha}] = \alpha \cdot H \; . \tag{19.12}$$

The root vectors together with $N_{\alpha+\beta}$ characterize the whole algebra. Actually, as we shall see below (subsection 20.8), the structure of the Lie algebras is so restrictive, that certain subset of the root vectors, called *simple roots*, completely determine the algebra (other roots and $N_{\alpha+\beta}$ follow from those). See below.

Ricapitulating, various $su(2)$ subalgebras (18.5), (18.6)

$$J_1 = \frac{1}{\sqrt{2\alpha^2}}(E_\alpha + E_{-\alpha}) \; , \qquad J_2 = \frac{-i}{\sqrt{2\alpha^2}}(E_\alpha - E_{-\alpha}) \; , \tag{19.13}$$

$$J_3 = \alpha^* \cdot \mathbf{H} \,, \qquad \alpha^* \; = \frac{\alpha}{\alpha \cdot \alpha} \,. \tag{19.14}$$

associated with *each nonzero root vector* are nested within any given Lie algebra [25]. The importance of this fact, in understanding the theory of Lie algebras, can hardly be overemphasized.

Applying the results such as (19.8), (19.9) to the adjoint representation (where $\mu = \alpha$ are the weight vectors) lead to some useful relations [26]

$$H_i|\alpha\rangle = \alpha_i|\alpha\rangle \,; \qquad H_i|\mathbf{0}\rangle = 0 \,; \qquad E_\alpha|\mathbf{0}\rangle = c\,|\alpha\rangle \,; \tag{19.15}$$

(c is a constant) so that

$$\begin{aligned} [H_i, E_\alpha]|\mathbf{0}\rangle &= H_i\, c\,|\alpha\rangle - E_\alpha H_i|\mathbf{0}\rangle = \\ &= \alpha_i\, c\,|\alpha\rangle = \alpha_i\, E_\alpha|\mathbf{0}\rangle \;: \end{aligned} \tag{19.16}$$

which is consistent with the algebra $[H_i, E_\alpha] = \alpha_i E_\alpha$.

From the theory of $SU(2)$ group, we know that

(i) The $SU(2)$ irreps are labelled by their Casimir, $\mathbf{J}^2 = j(j+1)$, j takes a nonegative integer or a semi-integer value only: $j = \frac{n}{2} = 0, \pm\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$;

(ii) The eigenvalues of J_3 are $j, j-1, j-2, \ldots, -j$;

(iii) $J_\pm$ are the raising and lowering operator of j_3 by one.

Applying these to one of the sub $su(2)$,

$$J_3 = \frac{\alpha \cdot H}{|\alpha|^2} \,; \qquad J_3\,|\mu\rangle = \frac{\alpha \cdot \mu}{\alpha^2}\,|\mu\rangle \,; \tag{19.17}$$

where α and μ are arbitrary root vector and weight vector. **Theorem 3** and hence **Theorem 1** follow from the properties (i) and (ii) of a general $su(2)$ algebra.

19.1 Proof of Theorem 2

The proof of Theorem 2 is also a straightforward application of these structures, applied to the adjoint representation. Start with

$$J_3^{(\alpha)}|\beta\rangle = \frac{\alpha \cdot \beta}{\alpha^2}|\beta\rangle = j_3|\beta\rangle \,. \tag{19.18}$$

[25] For precision these would have to be called $J_1^{(\alpha)}$, $J_2^{(\alpha)}$, etc., but the superscript (α) is omitted whenever the risk of misunderstanding is small.

[26] Read $|\alpha\rangle \equiv |\alpha, R^{adj}\rangle$, etc.

As

$$E_{\pm\alpha}|\beta\rangle \propto |\beta \pm \alpha\rangle \ , \tag{19.19}$$

$|\beta\rangle$ is a member of an $su^{(\alpha)}(2)$ multiplet. Indeed it easy to show that

$$J_3 E_{-\alpha}|\beta\rangle = (\frac{\alpha\cdot\beta}{\alpha^2} - 1)E_{-\alpha}|\beta\rangle \tag{19.20}$$

(the proof is left for the reader). From the known structure of an $SU(2)$ multiplet, we know that if $\frac{\alpha\cdot\beta}{\alpha^2} = j_3 > 0$, there must be a member with $j_3 = -\frac{\alpha\cdot\beta}{\alpha^2}$. Indeed acting $E_{-\alpha}$ on $|\beta\rangle$, $\frac{2\alpha\cdot\beta}{\alpha^2} \in \mathbb{Z}_{\geq 0}$ times, one gets

$$(E_{-\alpha})^{\frac{2\alpha\cdot\beta}{\alpha^2}}|\beta\rangle \propto |\beta - \frac{2\alpha\cdot\beta}{\alpha^2}\alpha\rangle \ , \tag{19.21}$$

i.e., $\beta - \frac{2\alpha\cdot\beta}{\alpha^2}\alpha$ is another root vector. If $\frac{\alpha\cdot\beta}{\alpha^2} = j_3$ is negative, we know that there must be a state with $j_3 = |j_3|$, which can be obtained by acting on β with E_α, $-\frac{2\alpha\cdot\beta}{\alpha^2} \in \mathbb{Z}_{\geq 0}$ times, reaching the same conclusion. *Q.E.D.*

19.2 Properties of the root vectors

A generic state (a weight vector) $|\mu, R\rangle$ is a member of certain $su(2)$ multiplet, (19.17). Let $j = \mathrm{Max}\{j_3\}$ of this multiplet. Thus for any $|\mu, R\rangle$ there is an integer $p \in \mathbb{Z}_{\geq 0}$ such that

$$J_+^p|\mu, R\rangle \neq 0 \ , \qquad J_+^{p+1}|\mu, R\rangle = 0 \ . \tag{19.22}$$

But since

$$J_3 J_+^p|\mu, R\rangle = (\frac{\alpha\cdot\mu}{\alpha^2} + p)J_+^p|\mu, R\rangle \tag{19.23}$$

this means that

$$\frac{\alpha\cdot\mu}{\alpha^2} + p = j \ , \tag{19.24}$$

with j either an integer or a half-integer. Similarly there is a nonnegative integer q such that

$$\frac{\alpha\cdot\mu}{\alpha^2} - q = -j \ . \tag{19.25}$$

By summing these two relations one finds that

$$\frac{2\alpha\cdot\mu}{\alpha^2} + p - q = 0 \ . \tag{19.26}$$

This is consistent with Theorem 3; the integers p and q label the positions of $|\mu, R\rangle$ within the multiplet $SU^{(\alpha)}(2)$.

Now we apply this important relation to a root vector β (which is a weight vector in

$(p-q)(p'-q')$	$\cos\theta_{\alpha\beta}$	$\theta_{\alpha\beta}$
0	0	$\pi/2$
1	$\pm 1/2$	$\pi/3,\ 2\pi/3$
2	$\pm 1/\sqrt{2}$	$\pi/4,\ 3\pi/4$
3	$\pm\sqrt{3}/2$	$\pi/6,\ 5\pi/6$

Table 12:

the adjoint representation), $\mu = \beta$, to get

$$\frac{\alpha \cdot \beta}{\alpha^2} = -\frac{1}{2}(p-q) \ , \tag{19.27}$$

and interchanging the roles of $su^{(\alpha)}$ and $su^{(\beta)}$, to find

$$\frac{\alpha \cdot \beta}{\beta^2} = -\frac{1}{2}(p'-q') \ , \tag{19.28}$$

so that

$$\cos^2\theta_{\alpha\beta} = \frac{(\alpha\cdot\beta)^2}{\alpha^2\beta^2} = \frac{(p-q)(p'-q')}{4} \ , \tag{19.29}$$

where

$$p, q, p', q' \in \mathbb{Z}_{\geq 0} \ . \tag{19.30}$$

Clearly this implies that

$$\frac{(p-q)(p'-q')}{4} \leq 1 \ . \tag{19.31}$$

But as $p, q, p', q' \in \mathbb{Z}_{\geq 0}$ only four possibilities are allowed for the angle between two root vectors: see Table 12.

20 Simple roots

Definition: A weight vector $(\mu_1, \mu_2, \mu_3, \ldots)$ is said to be *positive*, if the first nonvanishing component is positive.

This allows to order the weights: we say $\mu > \nu$, if $\mu - \nu$ is positive.

Definition: In particular, the *highest weight* of a representation is the one which is larger (higher) than all other weights.

Definition: Simple roots
Consider the roots $\{\alpha_1, \alpha_2, \ldots\} \in \Phi$ of a given Lie algebra. Some of them are positive. A positive root, which cannot be written as a vector sum of other positive roots, is *simple*. In a given algebra, there are exactly m simple roots, where m is the rank of the algebra. (Exercise: prove this). Let Λ denote the set of the simple roots, just as Φ denote the set of all roots. $\Lambda \subset \Phi$.

It follows from these definitions, that if a weight is annihilated by all the generators E_α associated with simple roots,

$$E_\alpha|\mu\rangle = 0 \; , \qquad \alpha \in \Lambda \; , \tag{20.1}$$

then it is a highest weight.

Problem: Find the highest weight vectors of representations $\underline{3}$, $\underline{8}$, $\underline{10}$ of $SU(3)$.

Problem: Find the simple roots of $SU(3)$ and $SU(4)$ algebras.

Theorem

If $\alpha \neq \beta$ are two simple roots, then $\alpha - \beta$ is not a root.

The proof is a simple reductio ad absurdum. Assume that $\alpha - \beta$ were a root, call it γ. Without losing generality we assume $\alpha - \beta > 0$. Then γ would be a positive root, and it would follow that $\alpha = \beta + \gamma$, i.e., α is a sum of two positive roots, which contradicts the supposition that α and β are both simple.

From this theorem it follows that

$$E_{-\alpha}|\beta\rangle = E_{-\beta}|\alpha\rangle = 0 \; , \qquad \alpha, \beta \in \Lambda \; . \tag{20.2}$$

Recalling that $E_{-\alpha}$ is a lowering operator $J_-^{(\alpha)}$ of $su^\alpha(2)$, and similarly for $E_{-\beta}$, the above means that a simple root is the lowest member of all $SU(2)$ multiplets associated with other simple roots. See Fig. 9. It follows that

$$\frac{\alpha \cdot \beta}{\alpha^2} + p = j \; ; \qquad \frac{\alpha \cdot \beta}{\alpha^2} - 0 = -j \; ; \tag{20.3}$$

that is,

$$\frac{2\alpha \cdot \beta}{\alpha^2} = -p \; ; \tag{20.4}$$

and similarly,

$$\frac{\alpha \cdot \beta}{\beta^2} + p' = j' \; ; \qquad \frac{\alpha \cdot \beta}{\beta^2} - 0 = -j' \; ; \tag{20.5}$$

that is,

$$\frac{2\alpha \cdot \beta}{\beta^2} = -p' \; . \tag{20.6}$$

Namely for a couple of simple roots α, β, there is a relation

$$\frac{\alpha \cdot \beta}{\alpha^2} = -\frac{p}{2} \; ; \qquad \frac{\alpha \cdot \beta}{\beta^2} = -\frac{p'}{2} \tag{20.7}$$

so that

$$\cos\theta_{\alpha\beta} = -\frac{\sqrt{pp'}}{2} \; , \qquad \frac{\beta^2}{\alpha^2} = \frac{p}{p'} \; . \tag{20.8}$$

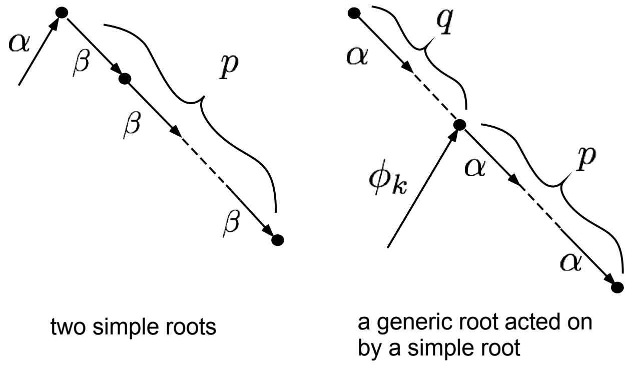


Figure 9:

Therefore,

$$\frac{\pi}{2} \le \theta_{\alpha\beta} \le \pi \ , \qquad \alpha \cdot \beta \le 0 \ . \tag{20.9}$$

Here are a few more theorems.

Theorem: The simple roots are linearly independent.

The proof is again straightforward.

$$\gamma = \sum_{\alpha \in \Lambda} x_\alpha \alpha = 0 \tag{20.10}$$

is clearly not possible if $x_\alpha > 0$, $\forall \alpha$. If some of x_α are negative, we may write the above as

$$\gamma = \sum_{\alpha_i \in \Lambda} x_i \alpha_i - \sum_{\alpha_j \in \Lambda} x_j \alpha_j \equiv \mu - \nu = 0 \qquad (x_i, x_j > 0) \ . \tag{20.11}$$

Again this is impossible, as

$$\gamma^2 = \mu^2 + \nu^2 - 2 \sum_{i,j} x_i x_j \alpha_i \cdot \alpha_j > 0 \ , \tag{20.12}$$

where use was made of (20.9).

Theorem: Any positive root can be written as a linear combination of simple roots, with positive integer coefficients,

$$\phi = \sum_{\alpha \in \Lambda} k_\alpha \alpha \ , \qquad k_\alpha \in \mathbb{Z}_{\geq 0} \ . \tag{20.13}$$

Theorem: The m simple roots form a complete set of vectors.

The proof again is done with a reductio ad absurdum: assume that there exists a vector $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$ which is orthogonal to all of the simple roots. It follows from the previous theorems that such a vector is orthogonal to all positive roots, and indeed to all roots,

$$\xi \cdot \alpha = 0, \qquad \forall \alpha \in \Phi \ . \tag{20.14}$$

Then

$$[\xi \cdot H, E_\alpha] = (\xi \cdot \alpha) E_\alpha = 0 \ ; \qquad [\xi \cdot H, H_j] = 0 \ : \tag{20.15}$$

i.e., $\xi \cdot H$ commutes with *all* generators, contradicting the hypothesis that the algebra is semi-simple.

20.1 Reconstruction of the algebra from the simple roots

We have learned above that all positive roots ϕ_k have the form,

$$\phi_k = \sum_\alpha k_\alpha \alpha \ , \qquad \alpha \in \Lambda \ , \qquad k_\alpha \in \mathbb{Z}_{\geq 0} \ ; \tag{20.16}$$

but it is clear that the opposite is not always true: not all of the expressions of this form give a root vector. In other words, the question is, given all $\alpha \in \Lambda$, which set $\{k_\alpha\}$ yields a root vector?

Let us proceed iteratively. If $k_\alpha = 1$ for an $\alpha \in \Lambda$, then $\phi_1 = \alpha$ is obviously a root (indeed it is a simple root). Suppose all of the roots which have the form of ϕ_k up to $k \leq \ell$ have been found. Consider $E_\alpha |\phi_\ell\rangle$: it would be proportional to the root $|\phi_\ell + \alpha\rangle$, if it does not vanish. Recall that a root vector ϕ_ℓ is in a position (p, q) specified by two integers,

$$\frac{2\alpha \cdot \phi_k}{\alpha^2} = -(p - q) \ , \tag{20.17}$$

within an $su(2)$ multiplet generated by a root vector α. See the right of Fig. 9, There will necessarily be a nonnegative integer p such that

$$(E_\alpha)^{p+1} |\phi_\ell\rangle = 0 \ . \tag{20.18}$$

If $p > 0$ $\phi_\ell + \alpha$ is a root; If $p = 0$ $\phi_\ell + \alpha$ is not a root.

For instance, for $\ell = 1$, $\phi_1 = \beta$ is a (simple) root. So $q = 0$, and with $p, p' \in \mathbb{Z}_{\geq 0}$

$$\frac{2\alpha \cdot \beta}{\alpha^2} = -p \; ; \qquad \frac{2\alpha \cdot \beta}{\beta^2} = -p' \; . \tag{20.19}$$

If $\alpha \cdot \beta = 0$, then $p = 0$ and $\alpha + \beta$ is *not* a root; otherwise, it is a root.

An example from $SU(3)$ The simple roots are $\alpha_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $\alpha_2 = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$. $\beta = (1, 0)$ is a positive, but not simple, root. As $(\alpha_1)^2 = (\alpha_2)^2 = 1$, and $\alpha_1 \cdot \alpha_2 = -\frac{1}{2}$, one verifies that

$$\frac{2\alpha_1 \cdot \alpha_2}{\alpha_1^2} = \frac{2\alpha_1 \cdot \alpha_2}{\alpha_2^2} = -1 \; , \qquad \therefore \quad p = p' = 1 \; , \tag{20.20}$$

so that $\alpha_1 + \alpha_2$ is a root (which is indeed equal to β). Similarly, it is easy to check that none of $\alpha_1 + 2\alpha_2$, $2\alpha_1 + \alpha_2$, $\alpha_1 + 3\alpha_2$, etc., belong to Φ. Adding the three negative and null roots the root diagram of $SU(3)$ is now complete.

We have not yet taken fully into account of the basic results on the simple roots, (20.8),

$$\cos \theta_{\alpha\beta} = -\frac{\sqrt{pp'}}{2} \; , \qquad |\frac{\beta}{\alpha}| = \sqrt{\frac{p}{p'}} \; . \tag{20.21}$$

As the two integers can take only the values $p, p' = 0, 1, 2, 3$ there are only several possibilities:

(i) $p = p' = 0$, $\quad \beta/\alpha$ indeterminate $(so(4))$;

(ii) $p = p' = 1$, $\quad \theta_{\alpha\beta} = 120°$; $\quad \beta/\alpha = 1$ $(su(3))$;

(iii) $p = 1$, $p' = 2$, $\quad \theta_{\alpha\beta} = 135°$; $\quad \beta/\alpha = 1/\sqrt{2}$ $(so(5) \sim usp(4))$;

(iv) $p = 1$, $p' = 3$, $\quad \theta_{\alpha\beta} = 150°$; $\quad \beta/\alpha = 1/\sqrt{3}$ (g_2);

where the names of the appropriate rank 2 algebras are shown. See Fig. 10.

21 Dynkin diagrams

In a more general semi-simple algebra, these properties are summarized very efficiently by the Dynkin diagrams. For any simple root we draw a little circle; for any pair of them they are connected (or unconnected) by lines, according to the possible relations indicated in Eq. (20.21), see Fig. 11.

Actually there are only two different magnitudes between the simple roots, in any algebra. It is customary to indicate the larger simple root by an empty circle and the smaller one by a full circle. The complete list of semi-simple, compact groups with their Dyinkin diagrams is shown in Fig. 12.

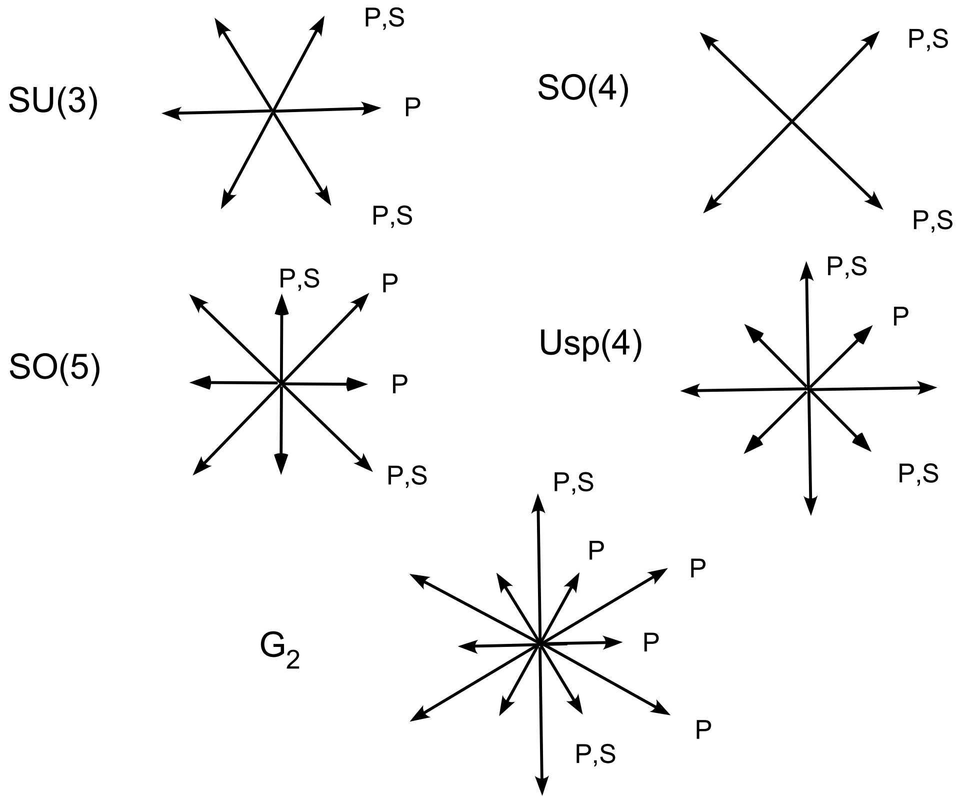


Figure 10: Root diagrams of rank 2 groups are shown. Positive (P) and simple (S) roots are indicated.

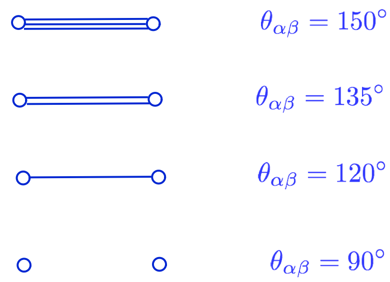


Figure 11:

Another, equivalent, way to summarize the structure of a given Lie algebra is the use of Cartan's matrix, defined by

$$A_{ji} \equiv \frac{2\alpha_j \cdot \alpha_i}{\alpha_i^2} \ , \tag{21.1}$$

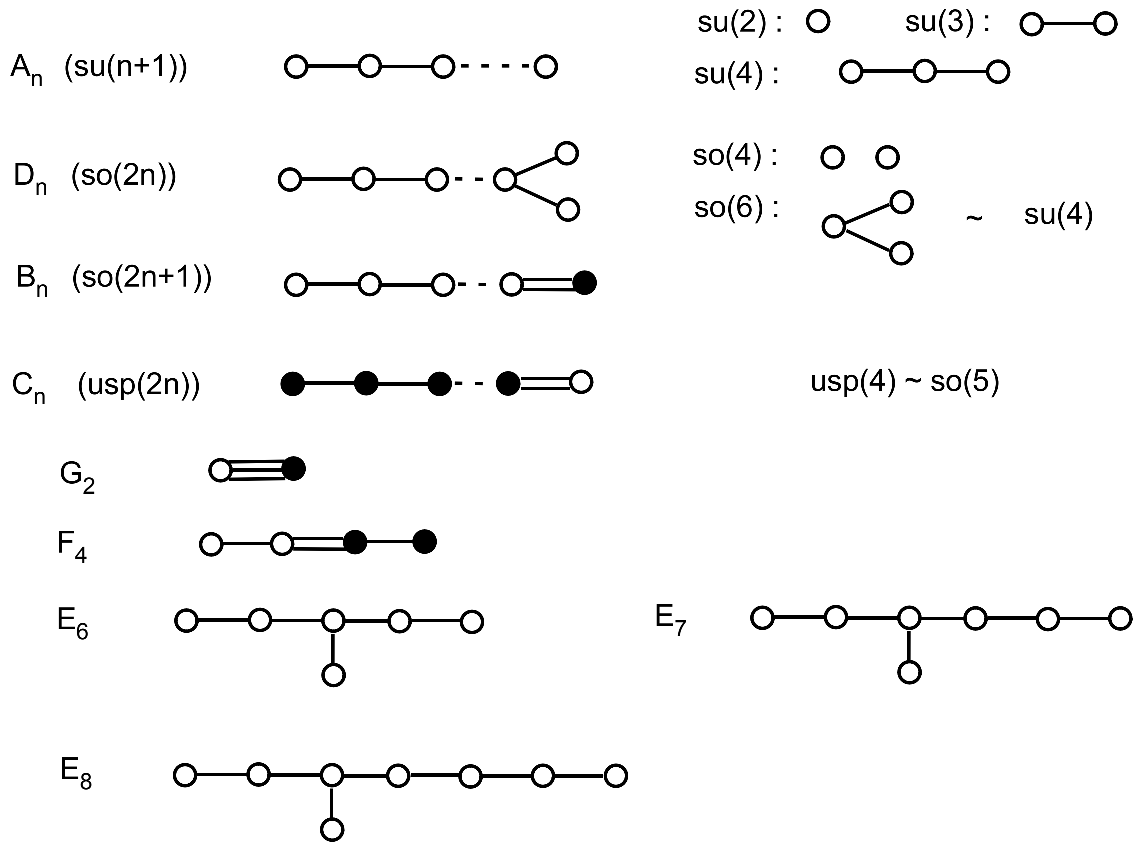
that is, the collection of the values of $2J_3^{(\alpha_i)}$ for α_j.

$$A_{ii} = 2 \qquad (\text{def}) \ . \tag{21.2}$$

Some examples of A_{ji} are:

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \ ; \qquad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \ ; \qquad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \ , \tag{21.3}$$

for $su(3)$, $su(4)$, and g_2, respectively.



A_n (su(n+1))
D_n (so(2n))
B_n (so(2n+1))
C_n (usp(2n))
G_2
F_4
E_6
E_7
E_8
su(2) :
su(3) :
su(4) :
so(4) :
so(6) :
~ su(4)
usp(4) ~ so(5)

Figure 12:

Part VI

Some applications in quantum mechanics

Group theory plays key roles in the understanding of symmetries and their manifestations in physics in general, and in particular, in quantum mechanics. Here three well-known applications of group theory in quantum mechanics will be discussed: the multi-dimensional harmonic oscillator, the Hydrogen atom spectrum, and the Wigner-Eckart theorem.

22 Multi-dimensional isotropic harmonic oscillators

The two and three (or more generally $D-$) dimensional harmonic oscillators

$$H = \frac{\mathbf{p}^2}{2m} + \frac{1}{2} m\,\omega^2 \mathbf{r}^2 = \omega\hbar \sum_{i=1}^{D} (a_i^\dagger a_i + \frac{1}{2}) \ , \tag{22.1}$$

can be easily solved by separation of variables in Cartesian coordinates, the wave functions reducing to the products of the one-dimensional harmonic oscillator states,

$$|\Psi\rangle_N = |n_1\rangle |n_2\rangle \cdots |n_D\rangle \ , \qquad N = n_1 + n_2 + \ldots n_D \ , \qquad n_i = 0, 1, 2, \ldots . \tag{22.2}$$

where

$$|n\rangle = \frac{a^{\dagger\, n}}{\sqrt{n!}} |0\rangle \ , \tag{22.3}$$

with the energy

$$E_N = \omega\hbar (N + \frac{D}{2}) \ . \tag{22.4}$$

The creation and annihilation operators satisfy the commutation relations,

$$[a_i, a_j^\dagger] = \delta_{ij} \ , \qquad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \ . \tag{22.5}$$

The degeneracy of the N-th energy level is then given by

$$G(N,2) = N+1 \quad (D=2) \ ; \qquad G(N,3) = \frac{(N+1)(N+2)}{2} = \frac{(N+2)!}{N!\,2!} \quad (D=3) \ ; \tag{22.6}$$

$$G(N,D) = \frac{(N+1)(N+2)\cdots(N+D-1)}{(D-1)!} = \frac{(N+D-1)!}{N!\,(D-1)!} \ , \tag{22.7}$$

as can be easily found (see Appendix A) by counting the sets of $\{n_i\}$'s which give the fixed total $N = n_1 + n_2 + \dots n_D$.

How can we understand this degeneracy formula? Clearly the Hamiltonian (22.1) is invariant under $SO(D)$ transformations, but this alone does not explain the degeneracy. For instance, for $D = 3$, the rotational invariance means that the $2L + 1$ states with the same angular momentum L are degenerate. This, complimented by the information that the states at a given level N are even (for N=even) or odd (for N=odd) under parity, and the fact that the maximum value of L at the level N is equal to N, shows that the total degeneracy is indeed given by the sum

$$\sum_{L \text{ even}}^{N} (2L+1) = \frac{(N+1)(N+2)}{2} = G(N,3) \ , \qquad N = \text{even} \ , \tag{22.8}$$

$$\sum_{L \text{ odd}}^{N} (2L+1) = \frac{(N+1)(N+2)}{2} = G(N,3) \ , \qquad N = \text{odd} \ , \tag{22.9}$$

but the *significance* of the degeneracy among the states with different L remains to be illucidated.

The crucial observation is that the Hamiltonian (22.1), written in terms of the creation and destruction operators, is actually invariant under a larger symmetry: $SU(D) \supset SO(D)$. Let us indicate the generators of $SU(D)$ as $D \times D$ matrices, T^a, $a = 1, 2, \dots, D^2-1$. For $D = 2$ and $D = 3$, T^a's are just (one half) the Pauli matrices and the Gell-Mann matrices, studied in Section 12 and Section 13, respectively. These satisfy the algebra

$$[T^a, T^b] = i f^{abc} \, T^c \ , \tag{22.10}$$

where f^{abc} are the structure constants of the $SU(D)$ group.

Define the operators

$$\mathbf{T}^a \equiv \sum_{i,j} a_i^{\dagger} (T^a)_{ij} a_j \ , \tag{22.11}$$

It can be easily verified that these satisfy

$$[\mathbf{T}^a, H] = 0 \ , \tag{22.12}$$

as well as the commutation relations

$$[\mathbf{T}^a, \mathbf{T}^b] = i f^{abc} \, \mathbf{T}^c \ . \tag{22.13}$$

(These follow from the definition of the $SU(D)$ generators, (22.10) and the commutation relations (22.5). The derivation is left to the reader as an excercise.) Note that even though (22.10) and (22.13) are formally identical, their meaning is different. The generators in (22.10) are the ones defined in the fundamental representation, $\underline{N}$: the matrices T^a's are $N \times N$. $\mathbf{T}^a$'s in (22.13) are defined in the infinite-dimensional space of the D- dimensional harmonic oscillator states,

$$|n_1, n_2, \ldots\rangle, \qquad n_i = 0, 1, 2, \ldots \ . \tag{22.14}$$

In particular, Eq. (22.12) shows that the action of $SU(D)$ does not modify the energy of the state

$$H\, e^{i\mathbf{T}^a\alpha^a}|N\rangle = e^{i\mathbf{T}^a\alpha^a} H|N\rangle = E_N\, e^{i\mathbf{T}^a\alpha^a}|N\rangle \ . \tag{22.15}$$

On the other hand, the action of $e^{i\mathbf{T}^a\alpha^a}$ on the states $|N\rangle = |n_1\rangle|n_2\rangle\cdots|n_D\rangle$ transform these among themselves, in other words, the states at a given level N transform according to an irreducible representation of $SU(D)$. The matrices

$$\langle n_1', n_2', \ldots | e^{i\mathbf{T}^a\alpha^a} | n_1, n_2, \ldots\rangle \ , \qquad \sum_i n_i' = \sum_i n_i = N \ , \tag{22.16}$$

form indeed a $G(N, D)$ dimensional irreducible representation of $SU(D)$ group.

Writing these as

$$|N\rangle = \prod_i \frac{a_i^{\dagger\, n_i}}{\sqrt{n_i!}}|0\rangle \ , \qquad n_1 + n_2 + \ldots n_D = N \tag{22.17}$$

these states can be interpreted as the states with N phonons (each with energy $\omega\hbar$), each photon one of D different types, without ordering, as all of $a_i^\dagger$'s commute with each other. Consequently, the states at the level N can be regarded as belonging to the symmetric representation of $SU(D)$, with N boxes in horizontal line of Young tableau

$$\square\square\square\square\square\square\square \tag{22.18}$$

Each box (each phonon) can be one of D types. Its multiplicity is given by $G(N, D)$, see Appendix A.

23 The spectrum of the Hydrogen atom and $SO(4)$ symmetry

Another interesting application of group theory to quantum mechanics is the spectrum of the bound hydrogen atom,

$$H = \frac{\mathbf{p}^2}{2m} - \frac{e^2}{r} \ . \tag{23.1}$$

The n-th energy level is given by the famous Bohr spectrum,

$$E_n = -\frac{e^2}{2\, r_B\, n^2} \ , \qquad r_B \equiv \frac{\hbar^2}{m\, e^2} \ , \tag{23.2}$$

furthermore the n-th level is

$$d(n) = n^2 \tag{23.3}$$

times degenerate. Even though, again, such a degeneracy is partially understood by the $SO(3)$ symmetry of the Hamiltonian, and from the fact that the orbital angular momentum takes the values $\ell = 0, 1, \ldots, n-1$, i.e.,

$$d(n) = \sum_{\ell=0}^{n-1}(2\ell+1) = n^2 \ , \tag{23.4}$$

the degeneracy among the states with different values of ℓ remains to be understood.

In order to gain more insight, we introduce the Lenz vector [27]

$$\begin{aligned} \mathbf{A} &\equiv \frac{e^2\mathbf{r}}{r} - \frac{\hbar}{2m}(\mathbf{p}\times\mathbf{L} - \mathbf{L}\times\mathbf{p}) \\ &= \frac{\mathbf{r}}{r} - \frac{1}{2}(\mathbf{p}\times\mathbf{L} - \mathbf{L}\times\mathbf{p}) \end{aligned} \tag{23.5}$$

where in the second line the dimensional constants are all set to unity ($e = \hbar = m = 1$). $\mathbf{L}$ is the orbital angular momentum operators,

$$\mathbf{L} \equiv \mathbf{r}\times\mathbf{p}/\hbar \ , \qquad \mathbf{p} = -i\hbar\nabla \ . \tag{23.6}$$

It is a straightforward exercise to show that all components of $\mathbf{A}$ commute with the Hamiltonian

$$[\mathbf{A}, H] = 0 \ . \tag{23.7}$$

Naturally H commutes with the angular momentum operators

$$[\mathbf{L}, H] = 0 \ , \qquad \therefore \quad [\mathbf{L}^2, H] = 0 \ , \tag{23.8}$$

but $\mathbf{A}$ and $\mathbf{L}$ do not commute,

$$[L_i, A_j] = i\epsilon_{ijk}A_k \ . \tag{23.9}$$

In particular,

$$[\mathbf{L}^2, A_j] \neq 0 \ : \tag{23.10}$$

This relation, together with (23.7), (23.8), implies further degeneracies among states in a given Bohr level with different angular momenta.

The components A_i satisfy the commutation relations

$$[A_i, A_j] = -2i\epsilon_{ijk}HL_k \ , \tag{23.11}$$

[27] Historically, the Lenz vector was introduced in classical mechanics in the study of the planetary motion.

and finally L_i's obey the known algebra

$$[L_i, L_j] = i\epsilon_{ijk} L_k \ . \tag{23.12}$$

The algebras among A_j's and L_i's, (23.9)-(23.12), are reminiscent of the $so(4)$ algebra. They in fact become isomorphic to the latter, if the operator factor H is replaced by a number. That is, these algebras become identical to $so(4)$, *if considered among the degenerate states of a Bohr level with a fixed energy:*

$$H|n,\ell,m\rangle = E_n|n,\ell,m\rangle \ , \qquad E_n < 0 \ , \qquad H \to E_n \ . \tag{23.13}$$

Define then

$$\mathbf{u} \equiv \frac{\mathbf{A}}{\sqrt{-2E_n}} \ ; \tag{23.14}$$

and

$$\mathbf{j}_1 \equiv \frac{\mathbf{L}+\mathbf{u}}{2} \ ; \qquad \mathbf{j}_2 \equiv \frac{\mathbf{L}-\mathbf{u}}{2} \ . \tag{23.15}$$

$\mathbf{j}_{1,2}$ satisfy indeed an $su(2) \times su(2) \sim so(4)$ algebra

$$[j_{1\,i}, j_{1\,j}] = i\epsilon_{ijk} j_{1\,k} \ ; \qquad [j_{2\,i}, j_{2\,j}] = i\epsilon_{ijk} j_{2\,k} \ ; \qquad [j_{1\,i}, j_{2\,j}] = 0 \ . \tag{23.16}$$

The irreducible representations of these algebras are well known:

$$|j_1, m_1\rangle, \qquad m_1 = j_1, j_1-1, \ldots, -j_1 \ ; \tag{23.17}$$

$$|j_2, m_2\rangle, \qquad m_2 = j_2, j_2-1, \ldots, -j_2 \ , \tag{23.18}$$

where

$$j_1, \ j_2 = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \ . \tag{23.19}$$

Actually an important restriction on the allowed representations exists, as

$$\mathbf{L}\cdot\mathbf{u} = 0 \ , \tag{23.20}$$

that is,

$$\mathbf{j}_1^2 = \mathbf{j}_2^2 \ : \tag{23.21}$$

only the multiplets with the same Casimir operator, $j_1 = j_2 = j$, are allowed.

Now a crucial passage in the discussion is the relation [28]

$$\mathbf{A}^2 = \frac{2\,H}{m}(\mathbf{L}^2+1) + e^4 \ , \qquad H = \frac{\mathbf{p}^2}{2m} - \frac{e^2}{r} \ , \tag{23.22}$$

which can be proven by straightforward, albeit somewhat lengthy, manipulations, by re-

[28]The factors m and e are restored briefly here.

peated use of the commutation relations among $\{L_i, r_i, p_i\}$. The derivation is left for the reader as an exercise. Setting again $m = e = 1$, one finds

$$\mathbf{L}^2 + \mathbf{u}^2 = -1 - \frac{1}{2H} = -1 - \frac{1}{2E_n} \; , \tag{23.23}$$

where in the final passage we restricted ourselves to a particular Bohr level, $H = E_n$. Finally one finds, by using (23.20) and this relation, that

$$\mathbf{j}_1^2 = \mathbf{j}_2^2 = \frac{\mathbf{L}^2 + \mathbf{u}^2}{4} = \frac{-1 - \frac{1}{2E_n}}{4} \; . \tag{23.24}$$

The energy eigenvalues can be found by setting $\mathbf{j}_1^2 = \mathbf{j}_2^2 = j(j+1)$,

$$E_n = -\frac{1}{2(2j+1)^2} = -\frac{1}{2\, n^2} \; , \tag{23.25}$$

where

$$n = 2j + 1 = 1, 2, 3, \dots \; . \tag{23.26}$$

By restoring all the dimensionful constants, one finds Bohr's energy levels

$$E_n = -\frac{e^2}{2\, n^2 r_B} \; , \qquad r_B \equiv \frac{\hbar^2}{m e^2} \; , \qquad n = 1, 2, 3, \dots \; , \tag{23.27}$$

and the Comlomb degeneracy

$$(2j_1 + 1)(2j_2 + 1) = (2j + 1)^2 = n^2 \; . \tag{23.28}$$

24 Wigner-Eckart theorem

A very important theorem that illustrates well the power of symmetry arguments in Quantum Mechanics is due to Wigner and Eckart. Consider matrix elements

$$\langle J, M; n' | T_q{}^p | j, m; n \rangle \; , \tag{24.1}$$

where $|j, m\rangle$ and $|J, M\rangle$ are two generic eigenstates of angular momentum (two $su(2) \sim so(3)$ representations), and n, n' stand for all other quantum numbers (the radial quantum number, type of particle, etc.). $T_q{}^p$ is the q-th component of a spherical tensor operator of rank p (see below).

The theorem:

$$\langle J, M; n' | T_q{}^p | j, m; n \rangle = \langle p, j; J, M | p, q, j, m \rangle \langle J, n' \| \mathbf{T}^p \| j, n \rangle \; . \tag{24.2}$$

Namely these matrix elements are *proportional* to the CG coefficients. The proportionality constant, indicated by $\langle J, n' \| \mathbf{T}^p \| j, n \rangle$, called the *reduced matrix element*, depends only on the absolute magnitude of the angular momenta, the rank of the spherical tensor, as well as other quantum numbers, but not on the azimuthal quantum numbers, M, q, m. All the dependence on the latter resides in the universal Clebsch–Gordan coefficients.

Equation (24.2) provides many nontrivial relations among those matrix elements which differ only in the azimuthal quantum numbers M, q, m. In particular, it leads to a set of *selection rules*: a necessary condition for a matrix element to be non-vanishing is that the Clebsch–Gordan coefficient in (24.2) is not null. We talk about *allowed transition*, a terminology borrowed from the analysis of electromagnetic transitions. When the reduced matrix element is the same, the ratio between two matrix elements is simply given by known CG coefficients, without any need of computation. More generally, many matrix elements may be expressed in terms of a much smaller number of independent amplitudes (the reduced matrix elements), and in terms of the known CG coefficients. This implies that one can derive various nontrivial relations among the matrix elements, without need of any calculations of the matrix elements themselves.

The spherical tensors are defined as follows. Under a three-dimensional rotation any operator O transforms as follows:

$$O \to e^{i\boldsymbol{\omega}\cdot\hat{\mathbf{J}}}\, O e^{-i\boldsymbol{\omega}\cdot\mathbf{J}} \tag{24.3}$$

while a state transforms like this:

$$|\ \rangle \to e^{i\boldsymbol{\omega}\cdot\hat{\mathbf{J}}} |\ \rangle \ . \tag{24.4}$$

We have already seen that certain states—those with definite angular momentum (J, M)—transform in a simple, universal way:

$$|J, M\rangle \to e^{i\boldsymbol{\omega}\cdot\hat{\mathbf{J}}} |J, M\rangle = \sum_{M'} D^J_{M',M}(\boldsymbol{\omega})\, |J, M'\rangle \ . \tag{24.5}$$

The rotation matrix for spin J, $D^J_{M',M}(\boldsymbol{\omega})$, is known once and for all: it depend only on J and does not depend on any other attributes of the particular system considered. From the group-theoretic point of view, $D^J_{M',M}$ is an irreducible representation of the $SO(3)$ ($SU(2)$) group.

Certain operators transform in simple manner. Operators such as $\mathbf{r}^2$, $\mathbf{p}^2$, $U(r)$ are all *scalars*: they are invariant under rotations; others, such as $\mathbf{r}$, $\mathbf{p}$, e $\mathbf{J}$, are *vectors*. Quantities transforming as products of vectors are generally known as *tensors*.

It turns out that it is convenient to reorganize the components of tensors so as to make them proportional to the components of some spherical harmonics—they are known as *spherical tensors*—rather than using Cartesian components. For instance, a spherical tensor of rank 1 is equivalent to a vector (A_x, A_y, A_z), but its components are called $T_{1,m}$, $m = 1, 0, -1$, where

$$T_{1,1} = -\frac{A_x + iA_y}{\sqrt{2}}; \qquad T_{1,0} = A_z; \qquad T_{1,-1} = \frac{A_x - iA_y}{\sqrt{2}} \ . \tag{24.6}$$

In the particular case of the position vector $\mathbf{r}$, the corresponding spherical tensor components are:

$$T_{1,1} = -\frac{x + iy}{\sqrt{2}}; \qquad T_{1,0} = z; \qquad T_{1,-1} = \frac{x - iy}{\sqrt{2}} \ . \tag{24.7}$$

They are proportional to the spherical harmonics [29] $Y_{1,1}, Y_{1,0}$, e $Y_{1,-1}$. The inverse of eqn (24.7) is

$$A_x = -\frac{T_{1.1} - T_{1,-1}}{\sqrt{2}} \ ; \qquad A_y = -i\,\frac{T_{1,1} + T_{1,-1}}{\sqrt{2}} \ ; \qquad A_z = T_{1,0} \ .$$

The components of a spherical tensor of rank 2 (of "spin 2") are related to those in Cartesian components as follows:

$$\begin{aligned} T_{2,0} &= -\sqrt{\frac{1}{6}}(A_{xx} + A_{yy} - 2A_{zz}) \ ; \\ T_{2,\pm1} &= \mp(A_{xz} \pm iA_{yz}) \ ; \\ T_{2,\pm2} &= \frac{1}{2}(A_{xx} - A_{yy} \pm 2iA_{xy}) \ . \end{aligned}$$

The higher spherical tensors can be constructed by using the composition rule (12.55):

$$T_Q^P = \sum_{q,q'} T_q^p T_{q'}^{p'} \langle p, q; p', q' | p, p', P, Q\rangle \ . \tag{24.8}$$

By construction the spherical tensor operator of rank ("spin") p with $2p+1$ components transforms as:

$$T_q{}^p \Rightarrow e^{i\boldsymbol{\omega}\cdot\hat{\mathbf{J}}}\, T_q{}^p\, e^{-i\boldsymbol{\omega}\cdot\hat{\mathbf{J}}} = \sum_{q'} D^p{}_{q'\,q} T_{q'}^p \ , \tag{24.9}$$

just as a state with angular momentum $|j, j_z\rangle = |p, q\rangle$:

$$|p, q\rangle \Rightarrow e^{i\boldsymbol{\omega}\cdot\hat{\mathbf{J}}}\, |p, q\rangle = \sum_{q'} D^p{}_{q'\,q} |p, q'\rangle \ . \tag{24.10}$$

This means that the action of $T_q{}^p$ on the state $|j, m; n\rangle$ produces a state

$$T_q{}^p\, |j, m; n\rangle \ , \tag{24.11}$$

[29] In the convention used by Landau and Lifshitz Eqs. (24.6) and (24.7) are multiplied by a factor i.

which transforms exactly as the direct product of two angular momentum eigenstates

$$|p,q\rangle \otimes |j,m\rangle \ , \tag{24.12}$$

because

$$\begin{aligned} T_q{}^p\,|j,m;n\rangle \Rightarrow e^{i\boldsymbol{\omega}\cdot\hat{\mathbf{J}}}\,T_q{}^p\,\,|j,m;n\rangle &= e^{i\boldsymbol{\omega}\cdot\hat{\mathbf{J}}}\,T_q{}^p\,e^{-i\boldsymbol{\omega}\cdot\hat{\mathbf{J}}}\,e^{i\boldsymbol{\omega}\cdot\hat{\mathbf{J}}}\,|j,m;n\rangle \\ &= \sum_{q',m'} D^p{}_{q'\,q}\,D^j_{m'\,m}\,T^p_{q'}\,|j,m';n\rangle \ . \end{aligned} \tag{24.13}$$

This fact makes the WE theorem intuitively quite plausible.

To prepare for the proof, let us remember an angular momentum state $|J,M\rangle$ transforms under a generic rotation as

$$|J,M\rangle \to U(\omega)|J,M\rangle = \sum_{M'} D^J_{M',M}(\omega)|J,M'\rangle \ , \qquad U(\omega) \equiv e^{i\boldsymbol{\omega}\cdot\hat{\mathbf{J}}} \tag{24.14}$$

where $D^J_{M',M}(\omega)$ is the rotation matrix (an irreducible representation of rank J). Now by using a decomposition (12.55) on both sides. The l.h.s. gives

$$\begin{aligned} U(\omega)|J,M\rangle &= \sum_{m_1,m_2} U(\omega)|j_1,m_1;j_2,m_2\rangle\langle j_1,m_1;j_2,m_2|J,M\rangle \\ &= \sum_{m_1,m_2}\sum_{m_1',m_2'} D^{j_1}_{m_1',m_1}D^{j_2}_{m_2',m_2}|j_1,m_1';j_2,m_2'\rangle\langle j_1,m_1;j_2,m_2|J,M\rangle \ , \end{aligned} \tag{24.15}$$

whereas the r.h.s. of (24.14) yields

$$\sum_{M'} D^J_{M',M}(\omega)\sum_{m_1',m_2'} |j_1,m_1';j_2,m_2'\rangle\langle j_1,m_1';j_2,m_2'|J,M'\rangle \ . \tag{24.16}$$

By comparing the coefficient of $|j_1,m_1';j_2,m_2'\rangle$ (by using the orthonormality of the states) in (24.15) and (24.16) one gets

$$\sum_{m_1,m_2} D^{j_1}_{m_1',m_1}D^{j_2}_{m_2',m_2}\langle j_1,m_1;j_2,m_2|J,M\rangle = \sum_{M'} D^J_{M',M}(\omega)\langle j_1,m_1';j_2,m_2'|J,M'\rangle \ . \tag{24.17}$$

By using the completeness / orthogonality among the CG coefficients,

$$\sum_{J,M}\langle j_1,m_1;j_2,m_2|J,M\rangle\langle J,M|j_1,\tilde{m}_1;j_2,\tilde{m}_2\rangle = \delta_{m_1\tilde{m}_1}\delta_{m_2\tilde{m}_2} \tag{24.18}$$

one gets

$$D^{j_1}_{m_1',m_1}D^{j_2}_{m_2',m_2} = \sum_{M',J,M} D^J_{M',M}(\omega)\langle j_1,m_1';j_2,m_2'|J,M'\rangle\langle J,M|j_1,m_1;j_2,m_2\rangle \ . \tag{24.19}$$

We now multiply both sides with $(D^J_{\tilde{M}',\tilde{M}}(\omega))^*$, integrate over the $SO(3)$ group, $dg = d\omega = \frac{1}{8\pi^2} d\phi d\theta d\psi$ in terms of the Euler angles (see Eq. (3.22)), and use the orthogonality theorem (10.15) of the irreducible representations:

$$\int d\omega \, (D^{\tilde{J}}_{\tilde{M}',\tilde{M}}(\omega))^* D^J_{M',M}(\omega) = \frac{1}{2J+1} \delta_{\tilde{M}',M'} \delta_{\tilde{M},M} \delta_{\tilde{J},J} \ , \tag{24.20}$$

which gives

$$\int d\omega (D^J_{M',M}(\omega))^* D^{j_1}_{m_1',m_1} D^{j_2}_{m_2',m_2} = \frac{1}{2J+1} \langle j_1, m_1'; j_2, m_2' | J, M' \rangle \langle J, M | j_1, m_1; j_2, m_2 \rangle \ . \tag{24.21}$$

We are now ready to prove the Wigner-Eckart theorem. By inserting

$$U^\dagger(\omega) U(\omega) = \mathbb{1}$$

twice, one gets

$$\begin{aligned} & \langle J, M; n' | T_q{}^p | j, m; n \rangle = \langle J, M; n' | U^\dagger(\omega) U(\omega) T_q{}^p U^\dagger(\omega) U(\omega) | j, m; n \rangle \\ = & \sum_{M',q',m'} (D^J_{M'M}(\omega))^* D^p_{q'q}(\omega) D^j_{m'm}(\omega) \langle J, M'; n' | T_{q'}{}^p | j, m'; n \rangle \ . \end{aligned} \tag{24.22}$$

Now integrate the both sides over the group manifold $\int dg = \int d\omega$, noting that the left hand side is independent of ω. By using the result (24.21) found above, we obtain

$$\begin{aligned} & \langle J, M; n' | T_q{}^p | j, m; n \rangle \\ = & \frac{1}{2J+1} \sum_{M',q',m'} \langle J, M'; n' | T_{q'}{}^p | j, m'; n \rangle \langle p, q'; j, m' | J, M' \rangle \langle J, M | p, q; j, m \rangle \ , \end{aligned}$$

which is nothing but the Wigner-Eckart theorem, Eq. (24.2), after identifying the factor

$$\frac{1}{2J+1} \sum_{M',q',m'} \langle J, M'; n' | T_{q'}{}^p | j, m'; n \rangle \langle p, q'; j, m' | J, M' \rangle \tag{24.23}$$

which does not depend on $\{M, q, m\}$, as the reduced matrix element and *calling* it

$$\equiv \langle J, n' \| \mathbf{T}^p \| j, n \rangle \ . \tag{24.24}$$

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A Counting the sets of nonnegative integers $\{n_i\}$'s giving a fixed $N = n_1 + n_2 + \dots n_D$

Let $G(N, D)$ be this number. This number obviously satisfies a recursion formula

$$G(N, D) = \sum_{n=0}^{N} G(N - n, D - 1) \, . \tag{A.1}$$

Now define a generating function

$$F(D, t) \equiv \sum_{N=0}^{\infty} G(N, D)\, t^N \, , \qquad 0 < t < 1 \, . \tag{A.2}$$

From the recursion formula for $G(N, D)$ (A.1) one gets

$$F(D,t) = \sum_{N=0}^{\infty} G(N,D)\, t^N = \sum_{N=0}^{\infty} \sum_{n=0}^{N} G(N-n, D-1)\, t^N \ . \tag{A.3}$$

By changing the summation indices from (N, n) to $M \equiv N - n$ and n, which now run independently from 0 to ∞, one gets

$$F(D,t) = \sum_{M=0}^{\infty} \sum_{n=0}^{\infty} G(M, D-1)\, t^{n+M} = \frac{F(D-1,t)}{1-t} \ . \tag{A.4}$$

This is a recursion formula for $F(D, t)$ in D. For $D = 1$ one has from $G(N, 1) = 1$

$$F(1,t) = \sum_{N=0}^{\infty} t^N = \frac{1}{1-t} \ . \tag{A.5}$$

The solution of the recursion formula (A.4) is simply

$$F(D,t) = \frac{1}{(1-t)^D} \ . \tag{A.6}$$

The coefficient of t^N in the Taylor expansion of $F(D, t)$ is

$$G(N,D) = \frac{(-D)(-D-1)\cdots(-D-N+1)}{N!}(-)^N = \binom{N+D-1}{N} = \frac{(N+D-1)!}{N!(D-1)!} \ , \tag{A.7}$$

which gives the degeneracy of the N-th energy level of isotropic D-dimensional harmonic oscillator.