The force between the plates of a capacitor

Find the force between the plates of a standard plane capacitor, neglecting boundary effects. Give the result both for an isolated capacitor as a function of the costant charge Q and for a capacitor connected to an ideal voltage generator as a function of the constant voltage V. Use *two* different methods.

Solution

This simple classic problem is presented in order to point out and prevent typical and recurrent mistakes about both the electrostatic pressure on a conductor and the calculation of the force from the energy of the system.

Let us first proceed via the electrostatic pressure. If S is the surface of the plates, the surface charge density (on the *inner* surfaces) is $\pm\sigma = \pm Q/S$, and the uniform field inside the capacitor is $E = \sigma/\epsilon_0$. This leads to an electrostatic pressure

$$P = \frac{1}{2}\sigma E = \frac{\sigma^2}{2\epsilon_0}. \tag{1}$$

The typical mistake is to forget the 1/2 factor and to write $P \overset{\text{w}}{=} \sigma E$ ("w" stands for wrong!). As the pressure is normal to the charged inner surfaces, the force F is attractive (positive and negative charges attract each other!), so we put a minus sign (F acts to reduce the distance between the plates) and given as a function of Q by

$$F = -PS = -\frac{(Q/S)^2}{2\epsilon_0}S = -\frac{Q^2}{2\epsilon_0 S}. \tag{2}$$

As a function of Q, the force is independent of the distance h between the plates.

We can use the above method also for a capacitor connected to a voltage generator: all we need is to replace Q by V via the $Q = CV$ where the capacity $C = \epsilon_0 S/h$. Thus

$$F = -\frac{(CV)^2}{2\epsilon_0 S} = -\frac{\epsilon_0 V^2 S}{2h^2}. \tag{3}$$

Now let us recover these results in a different way, i.e. by calculating the force as minus the derivative of the energy variation with respect to a displacement dh. In the case in which the capacitor is isolated and has a charge Q, all its energy is electrostatic energy

$$U_{\text{es}} = \frac{Q^2}{2C} = \frac{Q^2 h}{2\epsilon_0 S} \tag{4}$$

so that the force is given by

$$F = -\partial_h U_{\text{es}} = -\frac{Q^2}{2\epsilon_0 S} \tag{5}$$

in agreement with the previous expression (2)

Now, if we rewrite U_{es} for a constant voltage

$$U_{\text{es}} = \frac{1}{2}CV^2 = \frac{V^2\epsilon_0 S}{2h} \tag{6}$$

we find for the derivative with respect to h

$$F \overset{\text{w}}{=} -\partial_h U_{\text{es}} = +\frac{Q^2}{2\epsilon_0 S}. \tag{7}$$

The "+" means that the force, altough of equal strength to (3), would change sign and be repulsive now! The point with this very common mistake is that we must take into account the variation of the *total* energy of the system, that also includes the internal energy of the voltage generator. In fact, the latter has to do some work to keep the voltage drop constant when the capacity is changed by an infinitesimal amount dC. Since a charge dQ must be carried across the constant voltage V, the work done is

$$dW = VdQ = Vd(VdC) = V^2dC, \tag{8}$$

thus the internal energy of the generator (whatever its nature: mechanical, chemical, ...) must change by the amount

$$dU_{\text{gen}} = -dW = -V^2dC. \tag{9}$$

Since at the same time the electrostatic energy changes by $d(CV^2/2)$, we obtain for the *total* variation

$$dU_{\text{tot}} = dU_{\text{gen}} + dU_{\text{es}} = -V^2dC + (V^2/2)dC = -(V^2/2)dC = -dU_{\text{es}}. \tag{10}$$

Thus, the force is

$$F = -\partial_h U_{\text{tot}} = +\partial_h U_{\text{es}} = -\frac{Q^2}{2\epsilon_0 S} \tag{11}$$

that gives back the correct result.