

## A charge inside a conducting shell

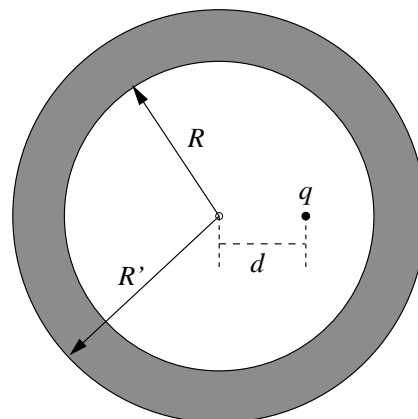
A point charge  $q$  is located inside a spherical conducting shell of internal radius  $R$  and external radius  $R'$ , at a distance  $d$  from the center. The shell is grounded, i.e. kept at zero potential.

**a)** Find the electric potential and the electric field in the whole space.

**b)** Find the force on the charge.

**c)** Show that the total charge induced on the sphere is  $-q$ .

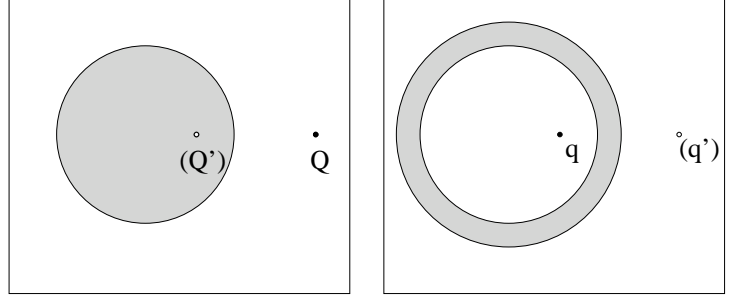
**d)** How does the answer to **a)** change if the shell is not grounded, but electrically isolated with a total charge equal to zero?



## Solution

a) Let us search for a system of image charges whose potential, summed up with the point charge potential, would yield a zero potential on the internal surface of the shell. Inside the shell the field and the potential will be those produced by  $q$  and the image charge(s). Outside the shell the field and the potential will be zero.

It is known that if a charge  $Q$  is located outside a conducting sphere of radius  $R$ , at a distance  $a$  from the center  $O$  of the sphere, placing on the axis from  $O$  the the position of  $Q$  an image charge  $Q' = -QR/a$  at a distance  $a' = R^2/a$  from  $O$ , the sum of the potentials of  $Q$  and  $Q'$  is zero at the surface of the sphere.



From such a case we can find a solution for our present problem by exchanging the roles of the image and the real charge.

We place an image charge  $q'$  outside the inner surface of the shell, on the axis trough the position of  $q$  and  $O$  and at a distance  $d'$  from  $O$ . Using the above mentioned result with the substitutions  $Q \leftrightarrow q'$ ,  $Q' \leftrightarrow q$ ,  $a \leftrightarrow d'$ ,  $a' \leftrightarrow d$ , if  $q = -q'R/d'$  and  $d = R^2/d'$  the potential on the sphere of radius  $R$  is zero. Thus we obtain  $d' = R^2/d$  and  $q' = -qd'/R = -qR/d$ .

b) The force is the Coulomb force between  $q$  and the image charge, and it is attractive:

$$F = k_0 \frac{qq'}{(d' - d)^2} = -k_0 \frac{q^2 R d}{(R^2 - d^2)^2}. \quad (1)$$

c) Since the field outside the inner cavity of the shell is zero everywhere, the flux of the field over any closed surface that contains the inner cavity vanishes trivially, thus due to Gauss's law the charge inside such surface is zero. Thus, the charge induced on the inner surface is  $-q$ , as may be verified directly calculating the surface charge and integrating over the whole surface.

d) The boundary conditions are now the following: the electric potential must be constant over the shell (but not necessarily equal to zero) and the total charge on the shell must be zero.

To satisfy these conditions, we first distribute an amount of charge  $-q$  on the inner surface in order to reproduce the field of the image charge, as in the above points. Now the potential of the sphere is zero, but we still have to distribute the remaining amount of charge  $+q$  somewhere on the shell and in a way that the potential of the shell is constant. The solution is to distribute such charge uniformly on the outer surface, since such distribution produces a zero field and a constant potential  $k_0 q/R_{\text{ext}}$  inside the surface, where  $R_{\text{ext}}$  is the outer radius of the shell.

Thanks to the superposition principle, the potential and the field are the sum of the potential and field found at point a), which are zero outside the inner shell, plus the potential and field produced by the charge  $q$  uniformly distributed on the outer surface, i.e. those of a point charge located in the center of the shell. Thus, outside the shell only such field is present, and we may not determine the position of the charge inside the shell only by measuring the field outside the shell: the field will always look like as if  $q$  were in the center of the shell. In the inner cavity the field is the same both in the case of a grounded or of an isolated shell.