Fluctuation and Noise Letters
Vol. 0, No. 0 (2001) 000–000

An algorithm for strongly correlated noise

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Received (received date)
Revised (revised date)
Accepted (accepted date)

An algorithm for the numerical integration of Langevin equations driven by green noise is proposed. The algorithm is local in time, hence it is particularly suitable for the integration when non equilibrium quantities (like, first exit times) are under investigation.

Keywords: Stochastic integration, algorithms, green noise

1. Introduction

It is widely accepted that the effect of noise in nonlinear systems can greatly change the dynamics, and lead to very complex behaviours [1–3]. It has also become clear in recent years that this complex behaviour is as much due to the intrisinc form of the dynamical system as to the details of the noisy source. For instance, taking an overdamped bistable system driven by noise, the dynamics changes dramatically as the noise, assumed to be gaussian, is changed from white to exponentially correlated, and to quasi harmonic (see, for example, [4]). In the case of harmonic correlated noise, for example, even a small bias in the bistable system leads to a nonanalytical change in the escape rates between different states [5].

A spectral density of noise which has received very little attention in the field of dynamical systems, despite its importance, for instance in modelling phonons spectral density, is the so called green noise (see [6, 7] for an explanation of the term "green"): noise with a spectral density which goes to zero for small frequency, of the form

$$S(\omega) = \frac{D}{2\pi}\frac{\omega^2}{\omega^2+\gamma^2}. \quad (1)$$

Algorithms for the numerical integration in the presence of a gaussian noise of arbitrary spectral densities are readily built through a Fourier transform (see, among others, [8, 9]): the idea is to build the needed noisy process (call it $\xi(t)$) using

$$\xi(t) = \sum_n a_n e^{in\Delta\omega t}. \tag{2}$$

With suitable choices of the coefficients a_n's, any spectral distribution can be generated. It is obvious that the sequence of $\xi(t)$ repeats itself after a time $\bar{t} = 2\pi/(\Delta\omega)$; and that the generation of the sequence is normally handled via an FFT, which means that the whole sequence is a priori generated, having extimated how long it needs to be. In a numerical simulation, this implies that using (2) may be far from optimal. For instance, in case of the passage time to a barrier, one would have to generate a very long random sequence to make sure that there are enough random terms to observe the escape, and yet, once the transition has taken place, what is left of the sequence will have to be rejected. Clearly, this is very inefficient.

A much better approach would be to work out an algorithm to generate the noisy sequence which is *local* in time. The characteristic of an algorithm which is local in time is that the algorithm requires only the noise value at the previous time step (or at a small and finite number of previous time steps) to generate the noise value for the following time step. In the spirit of [10–12], this letter deals with an algorithm, local in time, for the integration of a stochastic differential equation driven by green noise.

2. The algorithm

The stochastic differential equation which should be integrated has the form

$$\dot{x} = a(x) + f(t) \tag{3}$$

where $a(x)$ is a deterministic drift, and $f(t)$ is a stochastic random process, with gaussian statistics, zero average and a spectral density of fluctuations of the form (1). Eq. (1) implies that the moments of $f(t)$, averaged over the noisy realizations of $f(t)$ itself, should be

$$\langle f(t) \rangle = 0 \qquad \langle f(t)f(0) \rangle = D\left[\delta(t) - \frac{\gamma}{2}e^{-\gamma|t|}\right]. \tag{4}$$

In [6] a representation was found for the green noise $f(t)$, in the form

$$f(t) = \xi(t) - \gamma e^{-\gamma t}\int_{-\infty}^{t} e^{\gamma s}\xi(s)ds \tag{5}$$

where $\xi(t)$ is a white gaussian process with moments

$$\langle \xi(t) \rangle = 0 \qquad \langle \xi(t)\xi(0) \rangle = D\delta(t).$$

An algorithm for the integration of Eq. (3) would easily follow: introducing the integration time step h, and defining

$$g(t) = \int_0^t f(s)\, ds, \tag{6}$$

using, for instance, the Heun scheme [8] to step the variable $x(t)$ from $t = 0$ to $t = h$, one has to compute

$$\tilde{x} = x(0) + ha(x(0)) + g(h) \qquad x(h) = \frac{1}{2}\left[\tilde{x} + x(0) + ha(\tilde{x}) + g(h)\right]. \tag{7}$$

Although this approach is possible in principle, it is clear that it is very inefficient from a computational point of view. First, we would be forced to store the whole history of the process ξ; second, we would have to compute a lengthy time integral (see Eq. (6)) at each integration time step; and third, this integration would not be local in time.

We begin our derivation noting that the (stochastic) variable $g(t)$ is a linear combination of stochastic variables, so that we could approach the problem in a different way: instead of using the stochastic process $g(t)$, defined in (6), we could replace it with an appropriate surrogate stochastic process, characterized by the same statistics and correlation functions; incidentally, we wrote the Heun step between $t = 0$ and $t = h$: it will be clear from the following that there is no loss of generality.

Let us introduce the quantities

$$I(t') = e^{-\gamma t'} \int_{-\infty}^{t'} e^{\gamma s} \xi(s) ds \tag{8}$$

and the stochastic integrals

$$\begin{aligned} Z_0(t) &= \int_0^t \xi(s)\, ds \\ W_0(t) &= \int_0^t e^{\gamma(s-t)} \xi(s)\, ds \\ W_1(t) &= \int_0^t W_0(s)\, ds \end{aligned} \tag{9}$$

It follows that

$$\begin{aligned} I(t') &= e^{-\gamma t'} \int_{-\infty}^{t'} e^{\gamma s} \xi(s)\, ds = e^{-\gamma t'} \left(\int_{-\infty}^{0} e^{\gamma s} \xi(s)\, ds + \int_0^{t'} e^{\gamma s} \xi(s)\, ds \right) \\ &= e^{-\gamma t'} I(0) + W_0(t'). \end{aligned} \tag{10}$$

Recalling that in the Heun step we need to evaluate $g(h)$, we start rewriting

$$\begin{aligned} g(t) &= \int_0^t f(t')\, dt' = \int_0^t \xi(t')\, dt' - \gamma \int_0^t \left(e^{-\gamma t'} \int_{-\infty}^{t'} e^{\gamma s} \xi(s)\, ds \right) dt' \\ &= Z_0(t) - \gamma \int_0^t \left(e^{-\gamma t'} I(0) + W_0(t') \right) dt' \end{aligned} \tag{11}$$

By integration, we readily obtain

$$g(t) = Z_0(t) + \left(e^{-\gamma t} - 1\right) I(0) - \gamma W_1(t) \tag{12}$$

It is clear how to carry out the integration: at each time step one uses (10) and (12) to generate the appropriate random variables, which are in turn used in (7) to step the equation forward. At each time step, the previously computed $x(h)$ and $I(h)$ become the new $x(0)$ and $I(0)$, respectively, and so on.

The reader should appreciate that the memory of the stochastic process is basically restricted to the term $I(0)$: the quantities $W_0(t')$ and $W_1(t)$ are independent of the previous history of the stochastic process. The whole problem is to find a suitable representation of the stochastic integrals in (10), because clearly the various quantities are *not* independent of each other.

To find a representation of the stochastic integrals (10), we note first that they are linear combinations of gaussian variables, hence they can be represented via a suitable set of random numbers extracted from gaussian distributions of appropriate averages and standard deviations. This makes the problem much simpler to deal with. We only need to specify the first and second moments of these variables, and we should be able to easily generate them. Let us briefly show how to carry out the calculations, looking at the second moment of the variable $Z_0(t)$. We have, using $\langle\rangle$ to indicate averages taken over the noise realizations and recalling the definition of $Z_0(t)$ and the statistical properties of the process $\xi(t)$,

$$\langle Z_0(t)^2\rangle = \langle \int_0^t \int_0^t \xi(s)\xi(s')\, ds\, ds'\rangle = \int_0^t \int_0^t D\delta(s-s')\, ds\, ds' = Dt$$

.

It is clear by inspection that all stochastic integrals appearing in (10) have zero average. Their second moments are (we will omit to write the time dependence to keep the notation simple, and assume that $t = h$)

$$\langle Z_0^2\rangle = Dh$$

$$\langle W_0^2\rangle = \frac{D}{2\gamma}\left[1 - e^{-2\gamma h}\right]$$

$$\langle W_1^2\rangle = \frac{D}{2\gamma^3}\left[2\gamma h - 3 + 4e^{-\gamma h} - e^{-2\gamma h}\right]$$

$$\langle Z_0 W_0\rangle = \frac{D}{\gamma}\left[1 - e^{-\gamma h}\right]$$

$$\langle Z_0 W_1\rangle = \frac{D}{\gamma^2}\left[\gamma h - 1 + e^{-\gamma h}\right]$$

$$\langle W_0 W_1\rangle = \frac{D}{2\gamma^2}\left[1 - 2e^{-\gamma h} + e^{-2\gamma h}\right] \tag{13}$$

Suppose that Y_0, Y_1 and Y_2 are random gaussian variables of zero averages and standard deviation one, independent of each other, we can represent the stochastic integrals of (10) as

$$\begin{aligned} Z_0(h) &= \sqrt{\langle Z_0^2\rangle}Y_0 \\ W_0(h) &= b_0 Y_0 + b_1 Y_1 \\ W_1(h) &= c_0 Y_0 + c_1 Y_1 + c_2 Y_2. \end{aligned} \tag{14}$$

The coefficients appearing in (14) are easily obtained by combining the various quantities appearing in (14), with the help of (13): for example, if we wanted to work out the value for b_0, we would consider

$$\langle W_0(h) Z_0(h) \rangle = \frac{D}{\gamma} \left[1 - e^{-\gamma h}\right] = \langle (b_0 Y_0 + b_1 Y_1) \sqrt{\langle Z_0^2 \rangle} Y_0 \rangle = b_0 \sqrt{\langle Z_0^2 \rangle}$$

and so on. The analytic expressions for the constants appearing in (14) are cumbersome, and we will not write them here: it is clearly very easy to numerically compute them, given (14) and (13).

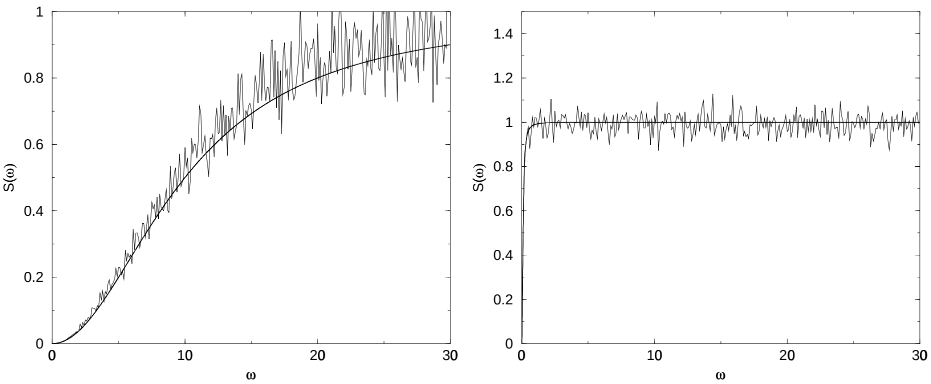

Fig 1. Spectral density of fluctuations for the quantity $\dot{g}(t) = f(t)$, obtained using Eq. (12), for $D = 2\pi$ and $\gamma = 10.0$ (left), and 0.1 (right). Smooth line: theory (Eq. (1)), jagged line: simulated spectral densities. Note that the agreement between simulation and theory at small frequencies is very good, and that the simulated spectral density does go to a constant for larger frequencies.

Fig. (1) shows that the noise generated using the proposed algorithm has the spectral density (1). Note that (1) refers to $f(t)$, whereas any integration scheme (like the Heun scheme) requires $g(t)$. Hence, we really generated $g(t)$, differentiated it (computing $(g(t+h) - g(t))/h$), and finally compute the spectral density of fluctuations of the time series obtained. In Figure (1) there are no adjustable parameters. The agreement between generated spectral density of fluctuations and theory is extremely good.

3. Conclusions and perspectives.

We derived an algorithm, local in time, for the integration of a stochastic differential equation driven by green noise. The algorithm opens the possibility of an efficient numerical integration of nonlinear dynamical systems driven by green noise. Some preliminary work on the models of [6] seems to indicate that the theory proposed in [6] agrees with the numerical simulations.

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