PHYSICAL REVIEW LETTERS

VOLUME 80 1 JUNE 1998 NUMBER 22

Field-Induced Stabilization of Activation Processes

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(Received 27 August 1996; revised manuscript received 15 October 1997)

An investigation of the noise activated escape from a metastable state and between attractors in a bistable system has been undertaken. It is demonstrated, both theoretically and by means of digital simulations, that the application of an external time-periodic field can lead to a significant increase in the lifetime (averaged over one period of the field) of a metastable state and in the residence time of a bistable system. In particular, it is shown that these characteristic times can be increased beyond the inverse Kramers rate calculated in the absence of the field. This effect is observed when the frequency of the external field is smaller than the unperturbed Kramers rate. [S0031-9007(98)06182-1]

PACS numbers: 05.40.+j, 02.50.Ey, 82.20.Pm

Thermal and stochastic activation processes are of considerable importance in many branches of the physical sciences and, as such, have been the focus of many theoretical studies [1,2]. Of particular importance in this context is the role of internal fluctuations of the system. The origins of internal fluctuations (noise) are numerous, although typically thermal in origin, but their effect in multistable systems is similar—they impose a lower bound on the stability of the system. In many cases it is either very difficult or fundamentally impossible to remove the source of internal noise—it is an intrinsic feature of the system. The stability (and hence the utility) of systems such as Josephson junctions, lasers, and semiconductor devices are all affected by the level of their internal noise and other systems, most notably chemical reactions, are driven by internal fluctuations. The ability to control the affect of these fluctuations is therefore of some importance.

In this Letter, we discuss a novel field induced effect that results in a net stabilization of metastable and bistable systems against fluctuations. Although thermal and stochastic activation processes occur in a diverse range of physical systems, they can generally be treated as the escape of a Brownian particle over a potential barrier. Therefore, it is hoped that the simple Brownian systems studied below, and the subsequent analysis, will have some generality.

For a Brownian particle the escape rate out of a metastable potential in the low temperature limit is given by Kramers rate [1] $W_0 \sim \exp(-\Delta E/D)$, where ΔE is the characteristic activation energy. The quantity D is related to the temperature through the Boltzmann constant k_B, but can more generally be thought of as a quantity which parametrizes the noise intensity of the system. A measure of the average lifetime of the state is given by W_0^{-1}. For a system in equilibrium (or quasiequilibrium within one well) this quantity is of fundamental importance as it gives a measure of the long-time stability of the system.

Recently, there has been a lot of interest in the dynamics of such systems in the presence of an additional periodic driving field [3,4]. The most widely studied case is for bistable systems which have been studied extensively in connection with the phenomenon of stochastic resonance [5]. The frequency of the driving field is often assumed to be small compared to the inverse relaxation time, τ_{rel}^{-1}, within one of the potential wells. This adiabatic approximation ensures that the system is in thermal equilibrium *within* one of the potential wells, and hence the escape rate out of the well, at any instant of time, is still given by Kramers rate.

The external field, which is modeled as an extra additive time-periodic force, can be viewed as giving rise to a slow modulation of the potential barrier height (activation energy), resulting in the modulation of the escape rates. Consequently, the system is more likely to make a transition when the Kramers rate is at its maximum—that is, when the potential barrier height is at its minimum. The larger the amplitude of the modulation, the shallower the minimum potential well depth becomes, and the greater the probability of escape. An increase in the magnitude of the external field is therefore expected to lead to an increase in the number of transitions that are made between the two wells, and hence to a reduction in the time the particle is resident in each well. This simple intuitive picture has been used extensively to explain the phenomenon of stochastic resonance. Similar arguments can also be applied to the case of a metastable state, and, consequently, one would generally expect an enhancement of the escape rate with increasing field strength in this case too.

However, it is important to note that the above arguments work only within a specific range of time scales. In general, they are valid when $T \ll W_0^{-1}$, where T is the period of the external force. If this is not the case, then transitions can occur with significant probability at points other than when the potential well depth is at its minimum and, in general, escape events occur at a faster rate than the external forcing frequency. Although a strong synchronization of the response to that of the driving field is not possible in this regime, we will show that the field can still exert sufficient influence to increase the characteristic lifetime (or residence time) of these systems beyond that of the inverse Kramers rate.

We will now discuss the metastable case in detail. For this case, as all cases discussed below, we will assume that the frequency of the external forcing is sufficiently small that the adiabatic approximation is valid. The lifetime of the metastable state can be characterized in terms of the mean first passage time, T_{MFPT}, for the particle to escape from the well. However, because of the presence of the external field—which will be taken to be $A\cos(\Omega t + \theta)$—the MFPT will depend on the initial phase, θ, of the field. The MFPT is obtained from the first moment of the escape time distribution $-\dot{w}(t,\theta)$, where $w(t,\theta)$ is the well population, and can easily be shown to be given by

$$T_{\mathrm{MFPT}}(\theta) = \int_0^\infty w(t,\theta)\, dt\,. \tag{1}$$

The well population can be calculated from the rate equation $\dot{w}(t,\theta) = -W(t,\theta)w(t,\theta)$, where $W(t,\theta)$ is the time dependent Kramers rate. The exact form of $W(t,\theta)$ will depend on the detail of the system under investigation, but can be obtained by calculating the time dependence of the potential barrier height. How the initial phase is taken into account depends on how the field is applied and how the ensemble of states is formed. One can envisage two different situations. First, the situation where the metastable state already exists when the field is applied—in this case the phase is known and can be treated as a parameter; or second, the case where the metastable state is created in the presence of the field. Given that the creation of a metastable state can be regarded as a random event, the phase of the field at $t = 0$ (time of creation of state) is not known. We will consider the second of these two cases, although the analysis for known phase follows exactly the same methodology. We will further assume that the state is not created at any preferential phase. With these considerations in mind the average lifetime $\langle T_{\mathrm{MFPT}}\rangle_\theta$ is found by averaging over the initial random phase

$$\langle T_{\mathrm{MFPT}}\rangle_\theta = \frac{1}{2\pi}\int_0^{2\pi} T_{\mathrm{MFPT}}(\theta)\, d\theta\,. \tag{2}$$

In the absence of the external field the Kramers rate $W_0 = T_{\mathrm{MFPT}}^{-1}$. Therefore, it is useful to introduce a rate $\mathcal{W} = \langle T_{\mathrm{MFPT}}\rangle_\theta^{-1}$ so that the affect of the field can be studied by direct comparison to the Kramers rate calculated in the absence of the field. We will refer to $\mathcal{W}$ as the field-induced decay rate (FIDR) to distinguish it from the standard "escape rate," as this term is usually reserved to describe the time dependent Kramers rate. It should be noted that our definition of a FIDR is not unique. In principle, one could take the phase average of the Kramers rate instead of the MFPT to give a quantity $\langle W\rangle_\theta$ and define this to be the FIDR. However, the stability of a metastable state is characterized by its lifetime. Generally, the quantity $\langle W\rangle_\theta^{-1}$ is not equal to the lifetime of the state. To illustrate this point, Fig. 1 shows the metastable decay of the population, $p(t) = \langle w(t)\rangle_\theta$, from one of the wells of the bistable system introduced below. Because of the affect of the field, the relaxation is nonexponential. For comparison, the approximations given by the solution of $\dot{w}(t) = -\mathcal{W}w(t)$ are shown. The dashed line was calculated using the definition

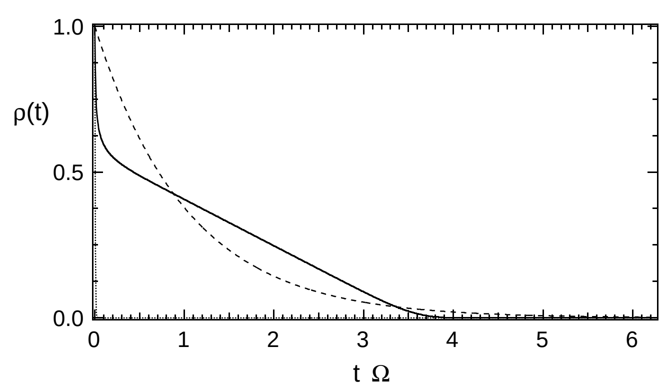

FIG. 1. The metastable decay of the phase averaged population from one of the wells of the quartic bistable potential for an external field strength of 0.2 and frequency 10^{-4}. The solid line is the actual relaxation calculated numerically; the dashed line and dotted lines are exponential approximations based on the phase averaged MFPT and Kramers rate, respectively.

$\mathcal{W} = \langle T_{\rm MFPT}\rangle_\theta^{-1}$ and the dotted line using $\mathcal{W} = \langle W\rangle_\theta$. Clearly, because of the highly nonexponential nature of the relaxation, neither of these two curves yields a good approximation, but it is important to note that the escape distribution derived from the dashed line does give the correct first moment, i.e., it gives the correct lifetime of the state. This is to be contrasted with the second approximation which underestimates the lifetime by a few orders of magnitude. We do not believe that, in general, $\langle W\rangle_\theta$ offers a useful physical characterization of the decay process over the whole frequency range of the external field. Although, if the probability of escape per cycle is sufficiently small, then $\langle W\rangle_\theta \sim \langle T_{\rm MFPT}\rangle_\theta^{-1}$ and therefore, in this limit, it once again gives a good approximation to the lifetime.

To illustrate the above arguments, we will now study the specific case of the quartic bistable potential, $V(x) = -x^2/2 + x^4/4$, by considering the decay of the population of one of the wells. The system is prepared at $t = 0$ with one of the wells fully populated and with a random initial phase. It is further assumed that after escape the particle is removed from the system, i.e., there exists an absorbing boundary—this ensures we are considering the metastable properties of the well. The calculation is greatly simplified if one considers the strong adiabatic limit which requires $\Omega \ll W_{\rm min}$, where $W_{\rm min}$ is the minimum Kramers rate attained during one forcing cycle. Within this approximation, the time dependence of the Kramers rate becomes negligible and integrals (1),(2) can be solved by a steepest descent method in the limit $A/D \gg 1$, yielding

$$\mathcal{W} = W_{\rm min}\sqrt{\frac{2\pi A(1 + 3A/2)}{D}}\,. \qquad (3)$$

Given that to second order in A, $W_{\rm min} = W_0 \exp[-A(1 + 3A/4)/D]$, Eq. (3) predicts an exponentially large decrease in $\mathcal{W}$ in comparison to W_0, and consequently, the FIDR has been reduced to below that of the Kramers rate. The lifetime of the metastable state has been considerably increased by the action of the field, and hence the system has experienced a net increase in stability.

We now show that this result is not restricted to metastable states by considering steady state switching between attractors in the quartic bistable potential. A useful way of characterizing the dynamics is in terms of the residence time—that is, the average time the system spends in each well. For the symmetric bistable system studied here, the residence time, $T_{\rm res}$, is obviously the same for both wells and, in the absence of an external field, is given by $T_{\rm res} = W_0^{-1}$. We therefore have a similar result as for the metastable case, and it is again useful to introduce the concept of a rate $\mathcal{W} = T_{\rm res}^{-1}$. Physically, $\mathcal{W}$ can be interpreted as the total probability current across the potential barrier, i.e., it is the actual rate at which particles make a transition from one well to the other—again this should not be confused with the Kramers rate. It is possible to use the same procedure for calculating the residence time as was used for calculating the lifetime of a metastable state (although one must now average the initial phase over the equilibrium phase distribution). However, it is easier to calculate $\mathcal{W}$ directly in terms of the total probability current across the barrier, which can be written as

$$\mathcal{W} = \frac{1}{T}\int_0^T [w_1(t)W_{12}(t) + w_2(t)W_{21}(t)]\,dt\,, \qquad (4)$$

where $w_1(t)$, $w_2(t)$ are the equilibrium well populations and $W_{12}(t)$, $W_{21}(t)$ are the instantaneous Kramers rates from well $1 \rightarrow 2$ and from $2 \rightarrow 1$, respectively. Physically, the quantities $w_n W_{nm}$ represent the average number of transition made per unit time from well $n \rightarrow m$.

The well populations are governed by the rate equation [6],

$$\dot{w}_1 = -[W_{12}(t) + W_{21}(t)]w_1 + W_{21}(t)\,, \qquad (5)$$

and $w_1 + w_2 = 1$. When the period of the drive is much longer than any other time scale in the system, it can easily be shown that

$$\mathcal{W} = \frac{2}{T}\int_0^T \frac{W_{12}(t)W_{21}(t)}{W_{12}(t) + W_{21}(t)}\,dt\,. \qquad (6)$$

For the specific case of the quartic double well system, the integral can be evaluated analytically in the parameter range $A/D \gg 1$ and for not too large A. Using the expressions for W_{nm} given in [7] we find by a steepest descent method,

$$\mathcal{W} \approx \frac{D}{A}\,W_0\,. \qquad (7)$$

This result not only predicts a field induced reduction of the total probability current, but also that the rate will be reduced to *below* the Kramers rate calculated in the absence of the field. Again, this can be interpreted as a net stabilization of the system, with the residence time being increased by a factor A/D by the action of the field.

It is interesting to note that the influence of the field is not as strong for the bistable system. The difference between these two cases can be easily understood by considering the dynamics within their strong adiabatic approximations and large A/D. For the metastable state the largest MFPT is attained when the potential well depth is at its maximum. Consequently, when computing $\langle T_{\rm MFPT}\rangle_\theta$, it is these times which dominate the average, and therefore $\mathcal{W}$ is of the order of $W_{\rm min}$. This is quite different from what occurs in the bistable case. At low frequencies the bistable system can always be regarded as being in equilibrium; that is, there is no *net* probability current across the barrier. For $A/D \gg 1$, the dynamics are dominated by the deepest potential well which is occupied with a probability close to unity. Only when the well depths are approximately equal, which occurs over

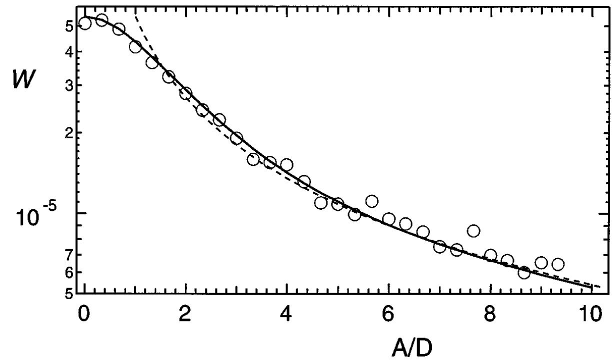

FIG. 2. Plot of $\mathcal{W}$ against A/D for $D = 0.03$ and $\Omega = 10^{-6}$. The data points are the results of the digital simulation and the solid and dashed lines are theoretical results based on Eqs. (6) and (7), respectively.

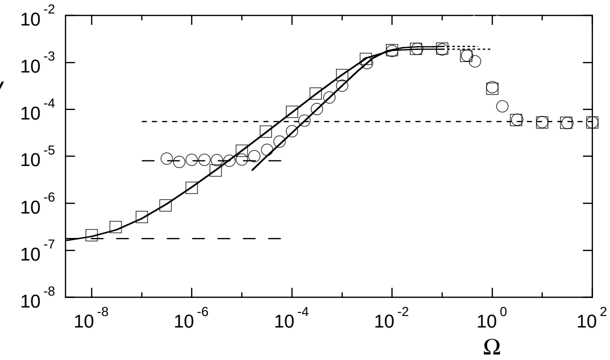

FIG. 3. Plot of $\mathcal{W}$ versus Ω for $A = 0.2$ and $D = 0.03$. The circular data points are the results of the digital simulation for the bistable case and the squares are for the metastable state. The long-dashed lines are the predicted asymptotic values for the two cases. The solid lines are given by Eq. (8) (bistable) and the numerical evaluation of Eqs. (1) and (2) (metastable). The short-dash line indicates the value of W_0.

a time interval δt, can the population switch to the other well—these transitions occur at the equilibrium rate W_0. Consequently, the transition rate averaged over the whole of the cycle is of the order of $W_0 \delta t/T$.

These theoretical results have been tested against digital simulations using the algorithm described in [8]. A comparison between the simulation and theoretical results for the bistable case is shown in Fig. 2. This figure shows the dependence of $\mathcal{W}$ on the parameter A/D. It is important to note that these results were obtained by fixing D and increasing A. The quantity A/D only acts as a scaling parameter over the range of validity of the steepest descent calculations and not for all values of A and D.

It can be seen that there is excellent agreement between the numerical solution of Eq. (6) (solid line) and the simulation results (circles). The steepest descent result (7) (dashed line) is also verified for a large range of A/D. These results confirm our predictions and indicate that the application of an external field can be to lead to a net stabilization of the system.

In Fig. 3 the dependence of $\mathcal{W}$ on Ω is plotted for both the metastable and bistable cases. Considering first the simulation results for the bistable case (circles), it can be seen that for $\Omega < W_0$, where the value of W_0 is indicated by the horizontal short-dashed line, $\mathcal{W}$ quickly tends to its asymptotic value. As Ω increases, $\mathcal{W}$ increases rapidly and is seen to be in good agreement with the relation

$$\mathcal{W} = \frac{\Omega}{\pi} \tanh\left[\frac{1}{2}\int_0^T W_{nm}\, dt\right], \qquad (8)$$

which was obtained by calculating the residence time from the residence time distribution (see Eq. 27 in [7]) in the parameter range $A/D \gg 1$. The calculation is based on an adiabatic approximation and therefore breaks down when $\Omega \sim \tau_{\rm rel}^{-1}$. The adiabatic results predict that $\mathcal{W}$ reaches a constant finite value at high frequencies (indicated by the dashed continuation of the solid curve), whereas, in practice, $\mathcal{W}$ should tend to W_0.

The simulation results for the metastable state (squares) are seen to follow closely those of the bistable case in the parameter range $\Omega > W_0$. The main difference is observed in the limit $\Omega \to 0$. In this limit a smaller asymptotic value is reached than for the bistable case. This value is accurately predicted by Eq. (3), which is shown by the long-dashed line. As predicted, the stabilization effect is much stronger for the metastable state.

The results presented in Fig. 3 illustrate nicely the different time scales involved in these systems and how, when these time scales are sufficiently well separated, analytical results can be obtained. However, more importantly, the results illustrate how the stability of these systems can be controlled by simply adjusting the frequency of the external field. Both enhancements and reductions in the characteristic rates are observed.

In conclusion, we would like to suggest that these results may have a practical use for the control of activation type processes in metastable and bistable systems. Possible applications are the control of reaction rates in chemical processes and the stabilization of SQUIDS against internal fluctuations.

The work was supported by EC TMR Grant No. Crbfmb 950386.

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[1] H. A. Kramers, Physica **7**, 284 (1940).
[2] For recent work, consult the following articles and references therein: V. I. Melnikov, Phys. Rep. **209**, 1 (1991); M. I. Dykman and K. Lindenberg, in *Some Problems in Statistical Physics,* edited by G. Weiss

(SIAM, Philadelphia, 1994); P. Hanggi and P. Jung, Adv. Chem. Phys. **89**, 239 (1995).

[3] P. Jung, Phys. Rep. **234**, 175 (1993); P. Jung, Z. Phys. B **76**, 521 (1989).

[4] R. N. Mantegna and B. Spagnolo, Phys. Rev. Lett. **76**, 563 (1996); B. Shulgin, A. Neiman, and V. Anishchenko, Phys. Rev. Lett. **75**, 4157 (1995).

[5] For references on stochastic resonance, consult the following workshop proceedings: *Proceedings of the NATO Advanced Research Workshop on Stochastic Resonance in Physics and Biology, San Diego, California, 1992* [J. Stat. Phys. **70** (1993)]; *Proceedings of the Workshop on Fluctuations in Physics and Biology: Stochastic Resonance, Signal Processing and Related Phenomena* [Nuovo Cimento Soc. Ital. Fis. **17D** (1995)].

[6] B. McNamara and K. Wiesenfeld, Phys. Rev. A **39**, 4854 (1989).

[7] N. G. Stocks, Nuovo Cimento Soc. Ital. Fis. **17D**, 925 (1995).

[8] M. I. Dykman, D. G. Luchinsky, R. Mannella, P. V. E. McClintock, N. D. Stein, and N. G. Stocks, in *Modern Nonlinear Optics, Part 3,* edited by Myron Evans and Stanislaw Kielich, Advances in Chemical Physics Vol. LXXXV (Wiley, New York, 1994).