Giant nonlinearity in the low-frequency response of a fluctuating bistable system

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The response of a bistable system to a low-frequency driving force is demonstrated to display a giant nonlinearity: a sinusoidal force can induce a nearly rectangular signal in the system even when its amplitude is small. The nonlinearity comes about through the force modulating the effective activation energies of fluctuational transitions between the stable states, which in turn can give rise to strong modulation of the transition probabilities and the populations of the states, for low noise intensities. Theoretical results are shown to be in good agreement with the results of analog and digital simulations.

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Several tens of papers have appeared in recent years dealing with fluctuation phenomena in bi- and multistable systems in diverse areas of science, ranging from condensed-matter physics (dynamics of bi- and multistable defects in crystals and glasses, in particular) through optical bistability to geophysics and neurophysiology. The problems that have been attacked most vigorously are the probabilities of fluctuational transitions between coexisting stable states (see Refs. [1,2] for recent reviews) and stochastic resonance (SR)—the noise-induced *increase* (followed by a decrease, for higher noise intensities) of the response of a system to a periodic force; SR was first considered in geophysics in the context of ice ages [3] and then observed in lasers [4] and other systems [5]. It was noticed when investigating SR in bistable systems [6,7] that their response can sometimes be nonlinear even for a weak driving force. In the present paper we investigate the nonlinear response of a bistable system to low-frequency forcing in the presence of Gaussian noise. We demonstrate theoretically, and by analog and digital experiments, that a *giant* nonlinearity of the response can occur under appropriate conditions.

To gain insight into the physics of the nonlinearity of the response we note first that fluctuations in noise-driven bistable systems are characterized by several time scales. These are the relaxation time(s) in the absence of noise τ_r, the correlation time(s) of the noise τ_c, and also specific times related to the interplay of the noise and the "internal" dynamics and equal to the reciprocal probabilities $(W_{12}^{(0)})^{-1}, (W_{21}^{(0)})^{-1}$ of the noise-induced transitions $1 \to 2$ and $2 \to 1$ between the stable states 1,2. These probabilities are physically meaningful when the characteristic noise intensity $\mathcal{D}$ is sufficiently small, and their dependence on $\mathcal{D}$ is then of the activation type

$$W_{nm}^{(0)} = C_{nm} \exp(-\mathcal{R}_n/\mathcal{D}), \quad \mathcal{R}_n \gg \mathcal{D}. \tag{1}$$

The quantity $\mathcal{R}_n$ may be reasonably called an effective activation energy of the transition from the stable state n. It is given by the solution of a variational problem [2] for the optimal path along which the system escapes from a given state, with an overwhelming probability. The prefactor C_{nm} in (1) is of order of $\min(\tau_r^{-1}, \tau_c^{-1})$.

The effect of a weak additional periodic force $A\cos\Omega t$ on the dynamics of the system is at its strongest in the range of small noise intensities (1) and for frequencies $\Omega \ll \tau_r^{-1}, \tau_c^{-1}$. Before the force is applied, the system mostly performs small-amplitude fluctuations about the stable states and occasionally switches from one stable state to another. The characteristic duration of the transitions is $\sim \tau_r, \tau_c$. Application of the periodic force gives rise to forced vibrations about the stable states. In addition it *modulates* (parametrically, for small Ω) the activation energies of the transitions $\mathcal{R}_{1,2}$. Therefore the escape probability is given by the *instantaneous* value of $\mathcal{R}_n$ (the adiabatic approximation [8,9]) and

$$\begin{aligned} W_{nm} &\equiv W_{nm}(t) = W_{nm}^{(0)} \exp(g_n \cos\Omega t), \\ g_n &= \tilde{g}_n A/\mathcal{D}, \quad \Omega \ll \tau_r^{-1}, \tau_c^{-1} \\ \tilde{g}_n &= -\left[\frac{\partial \mathcal{R}_n(A)}{\partial A}\right]_{A=0} . \end{aligned} \tag{2}$$

Here, $\mathcal{R}_n(A)$ is the value of the activation energy of the transition from the state n $(n = 1, 2)$ for a system driven by a *static* force A. Since the force is assumed weak (compared to the dynamical characteristics of the system in the absence of noise), only the term linear in A is taken into account in the expansion of $\mathcal{R}_n(A)$.

It is obvious from (2) that the effect of the field on the kinetics of a bistable system is determined by the *ratio* of A to the noise intensity $\mathcal{D}$, and $\mathcal{D}$ is itself small. Therefore the parameters $g_{1,2}$ can be arbitrarily large even for small A. If $g_{1,2}$ are large, the transition probabilities are strongly renormalized, and thus a "dynamically weak" force can nonetheless make a strong impact on the system as a whole. This is how the giant nonlinearity of the response arises.

The periodic modulation of the transition probabilities results in a modulation of the instantaneous values of the populations $w_{1,2}(t)$ of the stable states which is described by the balance equation

$$\begin{aligned} \dot{w}_1(t) &= -[W_{12}(t) + W_{21}(t)]w_1(t) + W_{21}(t), \\ w_2(t) &= 1 - w_1(t). \end{aligned} \tag{3}$$

For small $|g_{1,2}| \ll 1$ the stationary solution of Eqs. (2) and (3) is of the form of weak-field-induced sinusoidal oscillations of the populations of the states. The onset of such oscillations in narrow range(s) of the force frequency gives rise to super-narrow peaks in the spectra of susceptibility of bistable systems [9]; it is these peaks that are responsible for stochastic resonance [10].

An explicit solution of Eqs. (2) and (3) in the stationary regime can also be obtained in the opposite case $|g_{1,2}| \gg 1$ where the modulation of the populations by the weak periodic field is strong, which is the situation of primary interest for the present paper. To obtain the solution we note that, since, for large $|g_{1,2}|$, the transition probabilities vary *by orders of magnitude* within a period according to (2), the transitions *from* a given stable state n occur with overwhelming probability within *a short part* of the period when $W_{nm}(t)$ is close to its *maximum*. Correspondingly, transitions *to* the nth state are most likely to occur very close to the times at which $W_{mn}(t)$ is maximal. The behavior of the system is qualitatively different depending on whether the activation energies $\mathcal{R}_{1,2}$ are modulated by the periodic force "in phase" or "in contra-phase", i.e., whether the quantities g_1 and g_2 are of the same or of opposite sign. The first case is less interesting, since here the values of the populations $w_1(t), w_2(t)$ are nearly time independent: they are formed on balance when the probabilities of both transitions $1 \rightarrow 2$ and $2 \rightarrow 1$ are nearly maximal and then are "frozen" for the rest of the period of the force (for not too small frequencies, $\Omega \gg W_{nm}^{(0)}$).

For $|g_{1,2}| \gg 1$, $g_1 g_2 < 0$ the transitions from one state are most likely to happen just when those from the other one are most unlikely and can be ignored. The population of the nth state is at its maximal value $w_{n>}$ before $g_n \cos \Omega t$ nearly reaches its maximum, and then, after some time interval where the system is most likely to switch, $w_n(t)$ falls down to its minimal value, $w_{n<}$. The quantities $w_{n>}, w_{n<}$ are interrelated via the expression

$$w_{n<} \approx w_{n>} \exp\left(-\int_{t_0-\delta t_0}^{t_0+\delta t_0} dt\, W_{nm}(t)\right),$$

$$|g_n| \gg 1, \quad g_1 g_2 < 0 \tag{4}$$

$$t_0 = 2\pi k/\Omega \text{ for } g_n > 0, \quad t_0 = \pi(2k+1)/\Omega \text{ for } g_n < 0.$$

The integral (4) can be evaluated by the steepest descent method, yielding

$$\begin{aligned} w_{n<} &= w_{n>} \exp(-2\lambda_n), \\ \lambda_n &= W_{nm}^{(0)} \exp(|g_n|) \left(2|g_n|\Omega^2/\pi\right)^{-1/2}. \end{aligned} \tag{5}$$

The characteristic time interval that contributes to the integral (4) is $\sim \tau_n \equiv \Omega^{-1}(2/|g_n|)^{1/2}$, and λ_n just characterizes the probability of the transition during this time. In writing (4) it was assumed that $\pi\Omega^{-1} \gg \delta t_0 \gg \tau_n$ in which case the integral (4) is independent of the value of δt_0. The values of $w_{n>}, w_{n<}$ can easily be obtained from (5) by noticing that $w_{n>} + w_{m<} = 1$, $n, m = 1, 2$ $(n \neq m)$.

It follows from (5) and from the above arguments that the periodic time dependence of the populations in the stationary regime, on the time scale coarse grained over $\tau_{1,2}$, is described by a *square wave*:

$$w_1(t) \equiv 1 - w_2(t) \approx -2\Delta \sum_{k=-\infty}^{\infty} \left[\Theta\left(t - t_1 - \frac{2\pi k}{\Omega}\right) - \Theta\left(t - t_1 - \frac{\pi(2k+1)}{\Omega}\right)\right] + \bar{w}_1 - \Delta$$

$$t_1 = (\pi/\Omega)(|g_1| - g_1)/2|g_1|, \tag{6}$$

where $\Theta(t)$ is the step function, and

$$\Delta = \frac{\sinh \lambda_1 \sinh \lambda_2}{\sinh(\lambda_1 + \lambda_2)}, \quad \bar{w}_1 = \frac{\cosh \lambda_1 \sinh \lambda_2}{\sinh(\lambda_1 + \lambda_2)}. \tag{7}$$

The parameter Δ gives the amplitude of the square wave (it is of course the same for the populations of the both states), and $\bar{w}_1, \bar{w}_2 \equiv 1 - \bar{w}_1$ are the average values of the populations. It is seen from (4) that the decrease of $w_1(t)$ within a period is actually described, not by a discontinuous drop by 2Δ for $t - t_1 = 2\pi k/\Omega$, but by the expression $w_1(t) = (\bar{w}_1 + \Delta) \exp[-\lambda_1(1 + \operatorname{erf} \bar{t}_1)]$, $\bar{t}_1 = (t - t_1 - 2\pi k\Omega^{-1})/\tau_1$. The expression for the increase of $w_1(t)$, i.e., for the decrease of $w_2(t)$, is of the similar structure.

It is clear from (5) and (7) that, for the modulation of the populations to be strong, not only must $|g_{1,2}|$ be large but it is also necessary that two additional conditions be met. First, the frequency of the driving force must be *small*, because Ω has to be of order of the maximal transition probabilities (the transitions must have a chance to occur during a half-period of the force) which are themselves exponentially small for small noise intensity $\mathcal{D}$. Second, not only one but *both* of the parameters $\lambda_{1,2}$ must be of order 1 or more. The effect arises therefore in a narrow range of the parameters of the system, the dependence of the transition probabilities $W_{nm}^{(0)} \exp(|g_n|)$ on these parameters being exponentially sharp. On the other hand, in the appropriate parameter range the effect is really quite dramatic, with the value of Δ approaching its maximum (1/2) once $\lambda_{1,2}$ exceed ~ 1.5.

In the general case of arbitrary g_n Eqs. (2) and (3) can be analyzed numerically. The stationary solution for $w_{1,2}(t)$ is periodic in time, and to find it it is convenient to change from the differential equation (3) to the set of difference equations for the Fourier components w_{nk} of $w_n(t)$ ($w_{2k} \equiv \delta_{k,0} - w_{1k}$) using a standard [11] expansion of the exponent of a cosine in (2),

$$\left[ik\Omega + W_{12}^{(0)} I_0(g_1) + W_{21}^{(0)} I_0(g_2)\right] w_{1k} + \sum_{s\,(\neq 0)} \left[W_{12}^{(0)} I_s(g_1) + W_{21}^{(0)} I_s(g_2)\right] w_{1\,k-s} = W_{21}^{(0)} I_k(g_2), \quad w_1(t) = \sum_{k=-\infty}^{\infty} w_{1k} \exp(ik\Omega t). \quad (8)$$

Here, I_s are Bessel functions of imaginary argument [11].

Because of the nonsinusoidal time dependence of the populations of the states, the time dependence of the ensemble-average value $\langle q(t)\rangle$ of the coordinate $q(t)$ (and, indeed, of any dynamical variable) of the system is nonsinusoidal too. For a dynamically weak force and weak noise

$$\langle q(t)\rangle \approx \sum_{n=1,2} w_n(t)\{q_n^{(0)} + A\,\mathrm{Re}[\chi_n(\Omega)\exp(-i\Omega t)]\}, \quad (9)$$

where $q_n^{(0)}$ is the equilibrium position of q in the nth state and $\chi_n(\Omega)$ is the linear susceptibility of the system in the nth state in the neglect of fluctuations. Equations (6)–(9) provide a solution to the problem of the signal in a periodically driven bistable system with the account taken of the strong nonlinearity of the redistribution of the system over the stable states for low noise intensities. It is evident that signal can sometimes be of a nearly rectangular shape.

To seek evidence for this predicted giant nonlinearity in the response to a low-frequency driving force, analog electronic experiments and digital simulations were performed. The system simulated was an overdamped Brownian particle fluctuating in a symmetrical bistable potential. The equation of motion of the system was of the form of

$$\dot{q} = -U'(q) + A\cos\Omega t + \xi(t), \quad U(q) = -\frac{1}{2}q^2 + \frac{1}{4}q^4, \quad (10)$$

where $\xi(t)$ is white noise, $\langle\xi(t)\xi(0)\rangle = 2\mathcal{D}\delta(t)$. If the stable states 1,2 are chosen so that the equilibrium positions $q_1^{(0)} = -1$, $q_2^{(0)} = 1$, then the modulation parameters for the populations are $g_1 = A/\mathcal{D}$ and $g_2 = -A/\mathcal{D}$.

A sample of analog simulation data obtained from a circuit model of (10) of conventional design [12] is shown in Fig. 1. The results refer to comparatively small values of noise intensity ($\mathcal{D} = 0.0161$) and amplitude of the periodic force ($A = 0.1$). It is immediately evident that the signal is nearly rectangular in shape, in complete agreement with the above arguments. The departure of the observed shape from a rectangular one is due partly to the contribution from the vibrations about the equilibrium positions $q_{1,2}^{(0)}$ and partly to the fact that the characteristic halfwidth of the time interval τ within which the transition between the states is most likely to occur [$\approx \Omega^{-1}(2/|g_1|)^{1/2}$] makes up ~ 0.2 of the half-period of the force under the experimental conditions. Detailed

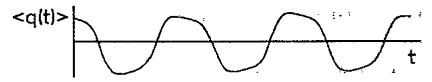

FIG. 1. The averaged coordinate $\langle q(t)\rangle$ measured for the electronic circuit simulating the overdamped system (10) with $\Omega = 1.9\times10^{-5}$, $A = 0.1$, $\mathcal{D} = 0.0161$. The results approximate a square wave.

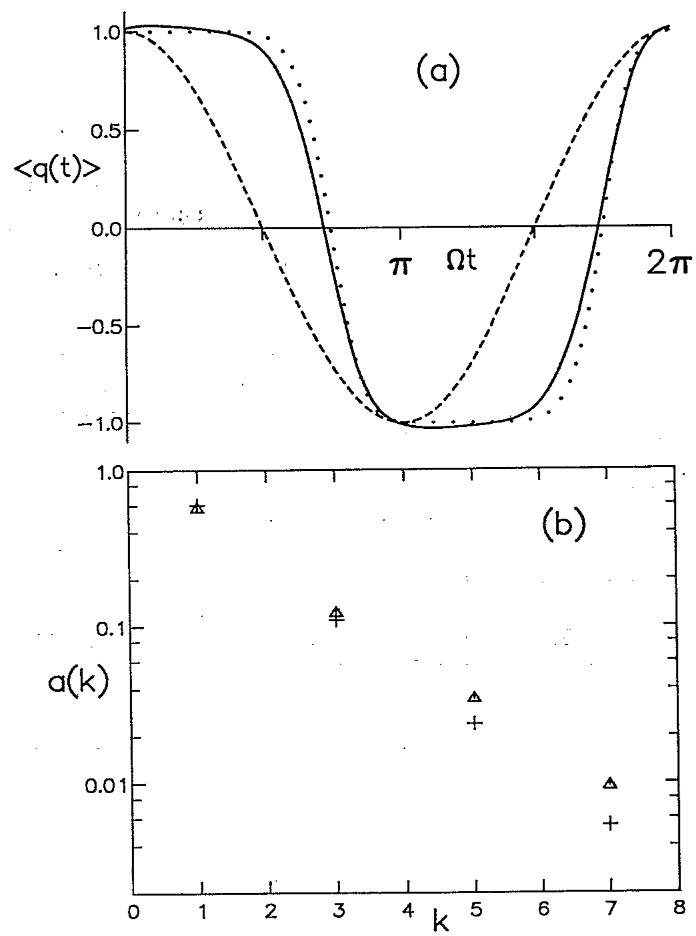

FIG. 2. The results of the digital simulation of the overdamped system (10) with $\Omega = 3\times10^{-4}$, $A = 0.12$, $\mathcal{D} = 0.03$ showing: (a) time-averaged signal $\langle q(t)\rangle$ (full curve) compared to the force scaled to the same amplitude (dashed curve); and (b) amplitudes of the lowest Fourier components of $\langle q(t)\rangle$ (triangles) as compared with the theoretical predictions (pluses) that follow from (8). The dotted line in (a) describes the theory of the square wave with account taken of the finite width of the distribution of the switching probability about its maximum as described below Eqs. (6) and (7).

analysis of the shape of the signal and of the fluctuations in the system, with account taken of τ being nonzero, will be given elsewhere [13].

Some digital simulation results for the same system (10) are shown in Fig. 2 (see Ref. [14] for the details of the numerical algorithm). Again, it is obvious from Fig. 2(a) that the signal is strongly distorted towards a rectangular shape, as compared to the sinusoidal drive. The value of the amplitude is very close to 1 and the signal is shifted in phase with respect to the forcing. It is seen clearly that the transitions occur just when the absolute value of the force is nearly at its maximum, in good agreement with the theory. A perfectly rectangular signal contains Fourier components with odd $k = 1, 3, \ldots$ only, and higher-order components are comparatively small, their absolute values being equal to $(2/k\pi)$ for unit signal amplitude. Correspondingly, the intensities of the Fourier components of the simulated signal, obtained by averaging over the time domain, are seen from Fig. 2(b) to decrease fast for small k. Their values, and also the values of the phases, are in a good quantitative agreement with the theory (6)–(9) (and the intensities of the even harmonics were negligibly small, as expected).

We note in conclusion that the giant nonlinearity observed and investigated in the present paper arises through an *interplay* between dynamics and fluctuations, for dynamically weak forcing. There is a sense in which it has much in common with the strong nonlinearity of the response of a system undergoing a phase transition. The price paid for the effect being so strong is the narrowness of the frequency range within which it occurs. This feature could, however, be helpful in relation to applications, in particular for a realization of noise-protected heterodyning.

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