Fokker-Planck description of stochastic processes with colored noise

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A detailed theoretical discussion is presented of the effect of colored noise on nonlinear dynamical systems. The ideas arising from it are applied to a particular example of such a system: a model of dispersive optical bistability, considered in the contexts of both additive and multiplicative forcing. Analogue experiments and digital simulation techniques are used to explore the applicability and range of validity of theories proposed to date. The physical phenomenon of bimodality induced by the finite bandwidth (alone) of additive forcing is demonstrated in detail for the first time. The three existing theories capable of accounting for this phenomenon can each be categorized in terms of a single (constant) value of a characteristic parameter $P_{\rm expt}$; but it is shown that, in reality, $P_{\rm expt}$ varies weakly with the correlation time τ of the noise. The variation of $P_{\rm expt}$ with τ is investigated in the currently accessible range of $0.1 \le \tau \le \infty$.

I. INTRODUCTION

Recent years have witnessed an increasing interest in the effects that arise from colored noise in nonlinear dynamical systems (see, for example, Refs. 1–11), a topic to which the present paper is also devoted. For the sake of definiteness, we will consider a one-dimensional stochastic differential equation of the form

$$\frac{dx}{dt}=\gamma[\phi(x)+\psi(x)\xi(t)] , \tag{1.1}$$

where x is a dimensionless variable; the rate constant γ sets the scale of the time evolution; $\phi(x)$ and $\psi(x)$ are arbitrary functions of x; and $\xi(t)$ is a dimensionless, stationary, Gaussian stochastic process with zero average, such that $\langle \xi(t) \rangle =0$. If the noise is white, i.e.,

$$\langle \xi(t)\xi(t') \rangle = 2\frac{Q}{\gamma}\delta(t-t') , \tag{1.2}$$

where the parameter Q specifies the noise level, it is well known that (1.1) leads to a Fokker-Planck equation (FPE) for the probability distribution $\sigma(x,t)$ of the stochastic variable x.[12,13] For the Stratonovich stochastic calculus, the FPE corresponding to (1.1) and (1.2) is

$$\frac{\partial}{\partial t}\sigma(x,t)=\gamma\left[-\frac{\partial}{\partial x}\phi(x)+\frac{Q}{\gamma}\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\psi(x)\right]\sigma(x,t) . \tag{1.3}$$

This type of equation is convenient in the sense that it allows for a straightforward calculation of the stationary probability distribution. If the noise is colored, however, the situation becomes very much more complicated.

Let us consider, again for the sake of definiteness, a stochastic process of the Ornstein-Uhlenbeck type. Thus, in place of the white-noise equation (1.2) we now suppose that

$$\frac{d\xi}{dt}=-\frac{1}{\tau}\xi+\frac{1}{\tau}\eta(t) , \tag{1.4}$$

where τ is the correlation time of the variable ξ and η is a Gaussian, stationary white noise of zero mean such that $\langle \eta(t) \rangle =0$ and

$$\langle \eta(t)\eta(t') \rangle = 2\frac{Q}{\gamma}\delta(t-t') . \tag{1.5}$$

Hence the variable ξ has the correlation function

$$\langle \xi(t)\xi(t') \rangle = \langle \xi^2 \rangle \exp\left[-\frac{t-t'}{\tau}\right] , \tag{1.6}$$

where the noise strength $\langle \xi^2 \rangle$ is given by

$$\langle \xi^2 \rangle = Q/\gamma\tau . \tag{1.7}$$

It may immediately be verified that, in the limit $\gamma\tau\to 0$, (1.6) reduces to the white-noise case of (1.2), and hence that the stochastic equation (1.1) leads to the one-dimensional FPE (1.3). On the other hand, for $\gamma\tau$ finite, the set of stochastic differential equations, (1.1) and (1.4), leads to a two-dimensional FPE for the joint probability distribution $P(x,\xi,t)$ in the variables x and ξ,[12,13]

$$\frac{\partial P(x,\xi,t)}{\partial t}=LP(x,\xi,t)$$

$$\equiv\left[-\gamma\frac{\partial}{\partial x}[\phi(x)+\psi(x)\xi]+\frac{1}{\tau}\left[\frac{\partial}{\partial\xi}\xi+\langle\xi^2\rangle\frac{\partial^2}{\partial\xi^2}\right]\right]P(x,\xi,t)\ . \quad (1.8)$$

Equation (1.8) is far less convenient than (1.3), for a number of reasons including, in particular, the following.

(i) For many purposes, the quantity of interest in practice is the probability distribution $\sigma(x,t)=\int d\xi\, P(x,\xi,t)$, and the additional information contained in $P(x,\xi,t)$ is irrelevant.[14]

(ii) The stationary solution of (1.8) cannot be calculated in closed form; only approximate expressions, or numerical results obtained, for example, by the matrix continued fraction method,[13] or the wideband perturbation expansion,[15(a)] are available.

These two factors have stimulated numerous efforts to investigate and approximate the equation which governs the time evolution of the reduced distribution $\sigma(x,t)$ and which follows from (1.8). As is also true in general of all reduction problems, the equation in question is not differential, but integrodifferential (in time); furthermore, it includes derivatives of all orders with respect to x. Consequently, there is no possibility that $\sigma(x,t)$ can obey *exactly* an equation of the Fokker-Planck type. This fact does not, however, exclude the prospect of the time evolution of $\sigma(x,t)$ being describable to a good approximation by a Fokker-Planck equation, at least in cases where the noise parameter Q is small. Several proposals of FPE's for $\sigma(x,t)$ have been advanced in this vein by a variety of authors in recent years.

The present paper has two main goals. The first of these is to discuss some of the more promising proposals, trying to identify the best candidate for the Fokker-Planck description of stochastic processes with colored noise. Our analysis will be based both on general considerations and also on the study of a specific example of a stochastic differential equation. The latter approach will allow us to obtain an estimate of the limitations of the Fokker-Planck approach.

The second purpose of this paper relates to the effect of noise color on *noise-induced transitions*.[15] Let us consider (1.1) in the absence of noise,

$$\frac{dx}{dt}=\gamma\phi(x,\lambda)\ , \quad (1.9)$$

where we have introduced explicitly a control parameter λ. We suppose also that, for all values of λ in the case of the system under consideration, the steady-state equation $\psi(x,\lambda)=0$ admits only one solution, $\bar{x}(\lambda)$. In the presence of noise $\xi(t)$, if this is white and additive [i.e., if $\psi(x)\equiv 1$ in Eq. (1.1)], the stationary solution of the FPE (1.3) is given by

$$\sigma_s(x)=N\exp\left[-\frac{1}{Q}\int\phi(x,\lambda)dx\right]\ , \quad (1.10)$$

where N is the normalization constant: (1.10) implies the existence of a single peak in the density located at exactly $x=\bar{x}(\lambda)$. When the noise is white but multiplicative [i.e., $\psi(x)$ is not a constant], on the other hand, the behavior of the system is strikingly different. For appropriate choices of $\phi(x,\lambda)$ and $\psi(x)$, and in a suitable interval of the parameter λ, an increase of Q alters the stationary distribution in a continuous fashion from one that has a single peak to one that has two peaks.[15,16] It is this phenomenon that is referred to as a *noise-induced transition.*

Rather similar behavior is found in the steady-state probability distribution in a nonequilibrium phase transition of the second order.[17] It must be noted, however, that the noise in the latter case is additive and that the behavior of the stationary distribution follows strictly that of the steady-state solutions of (1.9) which exhibit a pitchfork bifurcation when plotted as a function of λ. This is in contrast to the transitions considered in Refs. 14–16 where the bimodal structure of the stationary distribution is produced exclusively by the multiplicative character of the noise, whence the appellation of the phenomenon.

Kitahara, Horsthemke, and Lefever[18] showed that the phenomenon of noise-induced transitions persists when the noise is changed from white to colored. This change may also produce profound qualitative consequences in the behavior of the stationary probability distribution leading, for example, from the scenario of a second-order phase transition to one of the first order.[1,19]

In Refs. 1, 18, and 19, the noise is multiplicative in nature. We will show below that noise-induced transitions can arise, not only from multiplicative noise, but also from additive noise; provided, however, that it is colored rather than white. For the purpose of our demonstration we will focus our attention on a model that is usually studied in connection with optical bistability,[20,21] but considered here for values of the control parameter which do not produce bistability. The model in question will be used as a basis for the discussion of different Fokker-Planck descriptions for colored noise. The theoretical predictions are tested against analogue experiments and digital simulations of the same model; we will see that it provides a detailed and complete description of the noise-induced transition.

Because the parameter γ gives only the overall time scale of the dynamics, it is convenient to get rid of it by normalizing the time by γ^{-1}. Thus, the parameter γ drops from (1.1)–(1.5), (1.7), and (1.8), provided that we replace t by $\bar{t}=\gamma t$ and τ by $\bar{\tau}=\gamma\tau$. In what follows, for the sake of simplicity, we will always omit the overbar and write t and τ instead of $\bar{t}$ and $\bar{\tau}$, respectively.

In Sec. II, we review briefly some of the main proposals for the Fokker-Planck approximation of the equation which governs the time evolution of the reduced distribution $\sigma(x,t)$ in the case of colored noise. Section III is devoted to a short illustration of the projection operator method developed to fourth order in the perturbation, and continues the discussion of the best candidate for the Fokker-Planck description. In Sec. IV, we introduce the nonlinear passive optical system that we model extensively in this paper in its steady-state aspects. Section V describes the electronic analogue experiment undertaken to simulate this model; Sec. VI presents the digital simulation results and compares them in detail with the predictions of the various theories. The existence of a noise-induced transition due to additive colored noise is clearly evident in both the analogue and digital simulations. Section VII deals with the rather different case of multiplicative noise arising from fluctuations in the input intensity. Finally, Sec. VIII summarizes the main results and concludes the discussion of the Fokker-Planck approximation.

II. A DISCUSSION OF TWO CURRENT APPROACHES TO THE PROBLEM OF COLORED NOISE

Large efforts have been made in the recent past[1,2,6–9] to associate Eq. (1.1), supplemented by Eq. (1.4), with the Fokker-Planck equation

$$\frac{\partial}{\partial t}\sigma(x,t)=\left[\frac{\partial}{\partial x}\phi(x)+Q\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\Phi(x)\right]\sigma(x,t)\ , \tag{2.1}$$

where the function $\Phi(x)$ (which is precisely the object of these investigations) must satisfy the obvious white-noise-limit condition

$$\lim_{\text{white noise}}\Phi(x)=\psi(x)\ . \tag{2.2}$$

We shall focus on the following two theoretical proposals: (a) the best so-called Fokker-Planck approximation,[1,2,10] and (b) the Fox theory.[9]

According to the former theory,[1,2,10] $\Phi(x)$ has to be derived from the following differential equation:

$$\psi(x)=\Phi(x)+\tau[\phi(x)\Phi'(x)-\phi'(x)\Phi(x)]\ . \tag{2.3}$$

The technical difficulty associated with this equation is that, in general, no exact analytical expressions for $\Phi(x)$ can be derived from it (some special cases are discussed in Refs. 10, 22, and 23) and recourse has to be made to the assumption that τ is so small as to make it possible to solve Eq. (2.3) via a perturbation expression around $\tau=0$. By adopting this technique we obtain

$$\begin{aligned}&\Phi(x)=\psi(x)\ ,\\&\Phi(x)=\psi(x)+\tau[\phi'(x)\psi(x)-\phi(x)\psi'(x)]\ ,\\&\Phi(x)=\psi(x)+\tau[\phi'(x)\psi(x)-\phi(x)\psi'(x)]\\&\qquad+\tau^2[\phi'(x)\phi'(x)\psi(x)-\phi(x)\phi''(x)\psi(x)\\&\qquad\quad-\phi(x)\phi'(x)\psi'(x)+\phi(x)^2\psi''(x)]\ ,\end{aligned} \tag{2.4}$$

and so on.

The most recent version of the Fox theory,[9] on the other hand, leads to the following compact expression for $\Phi(x)$:

$$\Phi(x)=\frac{\psi(x)}{1-\tau\left[\phi'(x)-\dfrac{\psi'(x)}{\psi(x)}\phi(x)\right]}\ . \tag{2.5}$$

It is important to note that at the order τ the Fox theory (2.5) leads to the same result as the best Fokker-Planck approximation [i.e., the second of the set of equations (2.4)]. Since, as established by Faetti *et al.*[22,23] the Fokker-Planck equation (2.1) provides steady-state distributions correct up to the order τ [when the noise $\xi(t)$ is assumed to be Gaussian], this means that the Fox theory leads to equilibrium distributions *correct up to the order* τ *in the short-*τ *region.*

A further remarkable feature of the Fox theory is that it leads to *exact expressions* for the steady-state equilibrium distribution in the limit $\tau\to\infty$. This particular point, recently recognized also by Jung and Hänggi,[24] has already been pointed out, for the case of additive noise, by Faetti *et al.*[22] For the reader's convenience, we shall provide here a simple demonstration valid also in the more general multiplicative case (see Ref. 1 and references quoted therein).

When $\tau\to\infty$ the system has enough time to reach the equilibrium position determined by

$$\phi(x)+\psi(x)\xi=0\ , \tag{2.6}$$

which is obtained from Eq. (1.1) by assuming $\dot{x}=0$, before ξ is given a new value. On the other hand, the variable ξ is characterized by the equilibrium distribution

$$\rho_{\text{eq}}(\xi)d\xi=(2\pi\langle\xi^2\rangle)^{-1/2}\exp\left[-\frac{\xi^2}{2\langle\xi^2\rangle}\right]d\xi\ . \tag{2.7}$$

From Eq. (2.6) we get

$$d\xi=\left[-\frac{\phi'}{\psi}+\frac{\phi\psi'}{\psi^2}\right]dx\ . \tag{2.8}$$

By substitution of Eqs. (2.6) and (2.8) into the right-hand side (rhs) of Eq. (2.7) we derive

$$\rho_{\text{eq}}(x)dx=(2\pi\langle\xi^2\rangle)^{-1/2}\exp\left[-\frac{\phi^2}{2\psi^2\langle\xi^2\rangle}\right]\times\left|\frac{1}{\psi^2}(\phi\psi'-\phi'\psi)\right|dx\ . \tag{2.9}$$

It is straightforward to show that in the limiting case $\tau\to\infty$, Eq. (2.9) coincides with the equilibrium distribution corresponding to Eq. (2.1) with $\Phi(x)$ given by (2.5). This implies that the equilibrium distribution provided by the Fox theory coincides with the exact result of Eq. (2.9). Note that in this paper we shall not deal with the intriguing case where $(\phi\psi'-\phi'\psi)/\psi^2$ might also turn out to be negative in suitable intervals of the x variable (this is so because we shall not deal here with the fully bistable situation; rather, we shall limit ourselves to studying the bistability induced by the noise color).

We can thus remark that because the Fox theory leads to exact equilibrium distributions both in the short- and large-τ limits, it also has a good chance of providing satisfactory predictions even in the intermediate region. In other words, the equilibrium distribution provided by the Fox theory provides a natural interpolation formula between two exact limits. Similar conclusions have also been reached by Jung and Hänggi[24] and by Faetti and Grigolini.[25]

In anticipation of the developments in Sec. III, it is convenient to write the function $\Phi(x)$ to second order in τ, both for the best Fokker-Planck approximation and also for the Fox theory. Using Eqs. (2.4) and (2.5), one can obtain the compact expression

$$\Phi(x)=\psi(x)[1+\tau\Pi^{(1)}+\tau^2(\Pi^{(1)})^2+\tau^2 P_{\mathrm{expt}}\Pi^{(1)'}\phi(x)] , \tag{2.10}$$

where

$$\Pi^{(1)}=[\phi'(x)\psi(x)-\phi(x)\psi'(x)]/\psi(x) , \tag{2.11}$$

and one must select $P_{\mathrm{expt}}=-1$, in the case of the best Fokker-Planck approximation, and $P_{\mathrm{expt}}=0$, in the case of Fox's theory, respectively. Hence the former theory contains the additional term

$$-\tau^2\Pi^{(1)'}\phi(x)\psi(x) , \tag{2.12}$$

which would appear at first sight to imply that Fox's theory is the less accurate, at least when τ is small enough. The situation is, however, more subtle, as will be shown in Sec. III where, using the projection-operator method, we discuss the relationship of Fox's theory to the best Fokker-Planck approximation.

III. THE INFLUENCE OF HIGHER PERTURBATION ORDERS IN THE NOISE INTENSITY

The projection-operator approach[10] consists of dividing the operator L of Eq. (1.8) into a perturbation part, L_1, and an unperturbed one, L_0, and defining a corresponding projection operator. In the present case we adopt the following division of L:

$$L=L_0+L_1 , \tag{3.1}$$

$$L_0\equiv\frac{1}{\tau}\left[\frac{\partial}{\partial\xi}\xi+\langle\xi^2\rangle\frac{\partial^2}{\partial\xi^2}\right]-\frac{\partial}{\partial x}\phi(x) , \tag{3.1a}$$

$$L_1\equiv-\frac{\partial}{\partial x}\psi(x)\xi . \tag{3.1b}$$

The projection operator method[10] is basically a perturbation approach in the perturbation term L_1. At the order L_1^2 this method leads to a Fokker-Planck expression which coincides with the best Fokker-Planck approximation (see Sec. II).

At order L_1^4, one obtains nonstandard terms, with derivatives of order higher than second. These contributions can be replaced by terms of the standard type, i.e., with second-order derivatives, by adopting the renormalization of Ref. 22 extended to the multiplicative case. If, after the renormalization, one neglects the terms of order Q^2 or higher, together with those of order higher than second in τ, then following lengthy calculations[22,23] one arrives again at the expression (2.10) for the function Φ which appears in the Fokker-Planck equation (2.1). The value of the constant P_{expt} is now different, however, and is given by

$$P_{\mathrm{expt}}=\tfrac{1}{2} . \tag{3.2}$$

This result implies that the projection-operator method provides, at order L^4, a correction to the best Fokker-Planck approximation. This correction has *exactly* the same form as the difference (to order τ^2) between the best Fokker-Planck approximation and Fox's theory. Unfortunately, a direct comparison at order τ^2 between the various theories is not appropriate, owing to the almost unavoidable appearance of unphysical boundaries related to intervals of x within which the diffusion coefficient becomes negative. This artefact does not appear in theories which include contributions of all orders in τ: for example, in a large class of problems, the Fox theory[9] produces positive diffusion coefficients everywhere, provided that τ is kept below a certain critical value. On the other hand, a comparison to all orders in τ between the theories is also far from straightforward. This is because an expression for the diffusion coefficient in closed form is available only for the Fox theory, Even in the case of the best Fokker-Planck approximation, which is equivalent to the projection-operator approach to order L_1^2, the analytical expression for $\Phi(x)$ is, in general, an unsettled problem. In other words, the power expansion in τ (2.4) is generally not summable in closed form. Because, as already mentioned, a truncation of this expansion leads almost invariably to unphysical boundaries, the authors of Ref. 1 devised an appropriate exponential form to avoid this problem. Unfortunately, in the present case, there were computational difficulties which prevented us from adopting this technique.

In light of the above considerations, we decided to replace (2.1) and (2.10) by the following Fokker-Planck equation:

$$\frac{\partial}{\partial t}\sigma(x,t)=\left[-\frac{\partial}{\partial x}\phi(x)+Q\frac{\partial}{\partial x}\psi(x)\frac{\partial}{\partial x}\frac{\psi(x)}{(1-\tau\Pi^{(1)})}[1+\tau^2P_{\mathrm{expt}}\Pi^{(1)'}\phi(x)]\right]\sigma(x,t) , \tag{3.3}$$

where the factor $(1-\tau\Pi^{(1)})^{-1}$ introduces a "regularization" of the problem of negative diffusion coefficients, different from—but not necessarily worse than—the exponential form of Ref. 1. It should be noted that in the case of $P_{\mathrm{expt}}=0$, (3.3) coincides *exactly* with the Fox theory. For the cases $P_{\mathrm{expt}}=-1$ and $P_{\mathrm{expt}}=0.5$, however, (3.3) differs, respectively, from the best Fokker-Planck equation and from the projection-operator approach to order L_1^4 because they correspond only to a partial resummation in τ. Nonetheless, if we expand (3.3) to second order in τ, we recover Eqs. (2.1) and (2.10). Therefore, to order τ^2, (3.3) with $P_{\rm expt}=-1$ ($P_{\rm expt}=0.5$) does indeed coincide with the best Fokker-Planck approximation (projection-operator approach to order L_1^4). Strictly speaking, we should refer to (3.3) with $P_{\rm expt}=-1$ as the "modified best Fokker-Planck approximation," but for the sake of simplicity we shall omit the word "modified" in what follows; the distinction should, however, be borne in mind. We regard (3.3) as an equation which incorporates to order τ^2 the bulk of the best Fokker-Planck equation ($P_{\rm expt}=-1$) and of the projection-operator approach ($P_{\rm expt}=0.5$) and thus offers an opportunity to perform an otherwise impossible comparison between these two theories and the Fox theory.

In the bistable case $\phi(x)=\alpha x-\beta x^3$ [$\psi(x)=1$] it is straightforward to derive from a Fokker-Planck equation of the same type as Eq. (2.1) an analytical expression for the shift of the maximum of the probability distribution as a function of τ. It is easily shown that at the order τ^2 this shift does not depend on the fourth-order correction introduced by the projection-operator technique at order L_1^4. The resulting expression for this shift, on the other hand, is proven to coincide up to the order τ^2 with the theoretical prediction of Altares and Nicolis [Eq. (3.8) of Ref. 26]. This is quite encouraging as the analytical result of Altares and Nicolis is based only on the assumption that the noise intensity is weak, while setting no restrictions upon the noise color. However, this does not help us in supporting our prediction $P_{\rm expt}=0.5$ since (as already remarked) the term $P_{\rm expt}\Pi^{(1)'}\psi$ does not provide any contribution to the shift of the maximum of the probability distribution as a function of τ.

To monitor the correction to the Fox theory stemming from the term $P_{\rm expt}\Pi^{(1)'}\phi$ is quite a difficult matter. This is so because, when τ is very small, the order τ provides the predominant contributions to the effects of noise color. When the term in τ^2 begins to produce significant effects, so also do the terms of higher order in τ. This is the intermediate region where all theories are inaccurate. When the large-τ region is reached, the Fox theory leads to exact predictions for the steady-state distribution. It is therefore plausible that, in the intermediate region, the predictions of the Fox theory are at least fairly accurate.

IV. MODEL OF A NONLINEAR, PASSIVE, DRIVEN OPTICAL SYSTEM

In this section we consider a particular example of a stochastic nonlinear system, in the context of which some of the above ideas can be applied and tested. The example chosen is taken from optics. It is a bistable system which, in the last decade, has been the object of extensive investigations motivated both by its intrinsic theoretical interest and by practical considerations. The phenomenon of interest, called optical bistability,[20] occurs in an optical cavity filled with a material which has an intensity-dependent refractive index. The experimental arrangement is as shown in Fig. 1. A stationary, coherent beam, near to resonance both with the cavity and with the material, is injected into the cavity. The steady-state intensity of the beam transmitted by this system is a nonlinear function of the input intensity. By suitably adjusting the frequency of the incident field, the stationary curve of transmitted versus incident intensity becomes S shaped as shown in Fig. 2. The part with negative slope (dashed) is unstable, so that there is an interval of the input intensity within which the system is bistable. If the incident power is slowly increased from zero to beyond the bistable region, and then decreased back to zero, one obtains a hysteresis cycle. By varying continuously the input frequency, the size of the hysteresis cycle can be reduced to zero and the bistability disappears. Just at the boundary between bistability and monostability there is a "critical" situation (Fig. 3) on which we will focus in this paper.

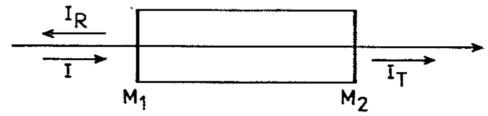

FIG. 1. Fabry-Pérot cavity. M_1 and M_2 are partially transmitting mirrors; I, I_T, and I_R are the incident, transmitted, and reflected intensities, respectively.

We will describe this phenomenon by means of a very simple model, the derivation of which is discussed in Ref. 5. We indicate by x the normalized, intensity-dependent part of the refractive index; it obeys the dynamical equation

$$\frac{dx}{dt}=-x+\frac{I}{1+(x-\theta)^2}\,, \qquad (4.1)$$

where the time t is normalized to the response rate γ of the material, I is the normalized input intensity, and θ is the detuning parameter, which arises both from the mismatch between the incident field frequency and the nearest cavity frequency, and from the constant (intensity-independent) part of the refractive index. The control parameters in this model are I and θ; in particular, I plays the role of the parameter λ mentioned above in the Introduction.

The normalized transmitted intensity I_T is given by

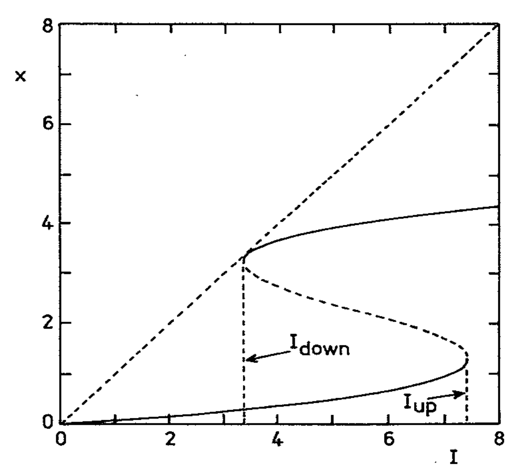

FIG. 2. Steady-state curve representing (4.3) with $\theta=2\sqrt{3}$; the part with a negative slope is unstable.

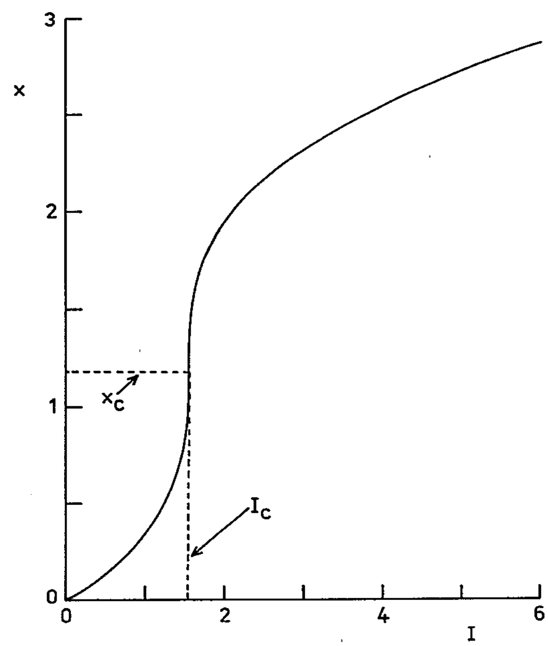

FIG. 3. Critical steady-state curve representing (4.3) with $\theta=\sqrt{3}$.

$$I_T=\frac{I}{1+(x-\theta)^2} \ . \tag{4.2}$$

The model (4.1),(4.2) holds in the limit of dispersion dominant over absorption, and requires that the cavity relaxation time is much smaller than the atomic time γ^{-1}; if any noise is present, the cavity relaxation time must be much smaller also than the correlation time of the noise. At steady state, Eq. (4.1) gives the cubic stationary equation

$$I=x[1+(x-\theta)^2] \ , \tag{4.3}$$

and I_T coincides with x. The plot of x as obtained from Eq. (4.3) exhibits bistability[21] for $\theta>\sqrt{3}$ (Fig. 2). The critical situation (Fig. 3) for which the curve displays a point of inflection with a vertical tangent corresponds to $\theta=\sqrt{3}$; the coordinates of the critical point are $x_c=2\sqrt{3}/3$ and $I_c=8\sqrt{3}/9$. Following Ref. 5, we will consider two kinds of noise: thermal noise in the material and fluctuations in the input intensity I. In the first case, Eq. (4.1) is replaced by a stochastic equation with additive noise

$$\frac{dx}{dt}=-x+\frac{I}{1+(x-\theta)^2}+\xi(t) \ , \tag{4.4}$$

where we assume that the noise is Gaussian and that the variable ξ obeys Eq. (1.4).

For $\tau \ll 1$, the noise is felt as white with

$$\langle \xi(t)\xi(t')\rangle=2Q\delta(t-t'), \quad Q=\gamma\langle \xi^2\rangle \ . \tag{4.5}$$

In the case of intensity noise, we make the substitution $I \rightarrow I+\xi(t)$, in (4.4) thereby obtaining the corresponding stochastic equation with multiplicative noise

$$\frac{dx}{dt}=-x+\frac{I}{1+(x-\theta)^2}+\frac{\xi(t)}{1+(x-\theta)^2} \ . \tag{4.6}$$

We will assume that the stochastic variable $\xi(t)$ is again Gaussian and characterized by Eq. (1.4). We now address ourselves to the solution of (4.4) and (4.6), corresponding to the cases of additive and multiplicative noise, respectively.

V. ANALOGUE SIMULATION

This section describes an experimental investigation of the system (4.4), introduced above, based on electronic analogue simulation. The analogue technique itself has already been described in connection with studies of other stochastic systems (see, for example, Refs. 8, 11, 15(b), and (16); more detailed discussions of the application of the technique and of its underlying philosophy will be found in Refs. 27 and 28. The essence of the method is that an electronic circuit is built to model as accurately as possible the system under investigation [in this case, Eq. (4.4)], and its statistical response to external forcing by the appropriate type of noise (in this case, colored Gaussian) is then analyzed in detail by means of a computer and/or data processor.

In the event, we found that it was extremely difficult to perform a satisfactory analogue simulation of (4.4) for the required critical values of $\theta=\sqrt{3}$, $I=\frac{8}{9}\sqrt{3}$. The reason for this becomes clear immediately on inspection of Fig. 3, and is associated with the extreme sensitivity of x to small changes in I when x is close to x_c. In electronic circuits of the kind under discussion, parameters can be set typically to an accuracy of ± 2 mV, but they are liable to drift by up to about 10 mV during the several minutes usually needed to complete the measurement of a statistical density. This implied in practice that the deterministic ($Q=0$) value of x could not be relied upon to be closer than $\pm 5\%$ to x_c, on average, during a measurement. Consequently, although the results of the direct simulation of (4.4) were in good qualitative agreement with the theoretical predictions, they tended to be rather irreproducible because of parameter drift, and the quality of the quantitative agreement with theory was only fair. In part, the stability problem arose from the existence of the quotient in (4.4) which exacerbated the tendency of the parameter I to drift.

In an attempt to improve the quality of the simulation, a second electronic circuit was built which contrived to model the system under investigation without any need explicitly to calculate a quotient.[29] The possibility of being able to do so depended on the realization that the equation under discussion (4.4) was originally obtained, by use of the technique of adiabatic elimination,[17] from the more complicated set of coupled differential equations normally used to model optical bistability,[20] namely,

$$\frac{dz}{dt}=-K[z-\sqrt{I}+iz(\theta-x)] \ ,$$
$$\frac{dx}{dt}=-\gamma(x-|z|^2)+\gamma\xi(t) \ , \tag{5.1}$$

where z is the transmitted field of intensity $|z|^2=I_T$, K^{-1} is the cavity relaxation time, and the variable $\xi(t)$ obeys (1.4). In the limit $K \gg \gamma$, which usually obtains in practice, it can be shown that (5.1) yields (4.4). Although equations (5.1) are, of course, very much more difficult to solve theoretically than (4.4), the corresponding analogue circuit is only slightly more complicated to connect up; and, as already mentioned, it has the considerable advantage in terms of stability that it does not include a quotient term.

A diagram of the second circuit, that is, of the circuit modeling (5.1), is shown in Fig. 4. The design is of the minimum component type[27] in order to reduce to the lowest possible levels nonidealities such as drift and internal noise; the necessary trimmers and offset adjustments are not shown. As usual, with this particular kind of circuit, the system actually simulated differed from (5.1) to the extent that some of the quantities appearing there had to be scaled by suitable factors: for the final results given below, however, the appropriate normalization factors have been taken into account.

An interesting feature of the use of this circuit (in effect) to model (4.4) is that the adiabatic elimination is being made to occur in practice in the circuit itself by setting $K \gg \gamma$. Thus, it is possible to check that this procedure works in reality. The ratio K/γ for the circuit was actually equal to 2000 (with $K=2\times10^5$ Hz, $\gamma=10^2$ Hz); it was found that, for the noise of correlation time of order γ^{-1}, the distributions obtained were almost indistinguishable from those for $K/\gamma=500$. This result can be construed as a rather direct and convincing demonstration that adiabatic elimination does work.

The circuit parameters were set as to operate the system at its critical point ($\theta_c=\sqrt{3}$, $I_c=\frac{8}{9}\sqrt{3}$: see Fig. 3) and the noise intensity was kept fixed at $Q=0.1$. Several equilibrium distributions of x were then acquired for different values of τ. These are shown by the jagged lines in Fig. 5, where τ is stated in each case in units of γ^{-1}. It is immediately evident that, when the noise is effectively white (small τ), the distribution is monomodal; but, as τ is increased, the distribution broadens and becomes bimodal; and the cleavage between the corresponding twin maxima steadily deepens with increasing τ. The change from a monomodal to a bimodal distribution represents a noise-induced transition, and it is important to note that, unlike the noise-induced transitions[15,16] previously reported, this transition is one that can be affected with *additive* noise as a result of changing only the noise *color*, with all other parameters held constant.

The behavior to be expected of (4.4) theoretically is readily calculated on the basis of the methods discussed in the preceding sections. Taking (3.3) with $\psi(x)=1$ and [from Eq. (4.4)] setting

$$\phi(x)=-x+\frac{I}{1+(x-\theta)^2}, \qquad \Pi^{(1)}=\phi'(x), \tag{5.2}$$

the equilibrium solution of x in (3.3) can be shown to be

$$\sigma_{\text{theor}}(x)=\mathcal{N}\left|\frac{1-\tau\phi'}{1+\tau^2P_{\text{expt}}\phi\phi''}\right| \times\exp\int^x \frac{1}{Q}\frac{\phi(1-\tau\phi')}{1+\tau^2P_{\text{expt}}\phi\phi''}dy, \tag{5.3}$$

where $\mathcal{N}$ is the normalization constant, and the distribution $\sigma_{\text{theor}}(x)$ is set equal to zero in the interval where $(1+\tau^2P_{\text{expt}}\phi\phi'')$ is negative. To make a direct quantitative comparison of (5.3) with the measured distributions of Fig. 5 is less than straightforward, however. This is because of the extreme sensitivity of the circuit, when working at its critical point, to any small errors in the operating parameters, already alluded to above. For this reason, the comparison was carried out indirectly. That is, rather than simply assuming that all the parameters

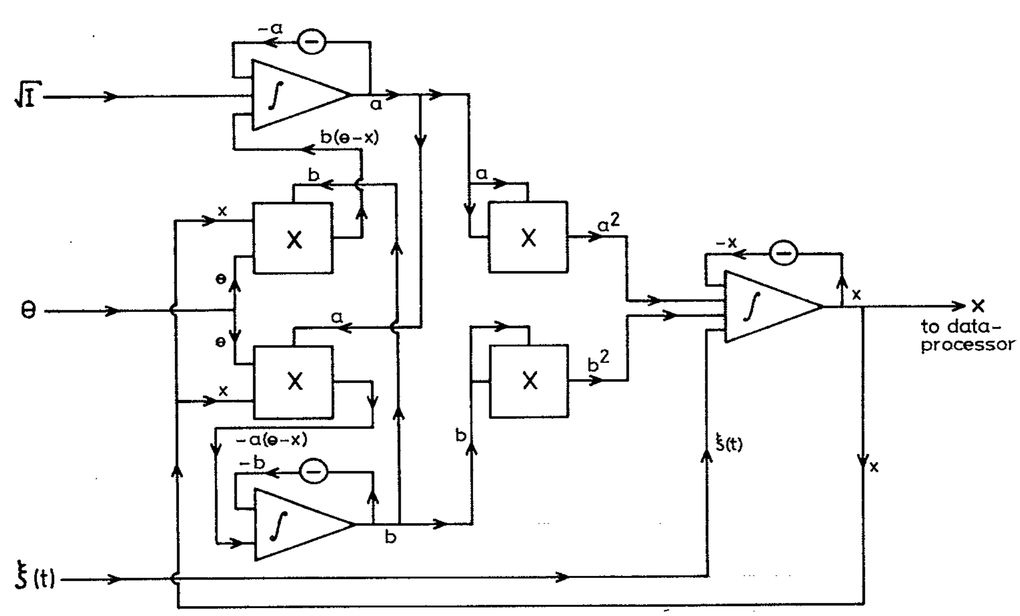

FIG. 4. Block diagram of the analogue electronic circuit used to model Eqs. (5.1), with $z=a+ib$. The multipliers form the product of the inputs shown at the top and at the left-hand side of the square in all cases, each input being differential.

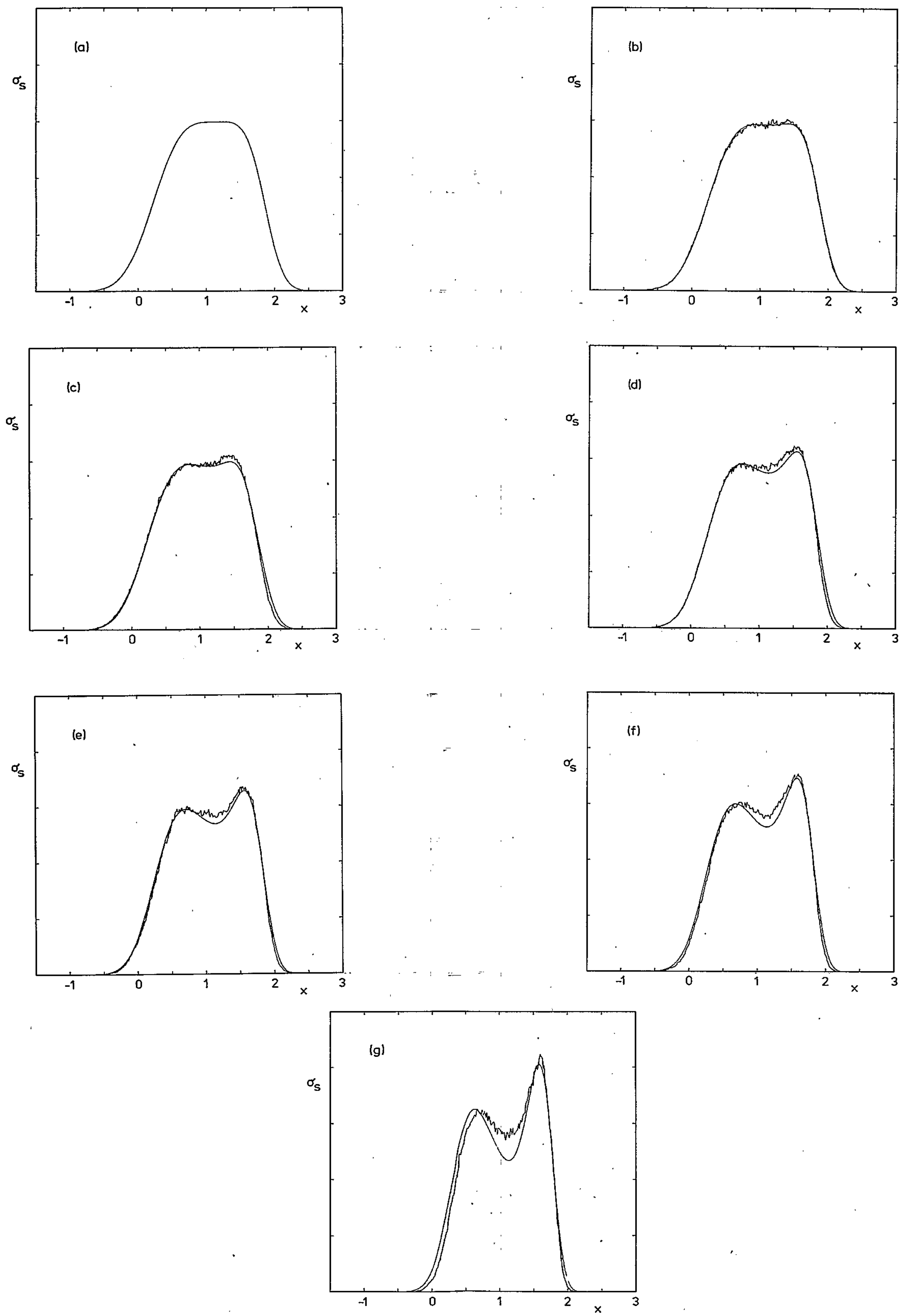

FIG. 5. Probability distributions of x for a range of different noise correlation times τ: (a) $\tau=0$; (b) 0.1; (c) 0.2; (d) 0.5; (e) 0.7; (f) 1.0; (g) 2.0. The jagged curves are the experimental results from the electronic circuit of Fig. 4. The smooth curves represent theory (see text).

possessed precisely their intended values, computing distributions from (5.3) and comparing the results directly with the experimental data, we attempted to answer a rather different question: Does there exist a set of parameters, equal within experimental error to those set in the circuit, for which good agreement can be demonstrated between (5.3) and the experimental data of Fig. 5?

In order to answer this question it was necessary not only to consider experimental errors in θ, I, Q, τ, and P_{expt}, but also to allow for the existence of a small constant c on the right-hand side of (4.4) and to allow for small errors in gain and offset in the input stage of the Nicolet 1080 data processor used to analyze the fluctuating voltage representing $x(t)$ for the determination of the experimental distribution, $\sigma_{\text{expt}}(x)$. In practice, the function

$$\int |\sigma_{\text{theor}}(x)-\sigma_{\text{expt}}(x)|^2 dx \qquad (5.4)$$

was minimized by use of the routine MINUIT.[30] Of course, not all of the eight parameters mentioned above are independent; and so the number of parameters used for the fit was reduced until the global correlation coefficients, measuring the internal correlation between a free parameter and a suitable linear combination of the other free parameters, were judged to be acceptable. It was found that three parameters were needed, chosen to be θ, I, and P_{expt}. The numerical results of an application of this procedure to the data of Fig. 5, for $\tau=0.1$ are summarized in Table I: for this small value of τ, the fit was found to be extremely insensitive to P_{expt} (see below) and, although the uncertainty limits are hard to estimate reliably, we would attach no significance beyond its order of magnitude to the fitted value of 2.883. The calculated distribution corresponding to these parameters is shown by the solid curve in Fig. 5(b). The distributions obtained for larger τ were fitted with θ and I fixed at the values shown in Table I, with P_{expt} as the only adjustable parameter, with the results shown by the smooth curves in Figs. 5(c)–5(g). The smooth curve of Fig. 5(a) for $\tau=0$ (white noise) was calculated for the same values of I and θ; there is no corresponding experimental curve because it is, of course, impossible to have $\tau=0$ in any real physical system.

It is evident that the agreement between the experimental and theoretical curves in Fig. 5 is good, even for the larger values of τ. Furthermore, the discrepancies between the fitted parameter values of Table I and the values that had been set experimentally are all in the range of a few percent; that is, they lie within the range of experimental uncertainty expected for the technique in question. It is safe to conclude, therefore, that the theoretical predictions are in satisfactory agreement with the results of the analogue experiment.

TABLE I. The parameter values used in the electronic circuit and in generating the theoretical curve of Fig. 5(b). The same values of θ, I, and Q were also used for generating the theoretical curves of Figs. 5(a) and 5(c)–5(g), changing τ to the experimental value and with P_{expt} as the only adjustable parameter.

Parameter	Value set in circuit	Value used for fit
θ	1.733	1.734
I	1.541	1.543
Q	0.1	0.1
τ	0.1	0.1
P_{expt}		2.883

VI. COMPARISON OF THEORIES

In Sec. V we showed that the system described by (4.4) exhibits bimodality induced by the finite bandwidth of the additive stochastic forcing and, furthermore, that the theory developed in Secs. II and III is able to account for the main features observed experimentally, *provided* that appropriate choices are made for the quantity P_{expt}. The present section is devoted to a detailed comparison of the relevant colored-noise theories that have been proposed, with each other, and with a digital simulation of the same system. We do not consider here the so-called mean-field theory,[6–8] which does not reproduce the particular bistability phenomenon under discussion.

As discussed in Sec. III, insertion of appropriately chosen values of P_{expt} into Eqs. (3.3) will yield the three (and, at the moment, only) theories able to describe bistability induced by noise color. It was evident from the analogue experimental results that P_{expt}, defined as the parameter optimizing agreement between theory and experiment in the sense of (5.4), changes with τ. A more detailed investigation of the variation of P_{expt} with τ therefore seemed desirable and to avoid possible problems with the inevitable nonidealities present in the electronic circuit, a digital simulation of (4.4) was undertaken. It was carried out on the basis of standard Monte Carlo techniques.[31]

The results of the digital simulation are shown for three values of τ by the slightly noisy curves (marked D) in Figs. 6(a)–6(c). The solid curves represent the theory (3.3) for different values of P_{expt}, as indicated in the caption. It is immediately evident that, for small τ [Fig. 6(a) for $\tau=0.2$], the theoretical curves are very insensitive to P_{expt}, falling almost on top of each other, in excellent agreement with the digital simulation.[32] As τ increases, however, the choice of P_{expt} clearly becomes more important: values of zero (Fox theory) or 0.5 (projection-operator method to order L_1^4) yield better agreement with the digital simulation than a value of -1 (best Fokker-Planck approximation).

Two comments should be made in relation to the comparisons of Fig. 6. First, we were unable to use the exponential forms of Refs. 1 and 2 for the equilbrium distribution function for the case of $P_{\text{expt}}=-1$. This was because no analytic expression was available for the equilibrium distribution itself, and no simple form for such exponentiation could be recognized. We would point out, however, that the renormalization incorporated in (3.3) already solves the problem of unphysical boundaries over a wide range of τ. Second, the distributions ought to be identically zero outside the boundaries, because no

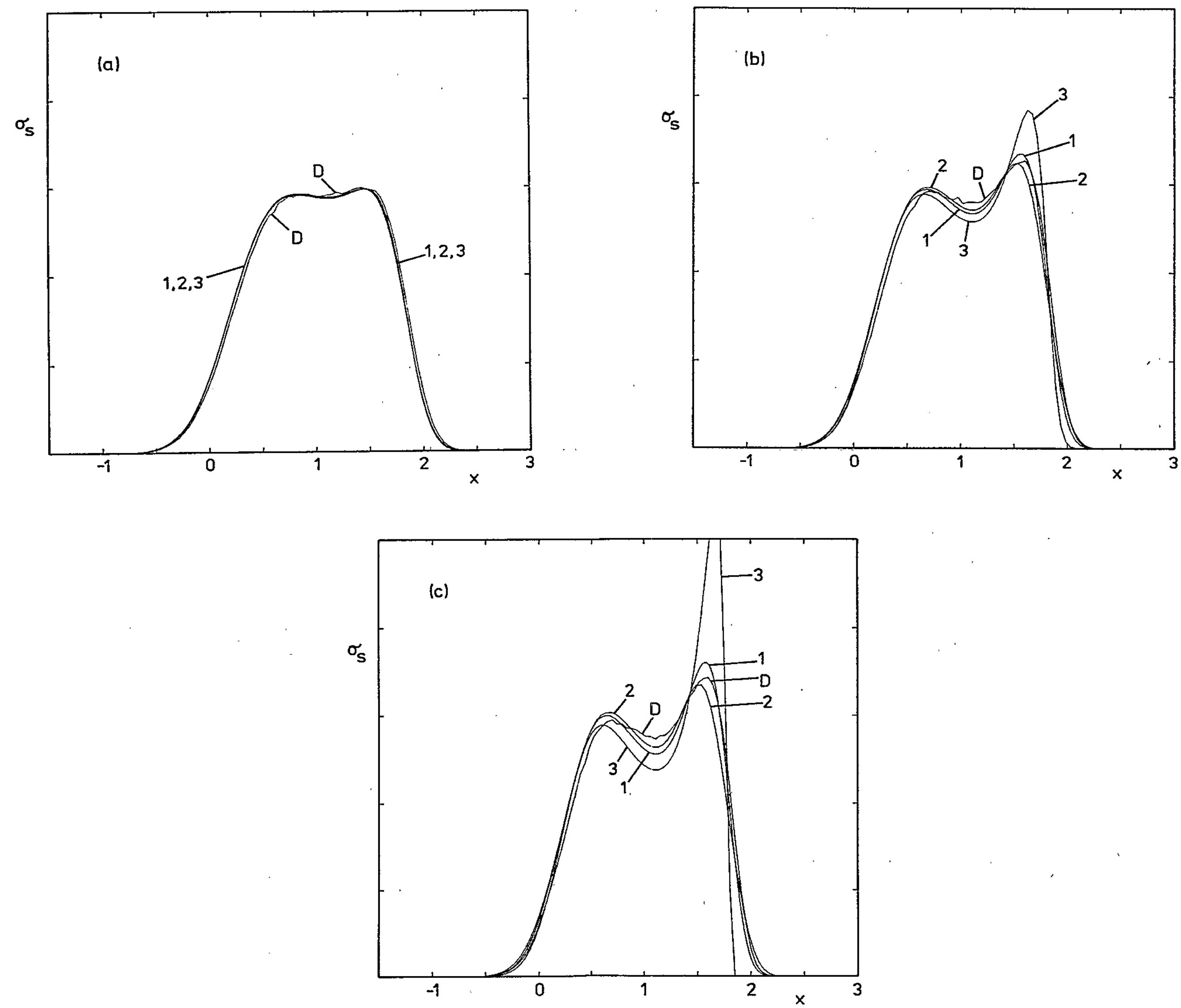

FIG. 6. Comparisons between a digital simulation (labeled D) of (4.4) and theories for which $P_{\rm expt}=0$ (Fox theory, curve labeled with a 1), $P_{\rm expt}=0.5$ (projection-operator method to order L_1^4 labeled 2), and $P_{\rm expt}=-1$ (best Fokker-Planck approximation, labeled 3), for different values of the noise correlation time τ: (a) $\tau=0.2$; (b) $\tau=0.7$; (c) $\tau=1.0$.

diffusion should occur beyond the points where the diffusion coefficient of the theoretical Fokker-Planck operator becomes negative. Third, we would like to point out that $P_{\rm expt}=0$ should be valid in both the limit $\tau\rightarrow 0$ and also $\tau\rightarrow\infty$.[24,25] For $\tau\rightarrow 0$, however, $P_{\rm expt}=-1$ should provide better results than $P_{\rm expt}=0$, since it seemingly includes terms of order τ^2 which are disregarded by the Fox approximation. As pointed out in Refs. 22 and 23, however, the terms of order L_1^n with $n>2$ can produce contributions of the same order of magnitude, and this is the reason why the Fox approximation can yield better agreement with experimental results than the seemingly more accurate theory of the best Fokker-Planck approximation. The exact evaluation of the effects of the terms of order L_1^4 with $n>2$ to order τ^2 lends to $P_{\rm expt}=0.5$, which gives predictions lying much closer both to those of the Fox theory and to the experimental results than in the case of $P_{\rm expt}=-1$.

As in the case of the analogue experimental results of Sec. V, we have attempted to investigate the variation of $P_{\rm expt}$ with τ by fitting the theory to the digital simulation distributions. For small values of $\tau<0.2$, the results are unreliable because, as already mentioned, the theoretical distributions are insensitive to $P_{\rm expt}$. The results obtained for $\tau\geq 0.1$ are shown by the circled points of Fig. 7. The crosses are taken from fitting the analogue data of Sec. V; they do not extend below $\tau=1$ because the data are relatively noisy and the values of $P_{\rm expt}$ obtained for smaller τ will be correspondingly unreliable. It is clear that the analogue and digital data are in reasonable agreement:

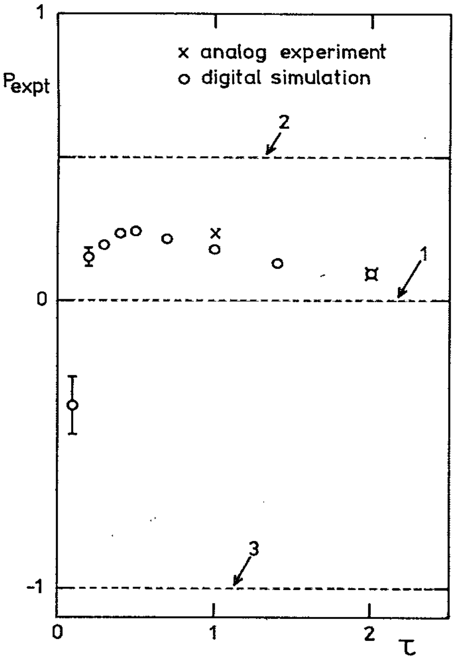

FIG. 7. Variation of $P_{\rm expt}$ with the noise correlation time τ as determined from the analogue experiment and by digital simulation. The dashed lines indicate the values of $P_{\rm expt}$ corresponding to the different theories: line 1, Fox theory; line 2, projection-operator method to order L_1^4; line 3, best Fokker-Planck approximation.

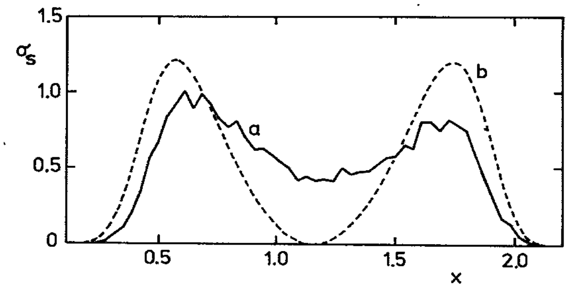

FIG. 8. Stationary probability distributions of x for (4.7), with noise on the input intensity. The other parameter values were $\theta=\sqrt{3}$, $I=I_c=8\sqrt{3}/9$, $\langle \xi^2 \rangle=0.06$. Curve a is a digital simulation of (4.7) with $\tau=10$; curve b is from (2.9) with $\tau\to\infty$.

$P_{\rm expt}$ is bounded between 0 and 0.5 for $\tau \geq 0.2$ and, for $\tau\to\infty$, $P_{\rm expt}$ appears to be tending towards zero. It would be interesting to study $P_{\rm expt}$ in the range of $\tau<0.1$, but difficult in view of the relative insensitivity of the distributions to the value of $P_{\rm expt}$ under these conditions [Fig. 6(a)]: a digital simulation with greatly improved statistics would be needed, or, alternatively, recourse might be made to the method of matrix continued fraction expansion.[13]

VII. A CASE OF MULTIPLICATIVE NOISE

In the preceding sections, we showed an example of a noise-induced transition originating from additive colored noise. If we start from the white-noise case ($\tau=0$) and increase τ, the steady-state distribution immediately splits into two peaks, and this bimodal structure becomes more and more pronounced as τ is increased (Fig. 5). From a straightforward analysis of Eq. (2.9) we realize that this kind of noise-induced transition caused by colored noise is not specific to the stochastic equation considered here, but has a general nature. In fact, it arises whenever the parameters are selected in such a way that there is a critical point such that $\phi(x_c)=\phi'(x_c)=\phi''(x_c)=0$. This follows from the fact that the distribution (2.9) then includes an exponential factor with a very flat maximum for $x=x_c$, and a prefactor that has a minimum and vanishes for $x=x_c$. Hence, if we increase τ enough while keeping $\langle \xi^2 \rangle$ fixed, the stationary distribution must certainly become bimodal, because it has this property in the limit $\tau\to\infty$.

Intuitively, it seems obvious that rather similar considerations will apply to any monotonically increasing S-shaped curve, with a point of inflexion and not just to the ones with a critical point as in Fig. 3. For a sufficient noise intensity of sufficiently colored noise, the distribution is bound to become bimodal; where, in addition, the deterministic system is at a critical point, any finite value of τ will induce bimodality even for arbitrarily small $\langle \xi^2 \rangle$.

It is clear that this result holds not only for additive but also for multiplicative noise, because Eq. (2.9) also covers the case $\psi(x)\neq 1$. An example of this type of system is provided by the case of input intensity noise, governed by Eq. (4.6). The stationary distribution for the variable x can be calculated analytically in the white-noise limit,[5] and numerically in the case of colored noise by solving the set of stochastic equations (4.6) and (1.4). In Fig. 8 we see the comparison between the steady-state distribution of the variable obtained for $\theta=\sqrt{3}$, $I=I_c$, $\langle \xi^2 \rangle=0.06$, and $\tau=10$ obtained by digital simulation, and the curve calculated from Eq. (2.9) ($\tau\to\infty$) for the

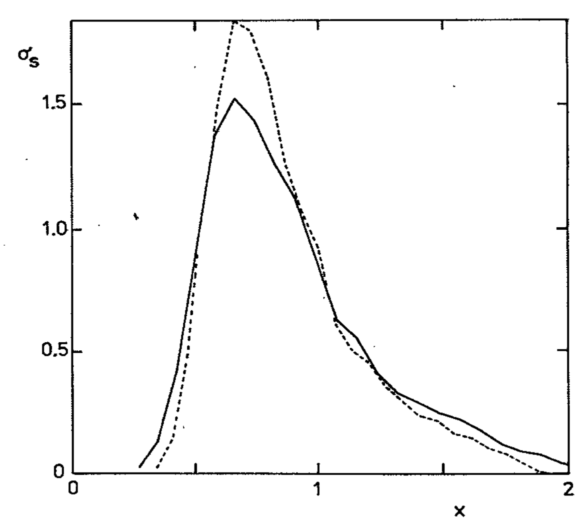

FIG. 9. Stationary probability distributions of x with noise on the input intensity; all parameters are as in Fig. 8, except that $\tau=1$. The distribution for the transmitted intensity (solid line) is compared with that for the material variable x (dashed line).

same values of θ, I, and $\langle \xi^2 \rangle$. The two curves exhibit a qualitative similarity, but the case $\tau=10$ still seems removed from the asymptotic situation $\tau \to \infty$. Finally, Fig. 9 shows the stationary probability distribution for the transmitted intensity, I_T, obtained numerically using Eq. (4.2) and replacing I by $I_0+\xi(t)$.

VIII. CONCLUSION

In this paper we have discussed and compared the available theories of colored noise in relation to nonlinear dynamical systems in general, and to the phenomenon of noise-induced optical bimodality in particular.[33] Optical bimodality can be induced by either additive or multiplicative noise, but we have focused our attention principally on the former case because of the interesting effect of bimodality induced by noise color alone, but in Sec. VII we showed that what we had found for additive noise is also true in the multiplicative case.

The most important conclusion to emerge from the work is that, of the applicable available theories,[1,2,9,10] that due to Fox[9] is the most successful for the case of the model that we have analyzed.[34] For the limits $\tau \to 0$ and $\tau \to \infty$, there is a characteristic quantity P_{expt} which becomes zero, and which is zero for all τ in the Fox theory. The behavior of P_{expt} between these limits, for finite τ, has been explored through the analogue and digital experiments; it has been found that P_{expt} apparently tends to zero as $\tau \to \infty$ and remains bounded within $0 < P_{\text{expt}} < 0.5$ for $\tau \geq 0.2$. The range of $\tau < 0.1$ cannot be investigated at the moment because of problems of precision and statistics in the digital simulation, but remains as an interesting problem to be tackled in the future.

The seemingly disconcerting result, that the Fox theory, which can be thought of as an approximation to the best Fokker-Planck approximation, turns out to be *better* than the best Fokker-Planck approximation itself, can be accounted for as follows. The best Fokker-Planck approximation consists of neglecting all the terms of order higher than L_1^2. The projection operator approach,[10,22,23,25] on the other hand, shows that we can get rid of these higher-order contributions if the physical conditions are right (i.e., very small diffusion coefficients[35]) for the "local linearization" $\Pi^{(1)\prime}(x)=0$ to take place; the corrections of order L_1^4 are, in fact, proportional[25] to $\Pi^{(1)\prime}(x)$. Furthermore, it is the case that, if the local linearization $\Pi^{(1)\prime}(x)=0$ holds, then the diffusion term proposed by the authors of the best Fokker-Planck approximation[1,2] can readily be resummated up to infinity (over all powers of τ); the resultant expression coincides with the diffusion term proposed by Fox. This point has been discussed in detail by Faetti and Grigolini.[25]

The regime of validity of the resultant Fokker-Planck equation extends far beyond that anticipated by Fox,[9] according to whom it should be confined to the short-τ region. In fact, the validity of the equation extends all the way from small τ to establish a natural contact with the sound physically motivated arguments that hold in the limit $\tau \to \infty$ [see Eq. (29)]. The unexpectedly extended range of validity is attributable in large measure to the role played by local linearization.[25] These conclusions are corroborated by the experimental results reported in this paper and, especially, by those of Fig. 7.

ACKNOWLEDGMENTS

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