Influence of the environment on anomalous diffusion

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(Received 18 May 1994)

We study the effect of weak environmental fluctuations on a deterministic map dynamics, which, in the unperturbed case, is characterized by anomalous diffusion. We show, with the help of numerical calculations, that there is a crossover time $1/\epsilon$, at which the waiting time distributions change from an inverse power-law distribution into an exponential behavior. We prove theoretically that the diffusion coefficient of the long-time process is proportional to $1/\epsilon^{\alpha}$, with α positive or negative, according to whether we consider the superdiffusive or the subdiffusive case. With very weak environmental fluctuations the diffusion coefficient of the former case becomes anomalously large and that of the latter case anomalously small. The theoretical predictions are confirmed by the numerical results.

PACS number(s): 05.40.+j, 05.60.+w

I. INTRODUCTION

Diffusion processes play a quite important role in many fields of chemistry, physics, and biology [1]. Processes of normal diffusion are those characterized by a variance increasing linearly in time [1]. In the last decade the attention of investigators has been attracted by the study of processes with a variance that is not a linear function of time but is either faster or slower than normal diffusion [2–18]. Anomalous diffusion is defined by observing the second moment of, say, the spatial coordinate of the diffusing particle, namely, looking at $\langle x^2(t)\rangle$, which generally has a time dependence of the form

$$\langle x^2(t)\rangle = Kt^{2H} \tag{1}$$

at long times. Anomalous diffusion corresponds to $H \neq 1/2$. Following the literature, we will term the case $H > 1/2$ superdiffusion (diffusion faster than the normal) and the case $H < 1/2$ subdiffusion (diffusion slower than the normal).

Typical models used to study anomalous diffusions are, for instance, random walks, or their dynamical representation in the form of a mapping. From a theoretical point of view [2–6], the origin of anomalous diffusion can then be traced back to the analytic form of the distribution of waiting times, $\psi(t)$, in one of the possible "states" of these mappings. These are the sojourn times in a state of the position or the velocity of the diffusing particle. Normally, in both cases (i.e., when looking at either the position or the velocity) the amplitude of the random jumps between each sojourn period is constant and finite (typically, the amplitude of the jumps is taken to be ± 1). Under these assumptions, anomalous diffusion has been shown [2–18] to be generated by a waiting time distribution with the inverse power-law structure

$$\psi(t) \sim \frac{1}{t^{\mu}} \quad \text{for large } t. \tag{2}$$

This is the long-time limit of a process that is triggered by microscopic dynamics, whose details become irrelevant asymptotically. It is arguable that if (2) is realized, then all physical observables will exhibit scaling. Broadly speaking, once this asymptotic regime is achieved, the system is considered to be in its stationary state.

However, the distribution function $\psi(t)$ should be integrable (the above expression for the function diverges for short times). A more realistic form for the waiting time distribution, for instance, is

$$\psi(t) = \frac{A}{(B+t)^{\mu}}, \tag{3}$$

with $\mu > 1$, where the constant A is related to the time constant B through a normalization condition to yield

$$A = (\mu - 1)B^{\mu-1}. \tag{4}$$

This means that there are essentially two significant time ranges. The former, with $t \ll B$, is the microscopic time region where the diffusing system is still far away from the stationary regime, which is only reached in the latter time region, $t \gg B$.

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A yet more realistic picture of diffusive processes, on the other hand, should take into account a third time range. It should be born in mind that (2) is an idealization, aiming at pointing out the essence of an anomalous diffusion process, and the consequent deviations from standard statistical mechanics. However, over a more extended time scale, standard statistical mechanics must be recovered. One could expect to find the following waiting time distribution instead of (2):

$$\psi_\epsilon(t) \sim \frac{1}{t^\mu} e^{-\epsilon t} \quad \text{for very large } t, \tag{5}$$

where the subscript ϵ on the distributions denotes the environmental perturbation. Setting ϵ equal to zero is equivalent to considering the hierarchical system unaffected by any sort of external (or "environmental") noise. Obviously, to extend the treatment to the short time dynamics we can combine (3) and (5) to obtain

$$\psi_\epsilon(t) = \frac{A}{(B+t)^\mu} e^{-\epsilon t}; \tag{6}$$

the normalization constant A is then

$$A = \frac{1}{e^{B\epsilon}\epsilon^{\mu-1}\Gamma(1-\mu, B\epsilon)}, \tag{7}$$

where $\Gamma(\alpha, x)$ is the incomplete gamma function. An illustrative example can be found in the field of weak chaos. It is well known that the regions on the border between the chaotic sea and the deterministic islands have fractal properties [19] and that this results in residence times characterized by the distribution (2). However, this property seems to be true only in the case of low-dimensional dynamical systems. Although the behavior of systems with a large but finite number of dimensions in a state of weak chaos is still poorly understood, according to Konishi [20], in such systems there exists a crossover time after which the system escapes from the local hierarchy, which is responsible for the inverse power-law behavior, and normal diffusive behavior is retrieved. More generally, in all situations where a deterministic dynamical approach leads to (2), weak fluctuations produced by the coupling of the system to the environment are expected to lead to a structure similar to (6). This means that (1) at microscopic times $t \lll B$ the system is expected to exhibit nonstationary behavior, strongly dependent on the details of the model examined and on the initial conditions; (2) in the time region $B \ll t \ll 1/\epsilon$ the system shows the properties of anomalous diffusion; (3) finally, in the region $1/\epsilon \ll t$ the process of standard diffusion, and hence ordinary statistical mechanics, is recovered.

The main purpose of the present paper is to show that in spite of its ordinary character, diffusion processes taking place in the third time region depend critically on the cutoff ϵ; the diffusion coefficient turns out to be proportional to ϵ^α with α positive or negative according to whether we consider the subdiffusive or the superdiffusive case. This means that in the case $H > 1/2$, the longer the third time scale is, the larger the standard diffusion coefficient becomes. For ϵ going to zero this standard diffusion coefficient becomes critically large. In the case $H < 1/2$, on the other hand, the diffusion process taking place with a speed slower than the standard one is replaced by a standard process with a diffusion coefficient whose intensity vanishes for ϵ going to zero. In this paper we have used numerical simulation to illustrate qualitatively the transition from anomalous (subdiffusion and superdiffusion) to normal diffusion. We have not yet approached a quantitative comparison with the theoretical predictions on the dependence of the intensity of this ordinary diffusion on the strength of the noise. This would require a theory for the derivation of (6) from the intensity of the perturbing noise, which is still not available in a form suitable for the specific purposes of this paper (but see [24]). Nevertheless, our theoretical predictions are exact, once given the waiting-time distribution (6).

The paper is organized as follows. Section II is devoted to illustrating the dynamical model used to produce anomalous diffusion. This model consists of the maps used in [7,8,16] to derive subdiffusion and superdiffusion. The maps are perturbed by a weak Gaussian stochastic force, and it is numerically shown that the resulting waiting-time distribution in the long-time limit has the form (5). Sections III and IV are devoted to studying the long-time limit in the superdiffusive and subdiffusive cases, respectively. Exact expressions for the corresponding diffusion coefficients are derived in the limiting case of extremely weak ϵ's, starting from the propagator of the dynamical system. In Sec. V we rederive the asymptotic behavior in the superdiffusive case using a dynamical calculation [8] based on the stationary autocorrelation function of the velocity $\dot{x}$. Section VI is devoted to discussing the main results of this paper and to formulating some conclusions.

II. THE DYNAMICAL MODEL UNDER STUDY

The dynamical models considered here were studied at some length by Geisel and co-workers [7,8] and by Zumofen and Klafter in [16]. The two maps have the general form

$$x_{n+1} = g(x_n), \tag{8}$$

where $g(x+n) = g(x) + n$, $g(x) = -g(-x)$, and x is assumed to be in the interval $[0, 1/2]$. For the superdiffusive case we use

$$g(x) = (1+\eta)x + ax^z - 1, \tag{9}$$

and for the subdiffusive case we use

$$g(x) = (1+\eta)x + ax^z. \tag{10}$$

The constant a has the value $a = 2^z(1-\eta/2)$. The small constant η is necessary to prevent "sticking" of the diffusive particle at $x = \pm n$. In the numerical calculation in the "unperturbed" case we used $\eta = 10^{-9}$. This would lead to a recovery of normal diffusion in the long-time limit [8]. However, when the environmental perturbation is switched on, the stickiness is destroyed by the environmental fluctuation itself; hence, our numerical calculations in the presence of the environmental fluctuation were done with $\eta = 0$.

In the continuous-time limit both maps are compatible with the dynamical picture

$$\dot{x} = \xi. \quad (11)$$

For both cases we assume that the "output" of the map (8) (i.e., x_n) is equivalent to x in (11). Basically we replace ξ in (11) by $x_{n+1} - x_n$ from (8). For the superdiffusive case, due to the phenomenology exhibited by the map, i.e., a dynamics characterized by a sequence of periods of laminar motion interrupted by short chaotic phases, in practice the velocity ξ shows a distinctly intermittent character, alternating in value between +1 and −1. The residence time in one of these two states is, to a good degree, given by the inverse power-law distribution (3), with $\mu = z/(z-1)$. Similarly in the subdiffusive case, using the appropriate map, the laminar phases correspond to the diffusive particle remaining located at a fixed position. Again, the residence times in one of these states is given by (3), still with $\mu = z/(z-1)$. The upper dots of Fig. 1 show that the numerical result is very close to the theoretical prediction (3).

We perturb the dynamics of the maps by replacing (8) with

$$x_{n+1} = g(x_n) + f_n, \quad (12)$$

where f_n is a standard stationary Gaussian process with zero average and autocorrelation function

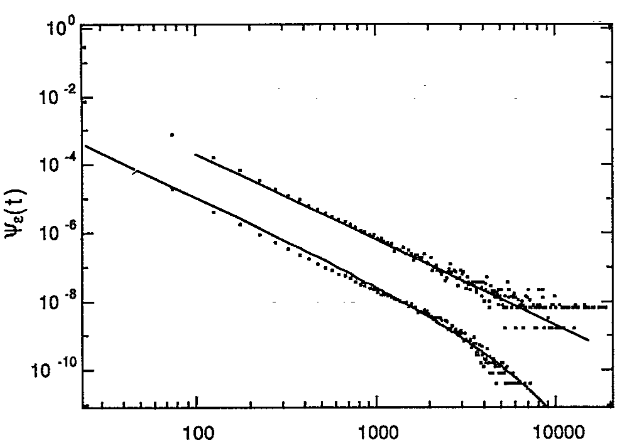

FIG. 1. Waiting-time distributions for $\mu = 2.5$. The upper dots and the upper solid line refer to the case with no environmental fluctuations. The dots refer to the map (9) and the solid line corresponds to the theoretical prediction (3). The lower dots and the lower solid line refer to the case of dynamics in the presence of environmental fluctuations. For the sake of a better visualization they are plotted dividing by 40 the corresponding values. The dots refer to the map (9) in the presence of noise, as in (12), with $T = 3 \times 10^{-13}$. The solid line is (6) and (7), with $\epsilon = 3.2 \times 10^{-4}$, as determined through a best fitting procedure. Note that we used dots and lines instead of histograms for the sake of clarity.

$$\langle f_n f_{n+m} \rangle = 2T\delta_{n,m+n}. \quad (13)$$

The noise is applied to the reduced version of the map, as defined, for instance, in [9]. To prevent the noise from making the trajectory escape from the reduced map, we set reflecting boundary conditions at the borders of the reduced map ($x = 0$ and $x = 1$). For small T's, the resulting equilibrium distribution for the waiting times (see Fig. 1, lower dots and curve) turns out to be (6). In this figure, the lower solid line is a best fit to the data using (6) and (7), and it is reasonably close to the simulated distribution. Incidentally, we note that the waiting-time distribution in the presence of a small but finite η, in the absence of noise, was derived in [8], and it is qualitatively similar to our Eq. (6), at least in the limits of small and large times. We conjecture that the results we obtain on anomalously large or small diffusion coefficients in the limit of a small ϵ apply also to the case treated in [8].

We see that environmental fluctuations play a role similar to that of Arnold diffusion [20]. They set an upper time limit on the validity of the inverse power-law nature of the waiting-time distribution, after which the exponential dominates. Consequently, we can argue that the lack of a time scale is perceived by the system only at times $t \ll 1/\epsilon$. At times much longer than $1/\epsilon$ the system cannot depart from the predictions of ordinary statistical mechanics. The transition from anomalous diffusion to the regime of ordinary diffusion is illustrated in Fig. 2, using map (9) (transition from superdiffusive to standard diffusion), and in Fig. 3 for map (10) (transition from subdiffusive to standard diffusion).

We are now in a position to predict, with intuitive arguments, that the "ordinary" diffusion recovered in the long-time limit as the effect of environmental fluctuations is characterized by unusually large or small diffusion coefficients. Let us integrate the equation of motion (11). Under the assumption that x is a stationary process, we

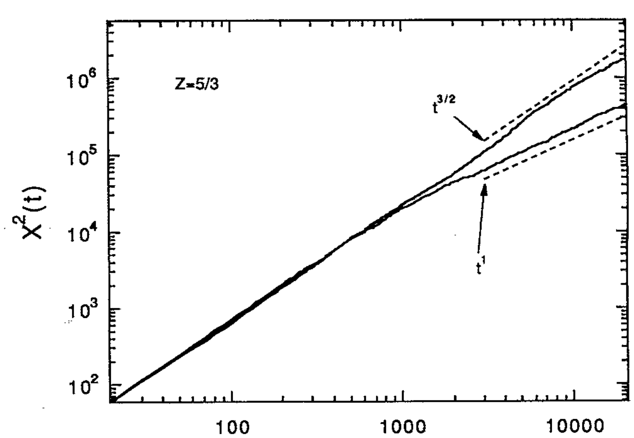

FIG. 2. Second moment $\langle x^2(t) \rangle$ vs time, for $\mu = 2.5$ in the superdiffusive case. The solid lines are the result of numerical simulations with (lower curve) and without (upper curve) external perturbation. The dashed lines show the expected asymptotic behaviors.

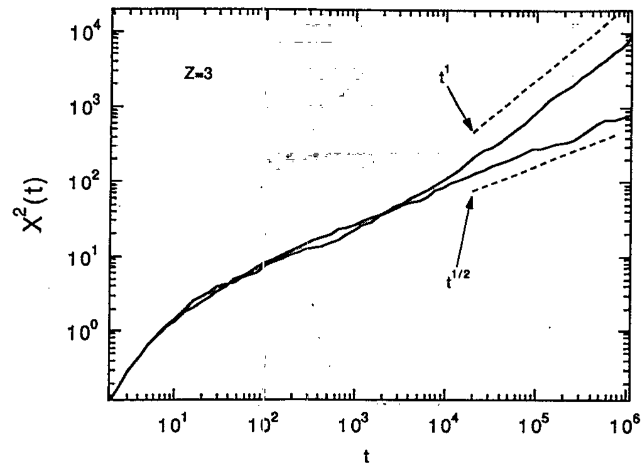

FIG. 3. Second moment $\langle x^2(t)\rangle$ vs time, for $\mu = 1.5$ in the subdiffusive case. The solid lines are the result of numerical simulations with (upper curve) and without (lower curve) external perturbation. The dashed lines show the expected asymptotic behaviors.

show [23] that

$$\frac{d}{dt}\langle x^2(t)\rangle = D(t), \tag{14}$$

where

$$D(t) \equiv 2\int_0^t \langle\xi(0)\xi(t')\rangle dt' = 2\langle\xi^2\rangle \int_0^t \Phi_\xi(t')dt', \tag{15}$$

and $\Phi_\xi(t)$ is the normalized autocorrelation function of the velocity $\dot{x}$,

$$\Phi_\xi(t) = \frac{\langle\xi(0)\xi(t)\rangle}{\langle\xi^2\rangle}. \tag{16}$$

Let us consider the case of superdiffusion. In the range of times $t < 1/\epsilon$ the second moment increases with a power-law index larger than one. Looking at (14), this is perceived as a transient process toward an extremely large value of the diffusion coefficient. The transition from anomalous to ordinary diffusion can be interpreted as $D(t)$ reaching a stationary value, which, after the anomalous increase, can only be extremely large. The same argument can be applied to the case of subdiffusion. In this case, $D(t)$ will move toward vanishing values and when ordinary diffusion is recovered this must be done through a diffusion coefficient that can only be extremely small.

We adopted, without loss of generality, the simplified form for $\psi_\epsilon(t)$,

$$\psi_\epsilon(t) = \frac{C_\epsilon}{(1+t)^\mu}e^{-\epsilon t} \tag{17}$$

in the theoretical calculation, with the normalization constant

$$C_\epsilon = \frac{1}{e^\epsilon \epsilon^{\mu-1}\Gamma(1-\mu,\epsilon)}. \tag{18}$$

This form is obtained from (6) after a proper rescaling of time.

III. RESPONSE TO WEAK FLUCTUATIONS OF SYSTEMS IN A STATE OF ACCELERATED DIFFUSION

The case of superdiffusion can be accurately described following three different approaches. The first two, the velocity model (VM) and the jump model (JM), have been developed by Zumofen and Klafter [16]. The third, based on a master equation (ME) approach, was developed by Trefan *et al.* [21] and proved to be very close, as far as its predictions are concerned, to the JM approach (with nonstationary initial conditions). The reader can find the illustration of these different theories in the quoted references. Here we shall limit ourselves to demonstrating that all three theories lead to an enhanced diffusion coefficient with the same functional dependence on ϵ, lending support to our conjecture that this is an exact asymptotic property.

A. Response to environmental fluctuations following the VM

The result in this case follows, after some manipulations, from a formal result obtained by Zumofen and Klafter [16], concerning the Laplace transform of the second moment $\langle x^2(t)\rangle$, denoted by $\langle\widehat{x^2}(s)\rangle$. This is

$$\langle\widehat{x^2}(s)\rangle = \frac{2[1-\hat{\psi}_\epsilon(s) + s\frac{d}{ds}\hat{\psi}_\epsilon(s)]}{s^3[1-\hat{\psi}_\epsilon(s)]}, \tag{19}$$

where $\hat{\psi}_\epsilon(s)$ is the Laplace transform of the waiting-time distribution (17). The validity of the expression (19) is independent of the explicit form of the waiting-time distribution. In Ref. [16] it has been applied to the unperturbed waiting-time distribution $\psi_0(t)$, but it can be applied to the case of the perturbed waiting-time distribution (17) with no restriction. Since the exponential factor makes all the moments of this distribution finite, we obtain

$$\hat{\psi}_\epsilon(s) = 1 - \bar{t}(\epsilon)s + \frac{\overline{t^2}(\epsilon)}{2}s^2 + \cdots, \tag{20}$$

where $\overline{t^n}(\epsilon)$ denotes the nth order moment of the perturbed waiting-time distribution. In the limit of long times, i.e., for s going to zero, we can limit ourselves to considering the first and the second moment of the perturbed waiting-time distribution, obtaining

$$\langle\widehat{x^2}(s)\rangle = \frac{\overline{t^2}(\epsilon)}{\bar{t}(\epsilon)s^2}. \tag{21}$$

This means that, as expected, in the limit of long times the diffusion process is normal,

$$\langle x^2(t)\rangle \sim Dt \quad \text{for large } t, \tag{22}$$

with the diffusion coefficient given by

$$D = \frac{\overline{t^2}(\epsilon)}{\bar{t}(\epsilon)}. \tag{23}$$

The determination of the first and second moment of the waiting-time distribution is derived from its Laplace transform as follows:

$$\bar{t}(\epsilon) = -\left.\frac{d\hat{\psi}_\epsilon(s)}{ds}\right|_{s=0} = -C_\epsilon \left.\frac{d\hat{\psi}_0(s+\epsilon)}{ds}\right|_{s=0} = -C_\epsilon \frac{d\hat{\psi}_0(\epsilon)}{d\epsilon}, \tag{24}$$

$$\overline{t^2}(\epsilon) = \left.\frac{d^2\hat{\psi}_\epsilon(s)}{ds^2}\right|_{s=0} = C_\epsilon \left.\frac{d^2\hat{\psi}_0(s+\epsilon)}{ds^2}\right|_{s=0} = C_\epsilon \frac{d^2\hat{\psi}_0(\epsilon)}{d\epsilon^2}. \tag{25}$$

Note that we consider the limiting case of environmental fluctuations of almost vanishing intensity, or the case of extremely small ϵ's. We can use the following series expansions of $\hat{\psi}(s)$

$$\hat{\psi}_0(s) \approx \begin{cases} 1 - cs^{\mu-1}, & 1 < \mu < 2 \\ 1 - \bar{t}s - cs^{\mu-1}, & 2 < \mu < 3 \\ 1 - \bar{t}s + \frac{\overline{t^2}}{2}s^2, & 3 < \mu. \end{cases} \tag{26}$$

We obtain, using (26) in (24) and (25),

$$\bar{t}(\epsilon) \sim \begin{cases} \epsilon^{\mu-2}, & 1 < \mu < 2 \\ \text{const}, & 2 < \mu < 3 \\ \text{const}, & 3 < \mu, \end{cases} \tag{27}$$

$$\overline{t^2}(\epsilon) \sim \begin{cases} \epsilon^{\mu-3}, & 1 < \mu < 2 \\ \epsilon^{\mu-3}, & 2 < \mu < 3 \\ \text{const}, & 3 < \mu, \end{cases}$$

resulting in the following dependence of the diffusion coefficients on the strength of the environmental fluctuation:

$$D \sim \frac{1}{\epsilon^{\alpha(\mu)}}, \tag{28}$$

with

$$\alpha = \begin{cases} 1, & 1 < \mu < 2 \\ 3 - \mu, & 2 < \mu < 3 \\ 0, & 3 < \mu. \end{cases} \tag{29}$$

This situation is illustrated in Fig. 4. Note that the region $3 < \mu$ refers to a case which would be normal even without perturbation: normal diffusion is unaffected by environmental fluctuations of weak intensity. In the region $2 < \mu < 3$ without perturbation we would have an asymptotic Lévy process [16,21]; clearly, there is an increasing sensitivity to the influence of weak fluctuations as μ moves from $\mu = 3$ to $\mu = 2$. The region of ballistic diffusion, with $1 < \mu < 2$, is that where weak environmental fluctuations have the greatest influence on the anomalous diffusion process.

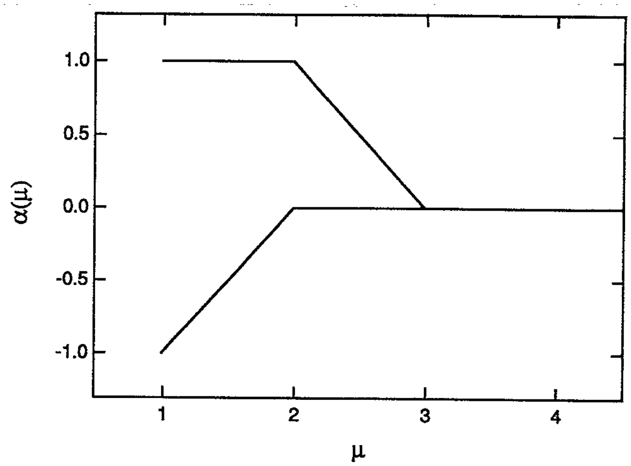

FIG. 4. Diagram of $\alpha(\mu)$ vs μ for the superdiffusive case (upper line) and for the subdiffusive case (lower line). The two cases coincide for $\mu > 3$.

B. Response to environmental fluctuations following the JM

In the nonstationary case [21], the JM model, although seemingly different from the VM [16], leads to the same asymptotic properties for the probability distribution $P(x,t)$ as the VM. There are minor differences in the time evolution of the second moment [21]. In the specific case of the region of Lévy processes, $2 < \mu < 3$, it was shown [21] that this approach coincides with the ME approach. It must be pointed out that although there are some minor differences concerning multiplicative constants, the power-law behavior of this model is exactly the same as that of the VM. Therefore, we expect this model to lead to the same inverse power-law dependence of the asymptotic diffusion coefficients on the strength of the environmental fluctuations.

Also, in this case the calculation is done using a relation provided by Zumofen and Klafter [16] [their Eq. (25)]. Adapting this relation to the problem under discussion here, we have

$$\langle \hat{x^2}(s) \rangle = -\frac{\hat{\Psi}_\epsilon(s)}{[1 - \hat{\psi}_\epsilon(k,s)]^2} \left.\frac{\partial^2 \hat{\psi}_\epsilon(k,s)}{\partial k^2}\right|_{k=0} = -\frac{\frac{d^2}{ds^2}\hat{\psi}_\epsilon(s)}{s[\hat{\psi}_\epsilon(s) - 1]}. \tag{30}$$

The function $\Psi_\epsilon(t)$ is defined by

$$\Psi_\epsilon(t) = \int_t^{+\infty} dt' \int_{-\infty}^{+\infty} \psi_\epsilon(x,t')dx, \tag{31}$$

where $\psi_\epsilon(x,t)$ is the probability density to move a distance x in time t, and reads

$$\psi_\epsilon(x,t) = \frac{1}{2}\delta(|x| - t)\psi_\epsilon(t). \tag{32}$$

In the limiting case of very small s, we get from (30)

$$\langle \hat{x^2}(s)\rangle = \frac{\overline{t^2}(\epsilon)}{\bar{t}(\epsilon)s^2}, \tag{33}$$

which coincides with (21). The prediction of this model is clearly identical, as far as the dependence of D on ϵ is concerned, with those of the VM.

C. Response to environmental fluctuations following the ME

This method was developed [21] for the specific purpose of studying the region of the Lévy processes, $2 < \mu < 3$. It is based on the master equation

$$\frac{\partial}{\partial t}P(x,t) = \int_0^t dt' \int_{-\infty}^{+\infty} K_\epsilon(x-x',t-t')P(x',t')dx'. \tag{34}$$

The kernel $K_\epsilon(x,t)$ has the following detailed balance structure:

$$K_\epsilon(x,t) = \Pi_\epsilon(x,t) - \delta(x)\int_{-\infty}^{+\infty} \Pi_\epsilon(x',t)dx'. \tag{35}$$

Using physical arguments inspired by the dynamics of the map for the accelerated diffusion illustrated in Section II, the authors of [21] set

$$\Pi_\epsilon(x,t) = \frac{1}{\bar{t}(\epsilon)}\psi_\epsilon(x,t), \tag{36}$$

where $\psi_\epsilon(x,t)$ is defined in (32), and $\bar{t}(\epsilon)$ is the first moment of the waiting time distribution. The Fourier-Laplace transform of the probability distribution $P(x,t)$ in (34) is

$$\hat{P}(k,s) = \frac{1}{s - \hat{K}_\epsilon(k,s)}, \tag{37}$$

where $\hat{K}_\epsilon(k,s)$ has the following explicit form

$$\hat{K}_\epsilon(k,s) = \frac{1}{\bar{t}(\epsilon)}[\hat{\psi}_\epsilon(k,s) - \hat{\psi}_\epsilon(s)], \tag{38}$$

where $\hat{\psi}_\epsilon(k,s)$ and $\hat{\psi}_\epsilon(s)$ follow from the JM approach. We use the property

$$\langle \hat{x^2}(s)\rangle = -\left.\frac{\partial^2 \hat{P}(k,s)}{\partial k^2}\right|_{k=0} = \frac{1}{\bar{t}(\epsilon)s^2}\frac{d^2\hat{\psi}_\epsilon(s)}{ds^2}, \tag{39}$$

to derive the prediction in the limit of small s

$$\langle \hat{x^2}(s)\rangle = \frac{\overline{t^2}(\epsilon)}{\bar{t}(\epsilon)s^2}, \tag{40}$$

which means standard diffusion with the diffusion coefficient D given by

$$D = \frac{\overline{t^2}(\epsilon)}{\bar{t}(\epsilon)}. \tag{41}$$

Note that the ME method was tailored for the specific purpose of studying Lévy processes [21]. Thus we must limit ourselves to exploring only the range $2 < \mu < 3$. In this case we obtain as before

$$D \sim \frac{1}{\epsilon^{3-\mu}}. \tag{42}$$

In conclusion, there is agreement on the inverse power-law dependence of the diffusion coefficient on the intensity of environmental fluctuations obtained following the three different theoretical approaches.

IV. RESPONSE TO WEAK FLUCTUATIONS OF SYSTEMS IN A STATE OF DISPERSIVE DIFFUSION

Years ago [22] Shlesinger evaluated the power H of (1) as a function of the power μ of the distribution of waiting times in one site. He found [see (1) for the definition of H]

$$H \sim \begin{cases} \frac{\mu-1}{2}, & 1 < \mu < 2 \\ \frac{1}{2}, & 2 < \mu. \end{cases} \tag{43}$$

These results have been confirmed by the theory and the numerical calculations of [7,16]. We see that the inverse power-law nature of the waiting-time distribution (3) only shows up in the region $1 < \mu < 2$, where it makes the diffusion process slower than standard diffusion. The map (10) is a suitable way to realize dynamics close to the continuous random walk of Shlesinger. In this specific case, borrowing from Zumofen and Klafter [16] the expression

$$\begin{aligned}\langle \hat{x^2}(s)\rangle &= \frac{1}{s[\hat{\psi}_\epsilon(s)-1]}\left.\frac{\partial^2\hat{\psi}_\epsilon(k,s)}{\partial k^2}\right|_{k=0} \\ &= -\frac{\hat{\psi}_\epsilon(s)}{s[\hat{\psi}_\epsilon(s)-1]},\end{aligned} \tag{44}$$

and making the usual calculation at small s, we obtain

$$\langle \hat{x^2}(s)\rangle = \frac{1}{\bar{t}(\epsilon)s^2}, \tag{45}$$

obtaining for the diffusion coefficient

$$D = \frac{1}{\bar{t}(\epsilon)}. \tag{46}$$

We finally have

$$D \sim \begin{cases} \epsilon^{2-\mu}, & 1 < \mu < 2 \\ \text{const}, & 2 < \mu, \end{cases} \tag{47}$$

namely, we can set it under the form of (28) with

$$\alpha = \begin{cases} \mu - 2, & 1 < \mu < 2 \\ 0, & 2 < \mu. \end{cases} \tag{48}$$

Again, the diffusion in the region of normal behavior is not affected by fluctuations of weak intensity. The diffusion in the region of anomalous (subdiffusive) behavior is increasingly affected by noise of weak intensity as we move from $\mu = 2$ to $\mu = 1$ (see Fig. 4). It is remarkable that the power-law dependence of D on the noise strength parallels the Shlesinger scheme (43).

V. CORRELATION FUNCTION APPROACH

In Sec. II we used (14) to show with intuitive arguments the effects of environmental fluctuations on the diffusional properties of a system which, in the absence of perturbation, exhibits anomalous diffusion. What if we directly used (14) for our theoretical predictions? This is possible for the superdiffusive case. In this case we need an explicit expression for $\Phi_\xi(t) = \langle \xi(0)\xi(t)\rangle$. A suitable expression was derived in [8], using an approach which is dynamically equivalent to the VM model. The main difference is that the origin for the time is chosen in the stationary regime. Here, by "stationary" it is meant that

$$\langle \xi(t')\xi(t'')\rangle = \langle \xi(0)\xi(t'-t'')\rangle, \tag{49}$$

a necessary condition for (14) to be applicable.

Using the hypothesis of independent laminar phases, and averaging over the different realizations, it can be shown that $\Phi_\xi(t)$ becomes identical to the probability that the times 0 and t belong to the same laminar phase. We then have [8]

$$\Phi_\xi(t) = \frac{1}{\bar{t}} \int_t^\infty (T-t)\psi(T)dT. \tag{50}$$

Recalling that $\langle \hat{x^2}(s)\rangle = 2\hat{\Phi}_\xi(s)/s^2$, we have that [8]

$$\langle \hat{x^2}(s)\rangle = \frac{2}{s^3}\left(1 + \frac{\hat{\psi}_\epsilon(s) - 1}{\bar{t}(\epsilon)s}\right). \tag{51}$$

This is distinctly different from (19), which is not at all surprising since, in the calculations leading to (19), it is assumed that at the initial time all particles are placed at the beginning of their respective laminar region, leading to a violation of the stationary conditions, necessary to derive (14). Incidentally, in the superdiffusive case, the inverse power-law tail of the correlation function leads to a strong correlation even at long times, with the consequence that the stationarity condition is reached very slowly. However, the asymptotic behavior of (51) (i.e., taking the lowest terms in s) coincides with the asymptotic behavior of (19), because in this limit the stationary condition is eventually realized. As a consequence, also in this case we recover the expression for D obtained earlier, $D = \overline{t^2}(\epsilon)/\bar{t}(\epsilon)$.

VI. CONCLUSIONS

We have examined a physical model, which seems to display a general property of nature. After a maximum time, the inverse power-law character of a waiting-time distribution is lost. This is caused by the presence of weak environmental fluctuations. With the help of numerical calculations, we have shown that the effect of external fluctuations is that of changing at long times the inverse power-law structure of the waiting-time distribution into an exponential function. The size of the cutoff of this exponential function, loosely referred to as the intensity of the environmental fluctuations, defines a crossover time before which the process of diffusion is anomalous and after which the diffusion becomes normal.

We have also seen that, although regular, the asymptotic diffusion process exhibits a dependence on the "fluctuation" strength, which becomes increasingly sensitive as we move away from normal diffusion; it becomes maximum in the ballistic regime, for the superdiffusive case, and in the localization state for the subdiffusive case. In our opinion, this property might have remarkable consequences in that it might allow the experimentalists to probe the underlying anomalous character of a diffusion process simply by detecting the direct or inverse power-law dependence of the diffusion coefficient on the strength of weak environmental fluctuations.

ACKNOWLEDGMENTS

We thank EC for financial support. B.J.W. thanks the Office of Naval Research for partial support of this research.

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