Fast and precise algorithm for computer simulation of stochastic differential equations

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We present and discuss an algorithm for integrating a set of stochastic differential equations driven by colored noise. The algorithm, being fully implicit for the stochastic differential equation governing the noise, is stable upon changing τ (the noise correlation time) to any desired value. In particular, the limit of vanishingly small τ can be safely taken, and the algorithm yields the corresponding white-noise quantities in a natural way.

I. INTRODUCTION

Recent years have witnessed an increasing interest in stochastic processes, and the idea of nondeterministic chaos has found interesting applications in many fields of science.[1-3] An important aspect of these studies is the actual solution of the nonlinear stochastic differential equations appearing in the theoretical models: a tool which has proved very valuable is the straight simulation (either digital[4-8] or analog[9]) of such stochastic differential equations.

In the past, some different algorithms have been proposed (see Ref. 4 for a review): it was also argued (Appendix of Ref. 4) that for particular applications ("colored" or finite bandwidth noise driving) these algorithms could be improved upon. Herein we present the final result of such speculations: Three different algorithms are presented: ALGO0, a basic finite-step-size expansion for the stochastic equations; ALGO1, a specialized version of ALGO0 for finite bandwidth of the noise (fully implicit in stochastic terms); and finally ALGO2, basically ALGO1 with a semi-implicit integrator for the deterministic flow. The algorithm ALGO1 has already been used in our simulations[5-6] and this paper is also meant to give a full account of the method exploited there.[10] Let us also stress that an algorithm that has been recently proposed[11] is a limiting case of the algorithm discussed here.

II. AN ALGORITHM FOR WHITE NOISE

First of all, let us briefly recall the white-noise algorithm developed in Ref. 4. The algorithm is a one-step collocation scheme. Given the set of stochastic differential equations

$$\dot{x}_i = f_i(\mathbf{x}) + g_i(\mathbf{x})\xi(t), \quad i = 1, N$$
$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t)\xi(s) \rangle = \delta(t-s) \tag{1}$$
$$\xi(t) \text{ Gaussian },$$

noticing that

$$\int_0^h \dot{x}_i(t)dt = x_i(h) - x_i(0) = \int_0^h f_i(\mathbf{x}(t))dt + \int_0^h g_i(\mathbf{x}(t))\xi(t)dt \tag{2}$$

with h the integration time step, Taylor expanding the functions appearing on the right-hand side (rhs) of Eq. (2), we obtain

$$f_i(\mathbf{x}(t)) = f_i(\mathbf{x}(0)) + [x_J(t) - x_J(0)]\frac{\partial}{\partial x_J} f_i(\mathbf{x}(0)) + \tfrac{1}{2}[x_J(t) - x_J(0)][x_\kappa(t) - x_\kappa(0)]\frac{\partial^2}{\partial x_J \partial x_\kappa} f_i(\mathbf{x}(0)) + \cdots ,$$
$$g_i(\mathbf{x}(t)) = g_i(\mathbf{x}(0)) + [x_J(t) - x_J(0)]\frac{\partial}{\partial x_J} g_i(\mathbf{x}(0)) + \tfrac{1}{2}[x_J(t) - x_J(0)][x_\kappa(t) - x_\kappa(0)]\frac{\partial^2}{\partial x_J \partial x_\kappa} g_i(\mathbf{x}(0))$$
$$+ \frac{1}{3!}[x_J(t) - x_J(0)][x_\kappa(t) - x_\kappa(0)][x_l(t) - x_l(0)]\frac{\partial^3}{\partial x_J \partial x_\kappa \partial x_l} g_i(\mathbf{x}(0)) + \cdots . \tag{3}$$

We now introduce the notation

$$f_i(\mathbf{x}(0)) = f_i ,$$
$$\frac{\partial^2}{\partial x_J \partial x_\kappa} f_i(\mathbf{x}(0)) = f_{i,J\kappa} , \tag{4}$$

etc. We have (a sum over repeated indices is understood)

$$x_i(h)-x_i(0)=\delta x_i^{1/2}+\delta x_i^{1}+\delta x_i^{3/2}+\delta x_i^{2}+\cdots , \tag{5}$$

where the ellipsis represents terms of the order of $h^{5/2}$ or higher [terms are grouped according to the order of h they have in Eq. (5)]

$$\begin{aligned}
&\delta x_i^{1/2}=g_i Z_1 ,\\
&\delta x_i^{1}=f_i h+\tfrac{1}{2}g_{i,J}g_J Z_1^2 ,\\
&\delta x_i^{3/2}=(f_{i,J}g_J-g_{i,J}f_J)Z_2+\frac{1}{3!}g_{i,J}g_{J,\kappa}g_\kappa Z_1^3+g_{i,J}f_J h Z_1+\tfrac{1}{6}g_{i,J\kappa}g_J g_\kappa Z_1^3 ;\\
&\delta x_1^{2}=\tfrac{1}{2}f_{i,J}f_J h^2+\tfrac{1}{2}f_{i,J}g_{J,\kappa}g_\kappa Z_3+\tfrac{1}{2}f_{i,J\kappa}g_J g_\kappa Z_3+g_{i,J}[(f_{J,\kappa}g_\kappa-g_{J,\kappa}f_\kappa)(Z_1Z_2-Z_3)+\tfrac{1}{2}g_{J,\kappa}f_\kappa(hZ_1^2-Z_3)]\\
&\qquad +\tfrac{1}{24}g_{i,J}g_{J,\kappa}g_{\kappa,l}g_l Z_1^4+\tfrac{1}{2}g_{i,J\kappa}g_\kappa[\tfrac{1}{2}f_J(hZ_1^2-Z_3)+\tfrac{1}{8}g_{J,l}g_l Z_1^4]+\tfrac{1}{24}g_{i,J\kappa l}g_J g_\kappa g_l Z_1^4+\tfrac{1}{24}g_{i,l}g_{l,J\kappa}g_J g_\kappa Z_1^4 .
\end{aligned} \tag{6}$$

Here, if Y_1, Y_2, Y_3 are three uncorrelated Gaussian variables with average zero and standard deviation one, so that

$$\begin{aligned}
&Z_1 \equiv \int_0^h \xi(t)dt=\sqrt{h}\,Y_1 ,\\
&Z_2 \equiv \int_0^h \left[\int_0^t \xi(s)ds\right]dt=h^{3/2}\left[\frac{Y_1}{2}+\frac{Y_2}{2\sqrt{3}}\right],\\
&Z_3=\int_0^h\left[\int_0^t \xi(s)ds \int_0^t \xi(y)dy\right]dt\\
&\quad \cong \frac{h^2}{3}(Y_1^2+Y_3+\tfrac{1}{2})
\end{aligned} \tag{7}$$

(note that the present Z_3 is defined differently from Z_3 in Ref. 4) Eqs. (3)–(7) will define the algorithm ALGO0.

For a one-dimensional system driven by additive noise we have

$$\dot{x}=f+\sqrt{2D}\,\xi(t) \tag{8}$$

and

$$\begin{aligned}
x(h)-x(0)=&\sqrt{2D}Z_1+fh+f'\sqrt{2D}Z_2\\
&+\tfrac{1}{2}ff'h^2+DZ_3f'' .
\end{aligned} \tag{9}$$

This coincides with the white-noise algorithm proposed in Ref. 8, apart from the term with Z_3.

III. THE ALGORITHM FOR COLORED NOISE

In the case where the noise appearing on the rhs of Eq. (1) [or Eq. (8)] is "colored" (finite bandwidth), the algorithm can be easily improved upon. Generally, the problem is cast in the equivalent form

$$\begin{aligned}
&\dot{x}_i=f_i(\mathbf{x})+g_i(\mathbf{x})y ,\\
&\dot{y}=-\frac{1}{\tau}y+\frac{\sqrt{2D}}{\tau}\xi(t) ,\\
&\langle \xi(t)\rangle=0,\quad \langle \xi(t)\xi(s)\rangle=\delta(t-s) ,\\
&\xi(t)\ \text{Gaussian} .
\end{aligned} \tag{10}$$

This yields for $y(t)$

$$\begin{aligned}
&\langle y(t)\rangle=0,\quad \langle y(t)y(s)\rangle=\frac{D}{\tau}e^{-|t-s|/\tau} ,\\
&y(t)\ \text{Gaussian} .
\end{aligned} \tag{11}$$

We could now apply Eqs. (5)–(7) to the system described by Eq. (10), but as pointed out in Ref. 4 (see also Ref. 11), an implicit (or integral) method, if it can be found, is surely more stable and has a wider range of convergency.

Focus the attention on

$$\dot{y}=-\frac{1}{\tau}y+\frac{\sqrt{2D}}{\tau}\xi(t) . \tag{12}$$

We have (exactly)

$$y(t)=e^{-t/\tau}y(0)+\frac{\sqrt{2D}}{\tau}\int_0^t e^{(s-t)/\tau}\xi(s)ds . \tag{13}$$

Define now

$$w_0\equiv\int_0^h e^{(s-h)/\tau}\xi(s)ds , \tag{14a}$$

$$w_1\equiv\int_0^h\left[\int_0^t e^{(s-t)/\tau}\xi(s)ds\right]dt , \tag{14b}$$

$$w_2\equiv\int_0^h\left[\int_0^t\left[\int_0^s e^{(-s+y)/\tau}\xi(y)dy\right]ds\right]dt . \tag{14c}$$

Also define

$$\alpha\equiv\frac{h}{\tau} . \tag{14d}$$

Notice that w_0, w_1, w_2 are Gaussian variables with zero average and still unknown standard deviations and cross correlations. By straightforward algebraic manipulation it is possible to derive these quantities and the results are summed up in Table I. Let us explain how to use it: Suppose we want to know the moment $\langle w_0 w_1\rangle$; moving along the line labeled $w_0/\tau^{1/2}$ until we get to the column $w_1/\tau^{3/2}$, we find $\frac{1}{2}(1-2e^{-\alpha}+e^{-2\alpha})$: noting that w_0 is "scaled" on $\tau^{1/2}$ and w_1 is "scaled" on $\tau^{3/2}$ we have

TABLE I. Autocorrelation and cross correlations of the stochastic variables appearing in ALGO1 [Eqs. (14)].

	$w_0/\tau^{1/2}$	$w_1/\tau^{3/2}$	$w_2/\tau^{5/2}$
$w_0/\tau^{1/2}$	$\frac{1}{2}(1-e^{-2\alpha})$	$\frac{1}{2}(1-2e^{-\alpha}+e^{-2\alpha})$	$\frac{1}{2}(1-2\alpha e^{-\alpha}-e^{-2\alpha})$
$w_1/\tau^{3/2}$		$\frac{1}{2}(2\alpha-3-e^{-2\alpha}+4e^{-\alpha})$	$\frac{1}{2}(1+\alpha^2-2\alpha+2\alpha e^{-\alpha}-2e^{-\alpha}+e^{-2\alpha})$
$w_2/\tau^{5/2}$			$\frac{1}{2}(1+2\alpha-2\alpha^2+\frac{2}{3}\alpha^3-4\alpha e^{-\alpha}-e^{-2\alpha})$

$$\frac{\langle w_0 w_1\rangle}{\tau^{1/2}\tau^{3/2}}=\tfrac{1}{2}(1-2e^{-\alpha}+e^{-2\alpha}) \tag{15a}$$

and

$$\langle w_0 w_1\rangle=\frac{\tau^2}{2}(1-2e^{-\alpha}+e^{-2\alpha}) \ . \tag{15b}$$

This way correlations and standard deviations can be easily derived.

Introduce

$$Z_0\equiv y(h)=e^{-\alpha}y(0)+\frac{\sqrt{2D}}{\tau}w_0 \ ,$$

$$Z_1\equiv\int_0^h y(t)dt=\tau(1-e^{-\alpha})y(0)+\frac{\sqrt{2D}}{\tau}w_1 \ , \tag{16}$$

$$Z_2\equiv\int_0^h\left[\int_0^t y(s)ds\right]dt=\tau^2(\alpha+e^{-\alpha}-1)y(0)+\frac{\sqrt{2D}}{\tau}w_2 \ .$$

Define Y_0,Y_1,Y_2 as three uncorrelated random Gaussian deviates with average zero and standard deviation one, and we have for w_0,w_1,w_2

$$w_0=(\langle w_0^2\rangle)^{1/2}Y_0 \ ,$$

$$w_1=\frac{\langle w_1 w_0\rangle}{(\langle w_0^2\rangle)^{1/2}}Y_0+\left[\langle w_1^2\rangle-\frac{(\langle w_1 w_0\rangle)^2}{\langle w_0^2\rangle}\right]^{1/2}Y_1 \ , \tag{17a}$$

$$w_2=AY_0+BY_1+CY_2 \ ,$$

where A,B,C are the solution of the following set of equations:

$$\frac{\langle w_2 w_0\rangle}{(\langle w_0^2\rangle)^{1/2}}=A \ ,$$

$$\frac{\langle w_1 w_0\rangle}{(\langle w_0^2\rangle)^{1/2}}A+\left[\langle w_1^2\rangle-\frac{(\langle w_1 w_0\rangle)^2}{\langle w_0^2\rangle}\right]^{1/2}B=\langle w_1 w_2\rangle \ , \tag{17b}$$

$$C=(\langle w_2^2\rangle-A^2-B^2)^{1/2} \ .$$

The values of A,B,C can be easily derived by inverting the system in Eq. (17b). The recipe is now to take the white-noise algorithm of Eqs. (5) and (6) [or, equivalently, Eq. (9)] and instead of using Z_0,Z_1,Z_2 from Eq. (7), use Z_0,Z_1,Z_2 from Eq. (16) (ALGO1). Note that there is no Z_3 in Eqs. (5) and (6). We have not yet been able to derive a suitable expression for Z_3 in the presence of colored noise and there is no "colored-noise" counterpart of Z_3 in Eq. (7) (but see Sec. IV). For this reason, we normally use ALGO1 only up to the term $\delta x^{3/2}$ included.

IV. DISCUSSION AND NUMERICAL SIMULATIONS

To gain more insight, let us now discuss the algorithm introduced in Sec. III in the limit $h\to 0$ and in the limit $\tau\to 0$, focusing on a one-dimensional system driven by additive colored noise (the analytic results will, however, be valid in general). First, consider $h\to 0$ $(\alpha\to 0)$.

We have

$$y(t+h)=e^{-\alpha}y(t)+\frac{\sqrt{2D}}{\tau}w_0 \ , \tag{18a}$$

$$x(t+h)=x(t)+\tau[1-e^{-\alpha}y(t)]+\frac{\sqrt{2D}}{\tau}w_1+hf(x(t))+\frac{\sqrt{2D}}{\tau}f'(x(t))\left[\tau^2(\alpha+e^{-\alpha}-1)y(t)+\frac{\sqrt{2D}}{\tau}w_2\right]+\tfrac{1}{2}h^2 f(x(t))f'(x(t))+\cdots \ , \tag{18b}$$

where the ellipsis represents terms with Z_3 and terms of the order of $h^{5/2}$ or higher. As a rule, we keep Eq. (18a) as it is, avoiding expanding it in powers of α, for it is exact. Let us only notice that in Eq. (18a) the first term is $O(h^0)$ $[O(h^n)$ representing the order of $h^n]$, and the second is $O(h^{1/2})$.

Using this in (18b) we have

$$\lim_{\alpha\to 0}\tau(1-e^{-\alpha})y(t)\approx h\left[e^{-\alpha}y(t-h)+\frac{\sqrt{2D}}{\tau}w_0\right]=O(h)+O(h^{3/2}) , \tag{19a}$$

$$\lim_{\alpha\to 0}\frac{\sqrt{2D}}{\tau}w_1=O(h^{3/2}) , \tag{19b}$$

$$\lim_{\alpha\to 0}\tau^2(\alpha+e^{-\alpha}-1)y(t)\cong\frac{h^2}{2}\left[e^{-\alpha}y(t-h)+\frac{\sqrt{2D}}{\tau}w_0\right]=O(h^2)+O(h^{5/2}) , \tag{19c}$$

$$\lim_{\alpha\to 0}\frac{\sqrt{2D}}{\tau}w_2=O(h^{5/2}) . \tag{19d}$$

For Z_3 we have

$$\langle Z_3^2\rangle\sim\left[\int_0^h Z_1^2(t)dt\right]^2\sim O(h^8)$$

and

$$\lim_{\alpha\to 0}(\langle Z_3^2\rangle)^{1/2}\sim O(h^4) . \tag{19e}$$

We are left with

$$x(t+h)=x(t)+\tau(1-e^{-\alpha})y(t)+hf(x(t))+\frac{\sqrt{2D}}{\tau}w_1+\tfrac{1}{2}h^2f(x(t))f'(x(t))+\frac{\sqrt{2D}}{\tau}f'(x(t))[\tau^2(\alpha+e^{-\alpha}-1)e^{-\alpha}y(t-h)]+\cdots , \tag{20}$$

$$y(t+h)=e^{-\alpha}y(t)+\frac{\sqrt{2D}}{\tau}w_0 ,$$

where the ellipsis represents terms of order higher than h^2. Note that if in Eq. (20) we take only terms at order h and use Eq. (19a) to simplify $\tau(1-e^{-\alpha})y(t)$, we have exactly the algorithm of Ref. 11.

In the other limit $\tau\to 0$ $(\alpha\to\infty)$ we have

$$\lim_{\tau\to 0}\tau(1-e^{-\alpha})y(t)=0 , \tag{21a}$$

$$\lim_{\tau\to 0}\frac{w_1}{\tau}=Z_1 \text{ of Eq. (7)} , \tag{21b}$$

$$\lim_{\tau\to 0}\frac{w_2}{\tau^2}=Z_2 \text{ of Eq. (7)} , \tag{21c}$$

$$\lim_{\tau\to 0}\tau(\alpha+e^{-\alpha}-1)y(t)=0 . \tag{21d}$$

In other words, for $\tau\to 0$, which corresponds to taking the limit of white noise, ALGO1 yields exactly (at order $h^{3/2}$) ALGO0. Let us also notice that the results found in the two limits are the white-noise algorithm and the colored-noise algorithm first introduced in Ref. 8.

Given that ALGO1 gives the "right" prescription both for $\alpha\to 0$ and $\alpha\to\infty$, we believe that it should be very accurate even for finite nonvanishing α. To check this, we tested the ALGO1 algorithm computing the mean-first-passage time (MFPT) in a double-well potential. Specifically [Eq. (10)], we have $f(x)=x-x^3$, $g(x)=1$ with $\tau=1.0$ (Fig. 1) and $\tau=0.1$ (Fig. 2).

For both cases, $D=0.1$, and we plot the MFPT as a function of h. Given that no explicit expression is yet available for Z_3 we, however, restricted the algorithm only up to the term $\delta x^{3/2}$. The time taken to reach the boundary $\{x=0\}$ starting from $\{x=-1, y$ randomly distributed according to the appropriate Gaussian distribution$\}$ was computed and averaged over 1000 distinct trajectories, both with our algorithm (crosses) and with the algorithm of Ref. 11 (circles). Given the number of realizations, we expect an error on the MFPT roughly around 3%. It is clear from the figures that if ALGO1 becomes unstable for values of h smaller than for the algorithm of Ref. 11, the latter nevertheless seems to have an overall poorer convergency when h is decreased: in particular, the fluctuations for $h\to 0$ are far larger than the expected 3% standard deviation. Also, for intermediate-to-large values of h it seems to show a trend: at $\tau=1.0$ (Fig. 1) the MFPT computed for $h=0.005$ is 10% larger than the MFPT computed for $h=0.05$ and for larger h the MFPT keeps decreasing. It is indeed true that the algorithm of Ref. 10 is fast (about three times faster than our algorithm and about two times faster than the algorithm of Ref. 8): as such, it might be argued that it allows us to compute more trajectories in the same time. On the other hand, however, the surprisingly large fluctuations signal that the algorithm of Ref. 11 somehow produces "noisier" trajectories [i.e., the individual $x(nh)$ are slightly more scattered than they ought to be]: As such, misleading information might be gathered when the actual $x(t)$ is important, as when correlation functions or power spectra are computed.

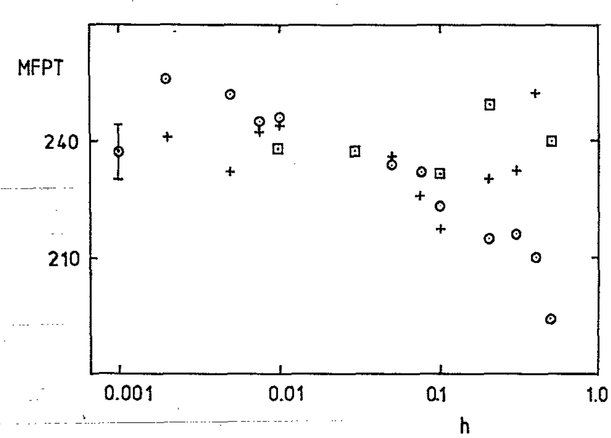

FIG. 1. MFPT vs h for $\tau=1.0$ and $D=0.1$. Circles, algorithm of Ref. 11; crosses, ALGO1; squares, ALGO2. *The bar* represents the one standard deviation error on the MFPT for the number of averages considered (1000).

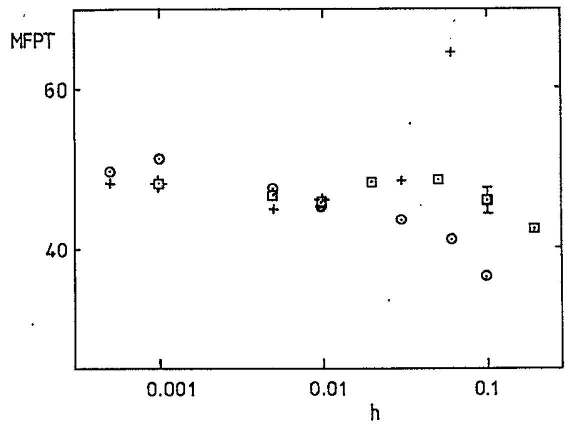

FIG. 2. MFPT vs h for $\tau=0.1$ and $D=0.1$. Circles, algorithm of Ref. 11; crosses, ALGO1; squares, ALGO2. The bar represents the one standard deviation error on the MFPT for the number of averages considered (1000).

On the other hand, ALGO1 is very stable even when the limit $\tau \to 0$ is taken, for $D=0.1$: theoretically we should have a MFPT of 30.8 (corrections due to finite $\Delta V/D$ taken into account). We find, at $\tau=10^{-4}$ a MFPT of 31.7 for both $h=10^{-2}$ and 5×10^{-3} and a MFPT of 30.7 for $h=10^{-3}$, well within 3% of the theoretical MFPT. The algorithm of Ref. 11, on the other hand, gives MFPT's around 5.

The result is particularly illuminating: it suggests that the poor convergency of ALGO1 for large h is not due to the finite value of the ratio h/τ, but to the fact that the values of h considered are probably too "large" to even adequately integrate the deterministic part of the stochastic equation. We have solved this problem by introducing a semi-implicit method (ALGO2) to integrate x (a sort of predictor corrector). Notice that ALGO0 and ALGO1 were derived evaluating f_i and g_i at the point $\mathbf{x}(0)$ [see Eqs. (3)–(6)]: however, it is possible to show that exactly the same algorithms can be obtained evaluating f_i and g_i at the end of point $\mathbf{x}(h)$. Looking at Eq. (9), to order $h^{3/2}$, it is clear that the deterministic flow is integrated via an Adams-Bashforth[12] (AB) integrator of order h. As far as the deterministic flow is concerned, a better convergency can be achieved introducing, for example, an Adams-Moulton[12] (AM) corrector. Basically, the AM corrector is given again by Eq. (9) but by using a guess for $x(h)$ in f and f' given by the AB step (see below). This trick "naturally" increases the stability and convergency of our algorithm [though the algorithm is still $O(h^2)$]. The stochastic part of Eq. (9) (see above remark) can be safely evaluated in $x(h)$. Let us now show how to implement ALGO2 focusing on

$$\begin{aligned}
&\dot{x}=f(x)+y\ ,\\
&\dot{y}=-\frac{1}{\tau}y+\frac{\sqrt{2D}}{\tau}\xi(t)\ ,\\
&\langle \xi(t)\rangle=0,\quad \langle \xi(t)\xi(s)\rangle=\delta(t-s)\ ,\\
&\xi(t)\ \text{Gaussian}\ .
\end{aligned} \tag{22}$$

We proceed as follows (ALGO2):

(1) Start from $x(0), y(0)$.

(2) Generate Z_0, Z_1, Z_2 as per Eqs. (16) and (17).

(3) Compute [Eq. (9), terms up to $\delta x^{3/2}$ included] (predictor step)

$$\begin{aligned}\bar{x}(h)=&x(0)+\sqrt{2D}Z_1+hf(x(0))\\ &+\sqrt{2D}Z_2 f'(x(0))\ .\end{aligned}$$

(4) Compute (corrector step)

$$\begin{aligned}x(h)=&x(0)+\sqrt{2D}Z_1+hf(\bar{x}(h))\\ &+\sqrt{2D}Z_2 f'(\bar{x}(h))\ .\end{aligned}$$

(5) Set $y(h)=Z_0$.

(6) Repeat from step 1 with $x(0)=x(h)$ and $y(0)=y(h)$, etc. Note that the Z_1 and Z_2 used in step (4) are the same Z_1 and Z_2 used in step (3).

The MFPT's obtained with ALGO2 are plotted in Figs. 1 and 2 as squares. It is evident that ALGO2 has a range of convergency for larger than either ALGO1 or the algorithm of Ref. 11.

A strong point in favor of the algorithm of Ref. 11 is speed (three times faster than ALGO1 or ALGO2, two times faster than the algorithm of Ref. 8, for the same h). On the other hand, ALGO2 allows a much larger time step (larger by more than a mere factor 3): when the need of a small of h is dictated by stiffness[13] in the system of differential equations, we believe that ALGO2 will be faster and more reliable than other existing algorithms. We stress again that also for ALGO2 the limit $\tau\to 0$, h finite ($h/\tau \to \infty$) can be safely taken.

V. CONCLUSIONS

We have derived a semi-implicit algorithm for integrating a set of stochastic differential equations driven by a finite bandwidth Gaussian process (noise). The algorithm is fully implicit for the noise: as such, any correlation time for the noise can be chosen, while the integration time step is kept constant, and no downgrading in the convergency is to be expected, neither will problems related to stiffness[13] be found. In particular, h (the integration time step) can be chosen to meet the required precision for the deterministic part of the set of differential equations (perhaps via an adaptive step-size Runge-Kutta method): for any such chosen value of h, the present algorithm will integrate over the stochastic forcing with no problem, whatever the value of τ (correlation time of the noise) is. Computing the mean first passage time in a bistable system we have shown that even the very stiff limit of white noise (in actual fact, four orders of magnitude) can be safely taken and that we do get very reliable results. Comparison with existing algorithms has been carried out.

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