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Mean first-passage time in a bistable system driven by strongly correlated noise: Introduction of a fluctuating potential

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The theories of Tsironis and Grigolini [Phys. Rev. Lett. **61**, 7 (1988); Phys. Rev. A **38**, 3749 (1988)] and of de la Rubia *et al.* [Phys. Rev. A **38**, 3827, (1988)] are discussed. A comparison with a digital simulation is made, and interesting conclusions on the application of a "fluctuating potential" to bistable systems driven by strongly correlated noise are drawn.

Very recently, Tsironis and Grigolini[1,2] (TG) introduced the idea of a fluctuating potential to study the mean first-passage time (MFPT) in a bistable system driven by additive colored noise. The idea, applicable when the correlation time (τ) of the noise becomes much larger than the deterministic relaxation time, is based on the observation that for strongly correlated noise the fluctuation is felt as a slowly varying constant by an otherwise deterministic motion. In particular, TG applied their idea to the flux

$$\dot{x}=x-x^3+\xi(t)\,, \tag{1}$$

where $\xi(t)$ is a Gaussian fluctuation with autocorrelation function given by

$$\langle\xi(t)\xi(s)\rangle=\frac{D}{\tau}\exp\left[-\frac{|s-t|}{\tau}\right]. \tag{2}$$

Following TG, for large τ, $\xi(t)$ will not change appreciably during the evolution of x and the latter will be found at one of the roots of

$$x-x^3+\xi=0\,, \tag{3}$$

where ξ is the instantaneous value of $\xi(t)$.

Three distinct regions can be identified solving Eq. (3) for different values of ξ: (i) for $|\xi|<\xi_c=\sqrt{4/27}$ we have three real solutions, corresponding to two stable states and an unstable one; (ii) for $|\xi|=\xi_c$ we have two coincident real solutions (a marginally stable state) and a (distinct) real solution (a stable state); (iii) for $|\xi|>\xi_c$ we have only one real solution, corresponding to an equilibrium state.

TG then propose to picture the escape from one to two minima of the deterministic potential in the following way: for $|\xi|<\xi_c$, x stays trapped into one of the minima, but as $\xi>\xi_c$ ($\xi<-\xi_c$) the well at negative (positive) x disappears and x is "forced" to make a transition into the other well. This led TG to identify the MFPT with the time taken by ξ, driven by the flux

$$\dot{\xi}=-\frac{1}{\tau}\xi+\frac{\sqrt{2D}}{\tau}f(t)\,, \tag{4}$$

with $f(t)$ a Gaussian fluctuation with autocorrelation function

$$\langle f(t)f(s)\rangle=\delta(t-s) \tag{5}$$

to reach the value ξ_c. From Kramer's theory[3] TG derived, for large τ,

$$T^{\infty}=\frac{(2\pi D\tau)^{1/2}}{\xi_c}\exp\left[\frac{\xi_c^2\tau}{2D}\right], \tag{6}$$

and finally TG proposed for the MFPT

$$\begin{aligned} T_{\rm TG}&=T_0+\exp\left[\frac{V_0}{D}\right]T^{\infty}\\ &=\exp\left[\frac{V_0}{D}\right]\left[\frac{\pi}{\sqrt{2}}+\frac{(2\pi D\tau)^{1/2}}{\xi_c}\exp\left[\frac{\xi_c^2\tau}{2D}\right]\right], \end{aligned} \tag{7}$$

where T_0 is the white-noise limit and V_0 ($=\frac{1}{4}$) is the potential barrier. The T^{∞} prefactor, $\exp(V_0/D)$, appearing in Eq. (7) is justified in Ref. 2. Unfortunately (Fig. 1),

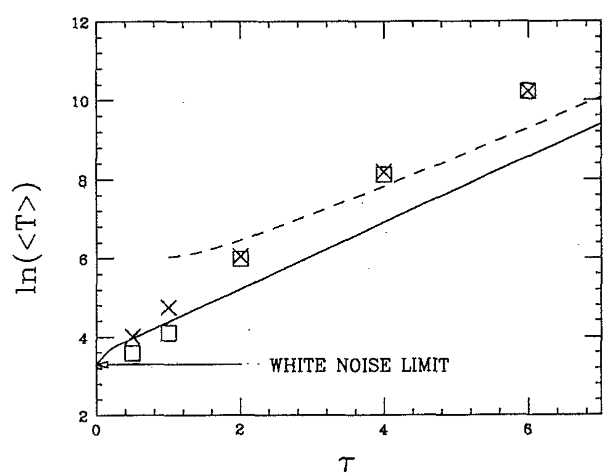

FIG. 1. The logarithm of the MFPT ($\ln\langle T\rangle$) as a function of τ for $D=0.1$. Solid line: Eq. (7); dashed line: Eq. (10); crosses: result of computer simulation; squares: theoretical predictions as deduced from the integration appearing in Eq. (13).

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comparison of T_{TG} (solid line) with computer experiments (crosses) is not very good, in the large-τ region accessible experimentally: from the figure, however, it is clear that for large τ's the slope of the MFPT is well reproduced by the theory, despite the bad agreement on the absolute value.

Since then, de la Rubia *et al.*[4] speculated on the idea of a "fluctuating potential" noticing that the deterministic time to make the transition would go to infinity if ξ equals ξ_c. But ξ is correlated on a time scale of order τ: it is then very probable that when ξ is close to ξ_c, in the very long time it takes to make the transition, ξ becomes appreciably smaller than ξ_c, thus "trapping" x again into the initial well. They proposed to keep Eq. (7) for the MFPT, substituting ξ_c with ξ_τ, where ξ_τ is the value of ξ for which

$$\int_{x_1}^{0} dx \frac{1}{x - x^3 + \xi_\tau} = \tau . \quad (8)$$

x_1 is chosen as the marginally stable solution of Eq. (3) for $\xi = \xi_c (x_1 = -1/\sqrt{3})$. This trick is meant to simulate the fact that the transition is probable only if the deterministic time taken for it is shorter than τ. In Ref. 4 an analytical expression for ξ_τ was derived which reads

$$\xi_\tau = \xi_c + \frac{\pi^2}{4\tau^2\sqrt{3}} + O(\tau^{-3}\ln\tau) . \quad (9)$$

Note that in the limit $\tau \to \infty$ ξ_τ goes to ξ_c, thus recovering Eq. (6) for large τ. Let us define for the MFPT of de la Rubia *et al.*:

$$T_{\mathrm{dlR}}^{\infty} = \frac{(2\pi D\tau)^{1/2}}{\xi_\tau} \exp\left[\frac{\xi_\tau^2 \tau}{2D}\right] ,$$
$$T_{\mathrm{dlR}} = T_0 + \exp\left[\frac{V_0}{D}\right] T_{\mathrm{dlR}}^{\infty} . \quad (10)$$

Comparing (Fig. 1) T_{dlR} (dashed line) with the numerical results, however, we still have a discrepancy, though the theoretical expression gets closer to the data in the "large-but-finite" τ region. Note also that it produces a relatively large variation in the value of the MFPT to replace ξ_τ with ξ_c.

To get a better understanding of the role played by the different ξ's, we thought it very appropriate to perform the digital simulation keeping track of the value of ξ at which the transition takes place.[5]

Define the value of $\xi(t)$ [Eq. (1)] when $x(t)$ goes through the boundary $x=0$ with ξ_{tr}^{τ}. The simulations gave distributions of ξ_{tr}^{τ} that we have plotted in Fig. 2 for different values of τ. Define such distributions with $P_\tau(\xi_{\mathrm{tr}})$ and

$$\overline{\xi_{\mathrm{tr}}^{\tau}} = \frac{\int \xi_{\mathrm{tr}}^{\tau} P_\tau(\xi_{\mathrm{tr}}^{\tau}) d\xi_{\mathrm{tr}}^{\tau}}{N} ,$$
$$\sigma_\tau^2 = \frac{\int (\xi_{\mathrm{tr}}^2 - \overline{\xi_{\mathrm{tr}}^{\tau}})^2 P_\tau(\xi_{\mathrm{tr}}^{\tau}) d\xi_{\mathrm{tr}}^{\tau}}{N} , \quad (11)$$

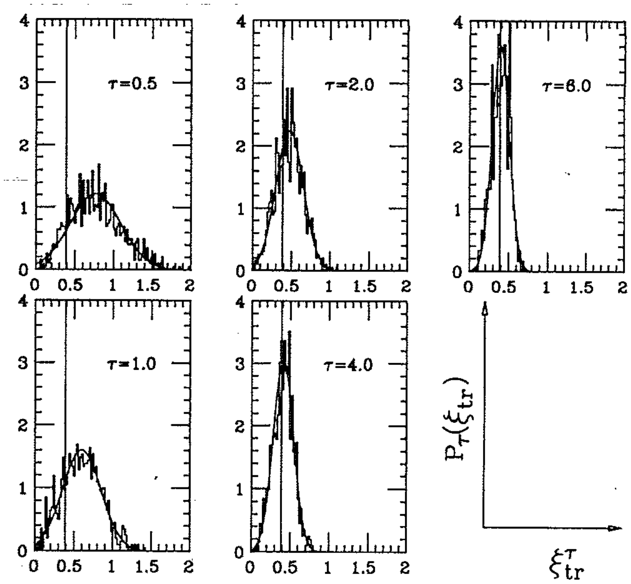

FIG. 2. Probability distribution $P_\tau(\xi_{\mathrm{tr}})$ of ξ_{tr}^{τ} vs ξ_{tr}^{τ} (see text) for different values of τ and $D=0.1$. The vertical line is drawn through ξ_c, the smooth solid curve is the best-fitting Gaussian.

where N is a normalization constant.

Note that $\overline{\xi_{\mathrm{tr}}^{\tau}}$ will be the "experimental" counterpart of ξ_τ. We are now in position to confirm the idea of de la Rubia *et al.*, namely that $\overline{\xi_{\mathrm{tr}}^{\tau}}$ is larger than ξ_c and that

$$\lim_{\tau\to\infty} \overline{\xi_{\mathrm{tr}}^{\tau}} = \xi_c . \quad (12)$$

We also notice that, for finite τ, $P_\tau(\xi_{\mathrm{tr}})$ has a relatively large σ_τ, though it seems that for $\tau \to \infty$, σ_τ goes to 0. Let us now go back to the idea of a fluctuating potential: Were it the correct model to describe the transition in the region of large τ, a better agreement between theory and experiment should be found integrating T^∞ [Eq. (6)] *with no prefactor over* $P_\tau(\xi_{\mathrm{tr}})$, i.e., we propose to consider the quantity

$$T_{\mathrm{MP}} = (2\pi D\tau)^{1/2} \int \frac{P_\tau(\xi_{\mathrm{tr}})}{\xi_{\mathrm{tr}}} \exp\left[\frac{\xi_{\mathrm{tr}}^2 \tau}{2D}\right] d\xi_{\mathrm{tr}} \times \left[\int P_\tau(\xi_{\mathrm{tr}}) d\xi_{\mathrm{tr}}\right]^{-1} \quad (13)$$

We have plotted in Fig. 1 T_{MP} as a function of τ (squares) and the agreement with the computer-simulation data (crosses) is astonishingly good, in the region $\tau > 1$, thus proving that the application of a fluctuating potential, supplemented by the knowledge of a quantity like $P_\tau(\xi_{\mathrm{tr}})$, gives the right description of the activation rate out of a potential well in the large τ region.

Notice the importance of integrating T^∞ over $P_\tau(\xi_{\mathrm{tr}})$: In fact, T^∞ depends *exponentially* on ξ_{tr} and even if $P_\tau(\xi_{\mathrm{tr}})$ went to zero like a Gaussian distribution, the integral would still get an appreciable contribution from ξ_{tr}'s much larger than $\overline{\xi_{\mathrm{tr}}^{\tau}}$. In Fig. 2 we have also shown for comparison the fit to $P_\tau(\xi_{\mathrm{tr}})$ with a Gaussian: despite the apparently good agreement, the MFPT obtained integrating T^∞ over the best fitting Gaussian is wrong, at

$\tau=6.0$, by a factor of 10^6. This should warn the reader that a theoretical derivation of the right prefactor for large τ might prove a very hard task, given that T^∞ depends dramatically on large ξ_{tr}. We have also found with our simulations that the inverse of the standard deviation squared of the best-fitting Gaussian is only slightly larger than τ/D which is the factor multiplying $\xi_{tr}^2/2$ in the exponent of T^∞, thus leading, in principle, to a slow convergency of the integral appearing in Eq. (13). On the other hand, we can give qualitative arguments in favor of a faster-than-Gaussian decrease of $P_\tau(\xi_{tr})$. First, note that large values of ξ_{tr} are distributed according to

$$\bar{P}(\xi_{tr}^\tau)\approx\exp\left[-\frac{(\xi_{tr}^\tau)^2\tau}{2D}\right] \qquad (14)$$

(equilibrium distribution of ξ). Second, starting from ξ_c, a fluctuation $O(\xi_c/e)$ will evolve on a time scale $O(\tau)$ [Eq. (4)]. Third, as ξ increases above ξ_c, the deterministic time [Eq. (8)] to make the transition decreases. Fourth, for $\xi\gg\xi_c$ such deterministic "transition" time can be very short, in particular shorter than τ for $\xi>\xi_\tau$. Finally, we recall that we defined ξ_{tr}^τ as the value of $\xi(t)$ when $x(t)$ reaches the boundary $x=0$. This implies, that for large τ's, $\xi(t)$ will not reach values much larger than ξ_τ given that such fluctuation would take $O(\tau)$ to develop while x will have reached the boundary in a time shorter than τ. This means that the dynamics will diminish $\bar{P}(\xi_{tr}^\tau)$ in Eq. (14) on the side of larger ξ_{tr}^τ, leading to a faster-than-Gaussian decrease of $P_\tau(\xi_{tr})$ for large ξ_{tr}.

A final point is the dependence of ξ_τ on the noise strength D. In Ref. 4, it was assumed that the value of ξ_τ was independent of D. In Fig. 3 we have plotted ξ_τ [solving numerically Eq. (8), solid line] together with the values of $\overline{\xi_{tr}^\tau}$ for three different values of D. It is evident that $\overline{\xi_{tr}^\tau}$ depends also on D though the qualitative behavior is well represented by the theoretical expression.

In conclusion, we have proved that the application of a fluctuating potential is correct to describe the activation rate out of a potential well in the presence of a strongly correlated fluctuation. The present theories based on this idea seem to reproduce fairly well the exponential dependence on τ of the MFPT, only failing to account for the prefactor. Furthermore, we showed that the prefactor depends strongly on the details of the probability distribution of the ξ_{tr}^τ's [in Refs. 1 and 2, $P(\xi_{tr}^\tau)\approx\delta(\xi-\xi_c)$, in Ref. 4, $P(\xi_{tr}^\tau)\approx\delta(\xi-\xi_\tau)$]: when the correct distribution is used, the theory is in very good agreement with the experiment. We also stressed that a theoretical derivation of $P(\xi_{tr}^\tau)$ seems to be a very hard task and gave some hints on how to proceed. Finally, studying its first moment, we showed that $P(\xi_{tr}^\tau)$ depends also on the noise intensity.

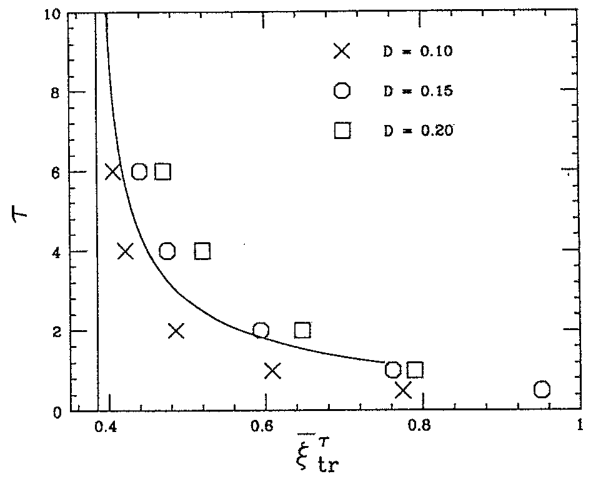

FIG. 3. Relation between τ and ξ_τ ($\overline{\xi_{tr}^\tau}$). Solid line: numerical solution of Eq. (8) ($\overline{\xi_{tr}^\tau}$ should be identified with ξ_τ). The data are the result of computer simulations for different τ's. The vertical line is drawn through ξ_c.

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[3]H. A. Kramers, Physica **VII**, 284 (1940).
[4]F. J. de la Rubia, E. Peacock-Lopez, G. Tsironis, K. Lindenberg, L. Ramirez-Piscina, and J. M. Sancho, Phys. Rev. A **38**, 3827 (1988).
[5]For the algorithm used, see R. Mannella and V. Palleschi, Phys. Lett. A **129**, 317 (1988), time step 0.01. The evolution was followed between -1 and 0, and the time taken to reach 0 was averaged to give the MFPT. Note that this time proved to be virtually the same as the time taken to reach the boundary of the basin of attraction, as defined by the deterministic motion.