Energy-fluctuation relaxation towards equilibrium in an infinite chain of anharmonic oscillators

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This study tries to extend results observed in finite chains, like slowing-down effects and stretched-exponential decays, into the thermodynamic limit. An infinite chain of nonlinear oscillators is related to pairs of coupled anharmonic modes, where each mode is coupled to an infinite number of harmonic oscillators (thermal baths). For the model of the two coupled infinite chains, the analytical treatment leads very naturally to a "critical" energy below which slowing-down effects and stretched-exponential decays should appear. Also, the model yields a zero-energy threshold for the onset of energy sharing between modes, suggesting that chaos should set in for arbitrarily small energies of the Hamiltonian. The theory is confirmed by digital simulations. These results seem to support the phenomenology observed in models with finite degrees of freedom, suggesting that this behavior survives in the thermodynamic limit and it is perhaps a common feature of chains of coupled anharmonic oscillators.

I. INTRODUCTION

The thermodynamic limit in a system of nonlinear oscillators has been a long-standing problem in statistical physics. Starting with the early work of Fermi, Pasta, and Ulam[1] (FPU), where for the first time a numerical study of a finite nonlinear chain was carried out, this problem has seen a number of publications both on the theoretical and on the experimental (mainly numerical) aspects. We refer the reader to a recent publication[2] for more details and a full bibliographic list. We would like only to point out Ref. 3, which somehow stimulated the work presented here. In those papers the evolution of a chain of nonlinear oscillators has been considered (the Fermi-Pasta-Ulam model).

One of the aims of Refs. 1 and 3 was to prove or disprove that there exists a threshold energy below which equipartition is never reached even when the number of oscillators in the chain goes to infinity. For the FPU model it is well known from the Kolmogorov-Arnold-Moser (KAM) theorem that for a very small number of oscillators this energy threshold is different from zero. It is also known, at least for some model systems, that the energy threshold for equipartition goes to zero quite rapidly as the number of degrees of freedom becomes large. There are also systems for which it is known that arbitrary small perturbation potentials will immediately yield a chaotic system, thus leading to a zero equipartition threshold.[4]

One amazing conclusion of Ref. 3 was that the FPU system seems indeed to show an energy threshold even when the number of oscillators goes to infinity. Of course, the proof being numerical, the number of oscillators had to be finite. However, when the number of oscillators was large enough, it was found that the equilibrium entropy per degree of freedom would be largely independent of the number of oscillators itself and would have values pointing to lack of equipartition when the average energy of the chain was small enough. Furthermore, this entropy seemed to show a critical behavior, with a rather distinctive threshold, as the energy of the chain was changed.

Clearly, this result came as a surprise. Subsequent numerical work (Ref. 2) has, however, shown that the critical behavior of the entropy was perhaps an artifact of the computer simulations. One important conclusion of Ref. 2 seems to be that, although very likely the energy threshold for equipartition goes to zero in the thermodynamic limit, the relaxation towards the final equilibrium state shows a rather interesting behavior. For large enough energy densities the relaxation time is only mildly dependent on the energy of the chain, whereas below what we will call a critical energy, the relaxation time starts to increase fairly sharply (though no critical slowing down seems to be present), and the relaxation process is no longer described by an exponential but rather by a stretched exponential. A very appealing physical interpretation was put forward in Ref. 2, based on the diffusion over restricted paths in the phase space, which led to decays described by a stretched exponential. These properties were numerically verified in a system with a finite number of degrees of freedom. An open question was then whether the same mechanism of restricted dynamics would still survive in the thermodynamic limit.

The aim of the present paper is to extend the analysis in the thermodynamic limit, i.e., when there is an infinite number of oscillators in the anharmonic chain. The focus will be mainly on the dynamic properties of a FPU-like model. We are going to show that indeed the system is characterized by energy flows between modes,

which is a necessary ingredient for equipartition, for any (arbitrarily small) energy of the chain (the theory yields actually a zero-energy threshold for this flow), and that a slowing down for the relaxation of the energy flow towards the final equilibrium appears when the energy of the chain is decreased and approaches a well-defined value (we will loosely speak of critical energy, though no critical behavior is really observed for the relaxation time), in a scenario very similar to the one of Ref. 2. The analogy with Ref. 2 is so close that, as we will show, the decay towards equilibrium is described by the same stretched exponential function used there, as we approach the critical energy.

The fundamental ingredient of our analysis is the replacement of a chain of nonlinear oscillators with a canonical fluctuation-dissipation process: This is a quite standard procedure in nonequilibrium stochastic physics, and we will review the basic ideas in Sec. II. Once the relation between a chain of oscillators and a canonical fluctuation-dissipation process has been established, in Sec. III we will relate the original problem of an infinite chain to the problem of two coupled nonlinear oscillators, each of which is coupled to the appropriate dissipation-fluctuation processes, following the line of reasoning of Ref. 5. For this simpler system it is possible to write a Fokker-Planck equation and to employ standard methods of nonequilibrium statistical physics. The result will be that the energy sharing between modes, which is a necessary condition for equipartition, is achieved for virtually any energy of the chain. Section IV is then devoted to studying the relaxation process towards equilibrium. We will derive theoretically the relaxation time for our reduced model. In Sec. V we will compare the theory with the results of computer simulations of our reduced model, and we will show that indeed the system exhibits a slowing down when the energy in the chain is decreased; we will also show that the relaxation process is well described by a stretched exponential. Section VI will then be devoted to our conclusions.

II. ON THE EQUIVALENCE BETWEEN AN INFINITE CHAIN OF OSCILLATORS AND A CANONICAL FLUCTUATION-DISSIPATION PROCESS

The system for which this equivalence can be shown is a model where an anharmonic oscillator (the oscillator of interest) is directly coupled to an infinite number of linear oscillators (bath). The couplings among the bath oscillators are supposed to be absent or linear, whereas the couplings between the bath and the oscillator of interest can be anharmonic (for the kind of "allowed" nonlinearities, see Ref. 6). Given that the dynamic matrix of this bath (a set of linear oscillators) can be cast in tridiagonal form (equivalent to a semi-infinite linear chain with nearest-neighbor couplings), the model of a nonlinear oscillator coupled (linearly or nonlinearly) to the first oscillator of a semi-infinite chain of linear oscillators is equivalent to the model of the same oscillator of interest coupled to an infinite number of linear oscillators. Now, let us forget the real (phase) space and instead let us carry out exactly the same considerations for a chain of modes, i.e., let us talk of a mode (the mode of interest) nonlinearly coupled to other modes (the bath modes), which themselves will be harmonic and so on, or, in other words, let us assume that our oscillators are oscillators in the mode space, not in the real space. For this system of coupled modes there will be a corresponding system in the real space, which, quite generally, will be described by an infinite number of nonlinearly coupled nonlinear oscillators. Obviously, the system in the real space will be quite complicated, possibly even with multineighbor couplings, etc., but it will have a fairly simple description in the mode space and it will be nevertheless a (perhaps very peculiar) chain of nonlinear oscillators.

Now we need to make a rather strong assumption: Given that our system, as nonlinear as it may appear in the real space, is really only mildly nonlinear (the nonlinearity being restricted to only two modes), we should be able to say that any chaotic behavior that it will exhibit should be of some relevance in similar (and perhaps even more nonlinear) systems, like the FPU chains. This assumption will be also justified *a posteriori,* when we will see that the relaxation process towards equilibrium is very similar to the relaxation process found in a FPU model,[2] to the point that even the physical mechanism invoked to explain the observed slowing down will be essentially the same. Finally, we have the suspicion that perhaps, as far as the relaxation towards equilibrium is concerned, all FPU models fall under the same universality class: Were this the case, given that our model is a sort of FPU model, it would be natural to find common behaviors with FPU systems. For all other assumptions that we will subsequently make, it should be understood that we are never weakening the basic ingredients, i.e., an infinite number of oscillators, some nonlinearity, and an underlying Hamiltonian description. These other assumptions will be needed to carry out the necessary algebra, and they will never increase the degree of chaos in our system, most often the opposite. In the end, the reduced model we will be working on will be barely nonlinear, but it will still be characterized by energy flows and slowings down in the relaxation towards equilibrium.

Now, let us briefly review the proof of the equivalence between an infinite number of coupled oscillators and a fluctuation-dissipation process. We are going to borrow the argument from Ref. 6. Start with the Hamiltonian system given by (x,v will be coordinate and velocity of the oscillator of interest)

$$H=H_0(x,v)+\sum_i \left[\frac{p_i^2}{2m_i}+\frac{m_i\omega_i^2 q_i^2}{2} \right]+kx\sum_i c_i q_i \ . \tag{2.1}$$

We assumed a linear coupling between bath and oscillator of interest, because this is the model we will be dealing with, but it is also possible to have nonlinear couplings (see Ref. 6 for details). It is now possible to integrate formally the equations of motion of q_i. One starts writing

$$\ddot{q}_i=-\omega_i^2 q_i+\frac{kc_i}{m_i}x(t) \ , \tag{2.2a}$$

$$\dot{v}=-\frac{\partial H_0}{\partial x}-k\sum_i c_i q_i \ ,$$
$$\dot{x}=v \ . \tag{2.2b}$$

Integrating Eq. (2.2a), one obtains

$$q_i(t)=\frac{kc_i}{m_i\omega_i^2}x(t)+\left[q_i(0)-\frac{kc_i}{m_i\omega_i^2}x(0)\right]\cos\omega_i t + \frac{p_i(0)}{m_i\omega_i}\sin\omega_i t - \frac{kc_i}{m_i\omega_i^2}\int_0^t d\tau\cos\omega_i(t-\tau)\frac{p_i(\tau)}{m_i} \ , \tag{2.3}$$

which, substituted into Eq. (2.2b), yields

$$\dot{v}+\frac{\partial H_0}{\partial Q}+\left[k^2\sum_i\frac{c_i^2}{m_i\omega_i^2}\right]x+k^2\int_0^t K(t-\tau)v(\tau)d\tau = kf(t) \ , \tag{2.4}$$

where

$$K(t-\tau)=\sum_i\frac{c_i^2}{2m_i\omega_i^2}\cos\omega_i(t-\tau) \ , \tag{2.5}$$
$$f(t)=\sum_i c_i\left[\left[q_i(0)-\frac{kc_i}{m_i\omega_i^2}x(0)\right]\cos\omega_i t+\frac{p_i(0)}{m\omega_i}\sin\omega_i t\right] \ .$$

If we assumed that at $t=0$ the system was in equilibrium, it would be possible to show[6] that a fluctuation-dissipation relation applies to $K(t)$ and $f(t)$, i.e., that

$$\langle f(t)f(\tau)\rangle=k_B T K(t-\tau) \ . \tag{2.6}$$

With this hypothesis there is equivalence between an oscillator coupled to an infinite number of other oscillators and an oscillator coupled to a standard fluctuation-dissipation process. Note that in Eq. (2.4) it appears that we have lost the determinism that was present in the original Hamiltonian. However, Eq. (2.4) is still deterministic and it would indeed show cycles [see also Eq. (2.5)], but due to the infinite number of oscillators in the bath, in general it could be expected that such cycles will reproduce after a very long (almost infinite) time. The idea is that if we want to study the dynamics of the system at relatively short times (much shorter than the repetition time of the bath) and we are not in the presence of critical slowing downs, the force of Eq. (2.5) will be indistinguishable from a Gaussian noise with the appropriate spectral density (at least, because of the central-limit theorem).

Now, if we assumed that the dynamics of the bath oscillators is much faster than the dynamics of the oscillator of interest, the fluctuation-dissipation process would become a δ-correlated process (white noise): Note that this is not weakening the fundamental ingredients, i.e., an infinite number of degrees of freedom, some nonlinearity, and an underlying Hamiltonian description.

To study the energy flow between modes, we now introduce the following model. We take two semi-infinite chains of linearly coupled oscillators, which are really meant to represent oscillations in the mode spaces: Let us stress that from now on we will understand "oscillations in the mode space" for "oscillators." For both chains, the end oscillator (the "first" oscillator) will be assumed to be characterized by a slow dynamics when compared to the other oscillators in the chain, so that for both chains the bath oscillators can be replaced by a white-noise fluctuation-dissipation process. For one of the two chains we also assume that the oscillator of interest in general evolves anharmonically. Then we couple the two oscillators of interest via a nonlinear coupling. Imposing the condition that the two chains have different energy densities (i.e., different temperatures), monitoring the energy flow through the oscillator of interest, we should be able to study the energy sharing between modes. We stress again that this model still retains the original hypothesis, infinite degrees of freedom, a fairly weak nonlinearity, confined to the evolution of two modes, and an underlying Hamiltonian description.

III. AVERAGE ENERGY OF THE MODE OF INTEREST

We will now concentrate on one of the two oscillators. In Refs. 2 and 3 most attention was devoted to the energy sharing between modes, the signal that the equipartition was taking place. In analogy, here we will consider the energy which is stored on the oscillator of interest. For a very general anharmonic force $F(x)$ (which will be specified later) and for an anharmonic coupling $yG(x)$ [$G'(x)$ will be the derivative of $G(x)$ with respect to x], we arrive at the differential equations,[5] where the two couples of variables (x,v) and (y,w) describe the evolution of the oscillators at the end of the two semi-infinite chains,

$$\begin{aligned}\dot{x}&=v \ ,\\ \dot{v}&=-\gamma v+F(x)+G'(x)y+f(t) \ ,\\ \dot{y}&=w \ ,\\ \dot{w}&=-\omega_w^2 y-\mu w+h(t)+G(x) \ ,\end{aligned} \tag{3.1}$$

where $f(t)$ and $h(t)$ obey a standard (δ-correlated) fluctuation-dissipation theorem, i.e.,

$$\begin{aligned}\langle f(t)\rangle&=0 \ ,\\ \langle h(t)\rangle&=0 \ ,\\ \langle f(t)f(s)\rangle&=2k_B\gamma T_A\delta(t-s) \ ,\\ \langle h(t)h(s)\rangle&=2k_B\mu T_M\delta(t-s) \ .\end{aligned} \tag{3.2}$$

At this level, T_A, T_M, γ, and μ are quantities related through Eqs. (2.5) and (2.6) to the exact coefficients appearing in the two (in general, different) semi-infinite chains: in the following, due to the arbitrariness of the chains, we will treat them as parameters.

The average energy of the oscillator of interest (x,v) will be defined as the harmonic energy, i.e., $2\langle E\rangle=v^2+\omega^2x^2$, where ω is the frequency of the small

oscillations around the equilibrium position of x: The reason for this choice will be clear further down. We need, however, yet another simplification: we will assume that μ is so large when compared to the frequency of oscillations of (y,w) as to allow for an adiabatic elimination of the motion of both variables y and w. Let us point out that this is merely an assumption of how strong the coupling is between the oscillator (y,w) and its chain of linear oscillators: To take the adiabatic limit it means that this coupling is fairly strong [see Eqs. (2.2b) and (2.4)]. However, given that this oscillator is linearly coupled with its bath, we are *not* increasing the degree of nonlinearity in the system. We need also to specialize $F(x)$ and $G(x)$. We will assume, after Ref. 5, that

$$F(x)=-x+GG' ,$$
$$G(x)=\frac{x^2}{2} , \tag{3.3}$$

which is a particularly useful definition, because the adiabatic elimination will yield a very simple set of equations. In passing, note that our choice has two purposes: It yields a nonlinear system that is still solvable, and decreases the degree of nonlinearity. Note also that we still have a nonlinear Hamiltonian system with an infinite number of degrees of freedom. Having carried out the adiabatic elimination, eventually we arrive at[5,7]

$$\dot{x}=v ,$$
$$\dot{v}=-\gamma v\left[1-\frac{\lambda}{\gamma}x^2\right]-x+\sqrt{2\gamma T_A}f(t)+x\sqrt{2\lambda T_M}h(t) , \tag{3.4}$$

where $\lambda=\mu/(\omega_w)^4$. We should now turn to the energy of the mode (x,v), defined as the harmonic energy of the small oscillations. It is clear from the structure of Eq. (3.4) why this choice is particularly convenient: Given that we will be working in the strongly underdamped limit, i.e., for very small γ, Eq. (3.4) reduces to a very weekly damped harmonic oscillator, for which the harmonic energy is a very "good" quantity, being almost conserved. To be more precise, we introduce a couple of action-angle variables (E,ϕ), where E is defined as the harmonic energy and ϕ contains all the nonlinear dynamics of the motion: Then, if γ is very small, E is a slow variable whereas ϕ is very fast. We then write the evolution equation for the new variables (E,ϕ), and obtain a one-dimensional equation for E, having averaged over the fast dynamics of ϕ. The necessary algebra was already carried out in Refs. 5 and 7, and one finds, assuming that γ is small (smaller than one),

$$\frac{\partial P(E,t)}{\partial t}=\gamma\frac{\partial}{\partial E}\left[E-T_A-2\frac{T_M}{T_C}E+\frac{E^2}{T_C}+\frac{\partial}{\partial E}\left[T_A E+\frac{T_M}{T_C}E^2\right]\right]P(E,t) , \tag{3.5}$$

where $T_C=(\omega_w)^4/\mu$ in the following will be treated as a parameter. Equation (3.5) will be the basis of our analysis. For the equilibrium distribution it yields[5]

$$P(E)=\frac{N}{(E+A)^B}\exp\left[-\frac{E}{T_M}\right] , \tag{3.6}$$

where we have defined

$$A\equiv\frac{T_C T_A}{T_M} ,$$
$$B\equiv\frac{T_C}{T_M}\left[1-\frac{T_A}{T_M}\right] , \tag{3.7}$$

where N is a normalization constant which has the value

$$N^{-1}=T_M^{1-B}\exp(A/T_C)\Gamma(1-B,A/T_C) , \tag{3.8}$$

and where $\Gamma(a,b)$ is the incomplete gamma function.[8]

Using Eq. (3.8), we obtain, for the average energy,

$$\langle E\rangle=T_M-T_C-\left[\frac{A}{T_M}\right]^{1-B}\frac{T_M\exp(-A/T_C)}{\Gamma(1-B,A/T_C)} . \tag{3.9}$$

Equation (3.9) is well defined for T_A different from zero and any value of T_M, or $T_A=0$ and T_M larger than T_C. In fact, if $T_A=0$ and $T_M<T_C$, Eq. (3.9) would yield a negative average energy, which is unphysical. We claim that this average energy should be identified with the average energy in the large time limit introduced in Ref. 3 (the quantity H_∞). We have already an interesting result—namely, that if we give no energy to half a chain, i.e., we only excite a well-defined group of modes and the nonlinearity is confined to a mode at the boundary of the excited modes (corresponding to the situation $T_A=0$), we would observe some sort of threshold when the energy flux through the oscillator of interest is considered as a function of the average energy per mode (T_M), or, in other words, no flux is present for small energies. This could be interpreted as some sort of remnant of the result found in Ref. 3, where only a finite number of modes was excited; however, bearing in mind that the actual models are very different. Note, however, that in any case the initial conditions leading to a null temperature for a semi-infinite chain have a measure zero in the space of all the possible initial conditions, and hence this threshold for the achievement of energy flux between modes has little weight in real physical systems (it would perhaps correspond to some pathological initial conditions).

Nevertheless, as clearly seen from Fig. 1, if $T_A\ll T_C$, we still observe some sort of rounded threshold around $T_M\sim T_C$ (see, for a similar point, Ref. 5). Obviously, the case $T_A\neq 0$ is of much more relevance in real systems, for which we must conclude that no threshold for energy flux between modes is present. Let us now turn to the dynamic aspects of our system. Loosely speaking, were we in the presence of some sort of slowing down for $T_M\leq T_C$, we would expect that if T_A were much smaller than T_C, $\langle E\rangle$ would reach the equilibrium value on a very long time scale, thus giving credit to the objections towards the numerical experiments of Ref. 3 put forward in Ref. 2. Indeed, the fact that if $T_M=T_C$ and $T_A=0$ the energy fluctuations exhibit a critical slowing down was shown perturbatively in Ref. 5. It will be the purpose of Sec. IV to prove that the critical slowing down for $T_A=0$ is ex-

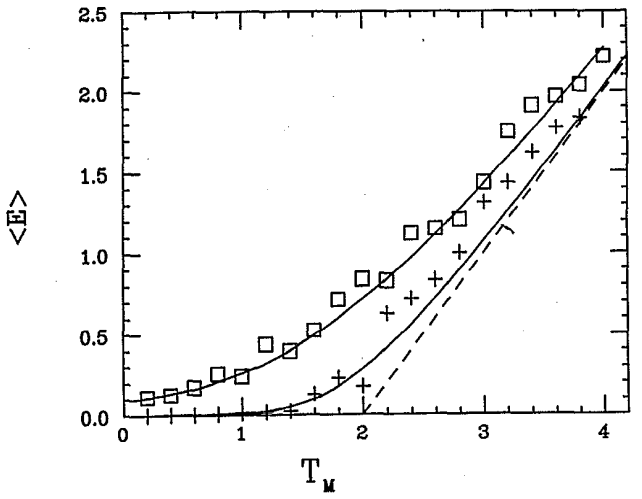

FIG. 1. The average energy $\langle E \rangle$ of the oscillator of interest vs intensity of the multiplicative noise (T_M) for different values of T_A and $T_C=2.0$. The solid lines are Eq. (2.7) for $T_A=0.1$ (top) and $T_A=0.001$ (bottom). The symbols are the corresponding numerical simulations. The dashed line is Eq. (2.7) for $T_A=0$.

act (i.e., it is present even when the problem is solved nonperturbatively), and to show that a (noncritical) slowing down is still present when $T_A \neq 0$, due to the diffusion over restricted paths even in the thermodynamic limit.

IV. RELAXATION OF THE ENERGY FLUCTUATIONS

To calculate the relaxation time of the energy fluctuations, we use a very general method[9] applicable to any one-dimensional Fokker-Planck equation such as the one of Eq. (3.5). We will briefly review the method. First of all, the energy fluctuations will relax on a time scale dictated by the relaxation of the energy autocorrelation function. Let us then consider the Fokker-Planck equation, where the term $f(x)$ is the drift and the term $g^2(x)$ is the diffusion,

$$\frac{\partial P(x,t)}{\partial t}=\frac{\partial}{\partial x}\left[-f(x)+\frac{\partial}{\partial x}g^2(x)\right]P(x,t)\ , \tag{4.1}$$

whose equilibrium distribution is (N is a normalization constant)

$$P_{ST}(x)=N\left[\exp\int^x\frac{f(s)}{g^2(s)}ds\right]\Big/g^2(x)\ . \tag{4.2}$$

We define the nonlinear relaxation time τ as

$$\tau\equiv\frac{\int_0^\infty C(t)dt}{C(0)}\ , \tag{4.3}$$

where the correlation function $C(t)$ is defined as

$$C(t)\equiv\langle[x(t)-\langle x\rangle][x(0)-\langle x\rangle]\rangle\ . \tag{4.4}$$

One finds that[9]

$$\tau=\frac{1}{C(0)}\int_0^\infty\frac{I^2(x)}{g^2(x)P_{ST}(x)}dx\ , \tag{4.5}$$

where $I(x)$ is defined as

$$I(x)\equiv-\int_0^x(y-\langle x\rangle)P_{ST}(y)dy\ . \tag{4.6}$$

For our system we have

$$I(E)=\frac{T_M e^{-A/T_M}}{\Gamma(1-B,A/T_M)}\left[\frac{A}{T_M}\right]^{1-B}e^{-E/T_M}\times\left\{\left[\Gamma\left[1-B,\frac{x+A}{T_M}\right]\Big/\Gamma\left[1-B,\frac{A}{T_M}\right]\right]e^{E/T_M}-\left[1+\frac{E}{A}\right]^{1-B}\right\}, \tag{4.7}$$

which inserted back into Eq. (4.5) allows the calculation of the relaxation time of the energy fluctuations, from the energy correlation function.

For the case of $T_A=0$ the calculation of T is tedious but analytical: The final result reads

$$\tau=\frac{T_C}{T_M-T_C}\ , \tag{4.8}$$

where, we like to remark, no approximation has been invoked. It is clear that in this case we indeed find a critical slowing down, though the behavior is somehow different from the one found in Ref. 5, where perturbative expansions of the Fokker-Planck operator of Eq. (3.5) for $T_A=0$ have been considered.

In general we have only been able to solve the integral appearing in Eq. (4.5) numerically; however, for $T_A/T_C \ll 1$ and small ratios T_M/T_C, it is possible to find the value of τ perturbatively. Incidentally, in this region the numerical integration of Eq. (4.5) is not very stable and the perturbative expansion is very useful to check the numerics. From the Fokker-Planck equation of Eq. (3.7), one can straightforwardly derive the evolution equation for the average energy,

$$\frac{\langle\dot{E}\rangle}{\gamma}=-\langle E\rangle+T_A-\frac{\langle E^2\rangle}{T_C}+\frac{2T_M}{T_C}\langle E\rangle\ , \tag{4.9}$$

which yields for τ,

$$\tau=\frac{1-2T_M/T_C}{[1-(2T_M/T_C)]^2+2T_A/T_C}\ , \tag{4.10}$$

if $T_A/T_C \ll 1$ and $T_M/T_C<1$. In the region $T_A/T_C \ll 1-2T_M/T_C$, τ diverges like $1/(1-2T_M/T_C)$.

V. COMPARISON BETWEEN THEORY AND NUMERICAL SIMULATIONS, AND NONEXPONENTIAL RELAXATIONS

We have also carried out a digital simulation of the system of Eq. (3.4). We recall that the system simulated is described by, if $f(t)$ and $h(t)$ are two uncorrelated white Gaussian processes with a standard deviation of 1,

$$\dot{x}=v\ ,$$
$$\dot{v}=-\gamma v\left[1-\frac{\lambda}{\gamma}x^2\right]-x+\sqrt{2\gamma T_A}f(t)+x\sqrt{2\lambda T_M}h(t)\ . \tag{5.1}$$

We have used the algorithm described elsewhere,[10] at first order in the integration time step, modified via a corrector of the Heung type.[11] The energy of the system has been defined as the harmonic energy, i.e., $2E=v^2+x^2$. For all simulations we have $\gamma=\lambda=10^{-2}$. We would like to point out that some disagreement between numerical simulations and theory could be due to the fact that the technique of averaging over the angle variable after the transformation from the variables $(x-v)$ to some action-angle variables to obtain a reduced equation for the energy is exact only when $\gamma=0$. Our γ is very small indeed [compared to the relevant time constant which is the coefficient of x in Eq. (5.1), equal to 1], surely small enough to guarantee that the static will be accurately described by an effective Fokker-Planck equation involving E only; however, we might expect some small discrepancies when we will consider the dynamics.

In Fig. 2 we present the comparison between numerical simulations and theory for the relaxation of the energy fluctuations: the solid line is the numerical integration of Eq. (5.1), the dashed line is Eq. (4.12), and the dot-dashed line is Eq. (4.10). We have chosen for the parameters $T_A=10^{-4}$ and $T_C=2.0$.

It is quite clear that for $T_M\sim T_C$ the system exhibits a slowing down, similar to the phenomenon observed in Ref. 2. However, in Ref. 2 not only was a slowing down for the energy relaxation observed, but, for values of the energy density below a critical value, there was also a change in the relaxation of the energy correlation function itself, from an exponential to a stretched exponential. Indeed, we do observe in our model long tail decays when $T_M\sim T_C$: In Fig. 3 we have plotted the correlation function of the energy fluctuations (solid curves) for three different values of T_M—namely, larger, order of, and smaller than T_C. For all cases, the dashed lines are the best fit with exponential functions. It is clear that in Fig. 3(b) the correlation function decays much slower than the best-fitting exponential and that a long-time tail is present. We believe that this already strongly supports the results of Ref. 2, in the limit of infinite degrees of freedom. To have a closer agreement with the results of

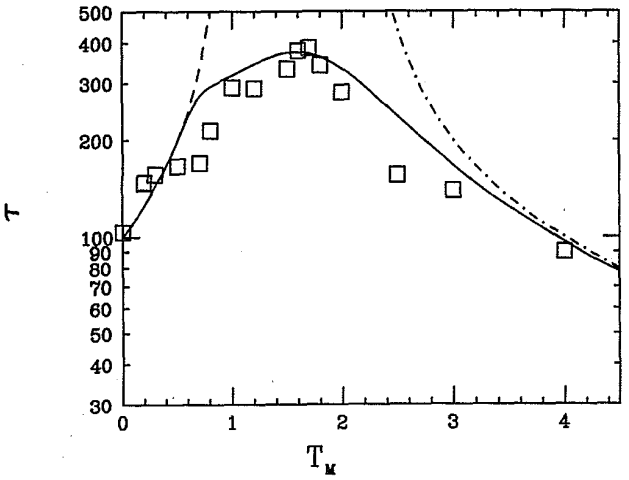

FIG. 2. Relaxation time of the energy fluctuations vs T_M for $T_A=10^{-4}$ and $T_C=2.0$. The solid line is Eq. (3.5), the dashed line is Eq. (3.10), and the dot-dashed line is Eq. (3.8). The symbols are the result of numerical simulations.

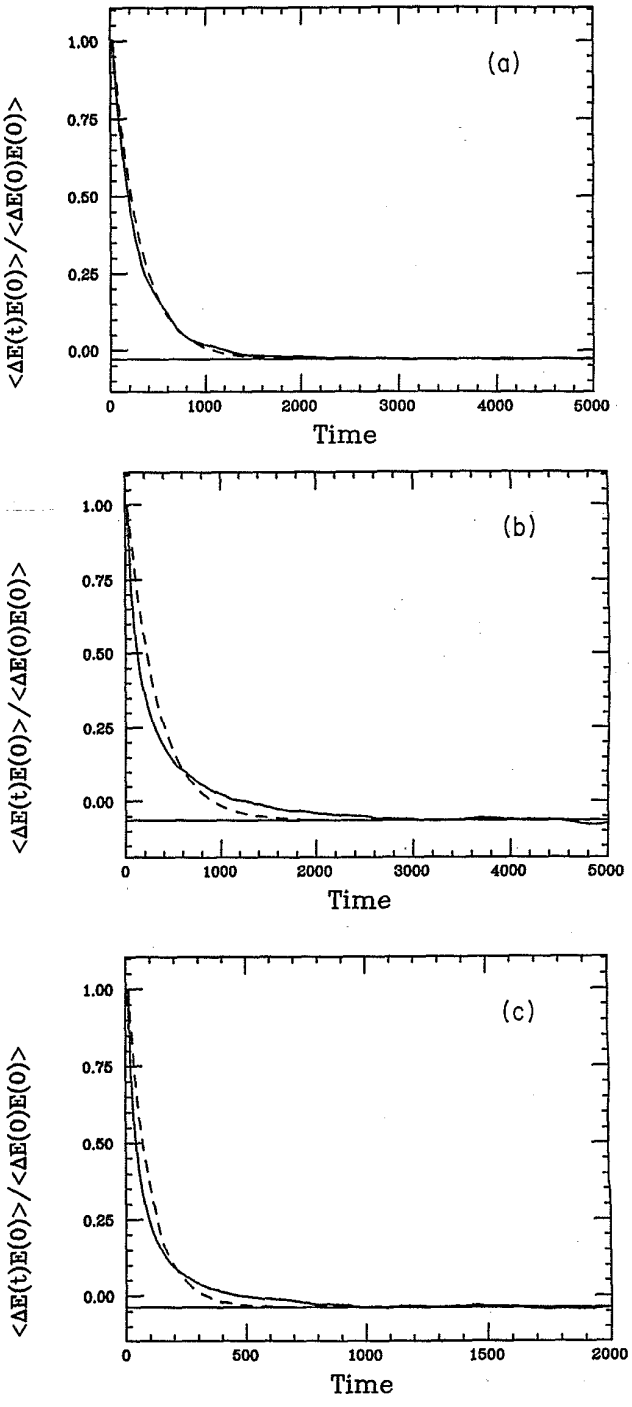

FIG. 3. Autocorrelation function of the energy fluctuations vs time (solid line). $\Delta E(t)$ is defined as $E(t)-\langle E\rangle$. The dashed line is the best-fit exponential. For all figures $T_A=10^{-4}$ and $T_C=2.0$, and (a) $T_M=4.0$, (b) $T_M=1.8$, (c) $T_M=1.0$.

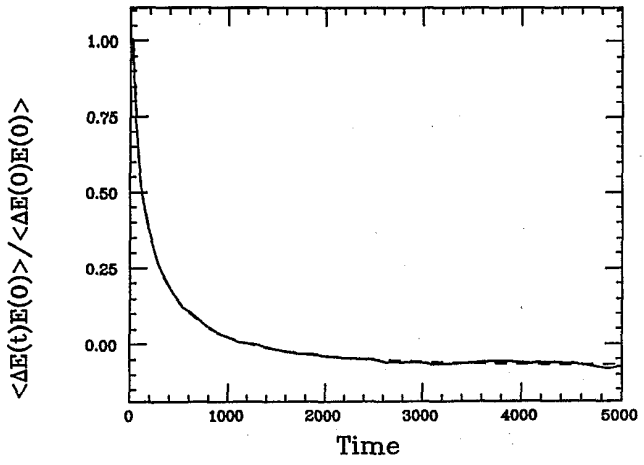

FIG. 4. Autocorrelation function of the energy fluctuations vs time (solid line). $\Delta E(t)$ is defined as $E(t)-\langle E\rangle$. The dashed line is the best fit with a stretched exponential. Parameters are as in Fig. 3(b).

Ref. 2, however, we should prove that our decay is described by a stretched exponential. That this is indeed the case is shown in Fig. 4, where we have plotted the results of numerical simulations together with the best-fitting stretched exponential: The agreement is astonishingly good.

Let us briefly discuss the other T_M regions: As expected, for $T_M \gg T_C$ the decay tends to be exponential [see Fig. 3(a)]. In the opposite limit [Fig. 3(c)] of $T_M \ll T_C$, bearing in mind also that $T_A/T_C \ll 1$, the decay tends to be exponential again because the system is essentially equivalent to a noisy harmonic oscillator with linear damping. Unfortunately, this seems to be beyond the possibilities of our simulations, because we still observed a stretched-exponential decay. However, given that for $T_M \ll T_C$ we assumed that the physics in our model changes quite radically (equivalence with a noisy linear oscillator with linear damping), we do not expect that the decay of the energy fluctuations in this region should bear much relevance to the model of Ref. 2.

VI. CONCLUSIONS

We have tried to extend the conclusions found for the finite though fairly large chain of nonlinear oscillators of Ref. 2 to the case of the thermodynamic limit, i.e., to a chain with an infinite number of nonlinear oscillators. The model used here, an infinite weakly nonlinear chain in the mode space with a Hamiltonian description, has been successfully simplified without lack of generality, i.e., keeping its Hamiltonian character, its nonlinearity, and an infinite number of degrees of freedom. For the reduced model we have been able to show that no energy threshold for the energy flow between modes is present, barring pathological initial conditions, thus suggesting that there is a transition to chaos for arbitrarily small energies of the Hamiltonian, and to draw interesting conclusions about the dynamics of the relaxation process. We think that our conclusions show fairly convincingly, in a model that is only very vaguely similar to the model of Ref. 2, that the conclusions presented there (the onset of nonexponential decays for energies below a certain critical average energy in the chain) are probably more general than expected: As such, there is some evidence that this behavior is probably universal, at least for a universality class of chains of coupled anharmonic oscillators. We also believe that the mechanism of restricted dynamics put forward to explain the observed anomalous relaxation is probably a sound one: In our language (stochastic differential equations) we would say that as we approach the critical density energy, we are in the presence of (more and more) rare fluctuations of the multiplicative force, which in turn will induce correlations at large times, thus leading to long-time tails. Finally, we would like to point out that our model suggests that a diffusion over restricted paths (Arnold web) can be present even in systems with infinite degrees of freedom.

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