Modulation-Induced Negative Differential Resistance in Bistable Systems

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This paper demonstrates that a wide range of driven, overdamped, nonlinear systems can exhibit negative differential resistance, where the equilibrium amplitude of oscillation decreases as the amplitude of the driving force increases. As an example, we examine the response of a parametrically driven bistable electronic circuit to both a dichotomous and a sinusoidal driving force. A general condition is derived for the onset of negative damping in the presence of a large amplitude modulation.

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Recently a great deal of attention has been focused on bistable systems which exhibit fluctuation-induced transitions.[1-5] Of particular interest are those systems which possess a macroscopic property x obeying a first-order evolution equation, in which the state of the environment is modeled by a randomly fluctuating parameter $\lambda(t)$, whose correlation function $g(t)=\langle[\lambda(t)-\langle\lambda\rangle][\lambda(0)-\langle\lambda\rangle]\rangle$ decreases with increasing time t. Although for any physical system, the correlation time[6] $T=\int_0^\infty dt \times t g(t)/\int_0^\infty dt\, g(t)$ is finite, most early studies were concerned with the limiting case of white-noise fluctuations, where $T=0$. This limit is convenient because equilibrium properties such as the probability density $P(x)$ can be obtained analytically from the appropriate Fokker-Planck equation, and for small but finite T the results form a starting point from which series expansions in T can be developed.[7,8]

For colored noise with a large T, few analytic results are available. With a view to probing the large-T limit, several workers have examined the response of periodically driven overdamped systems. Doering and Horsthemke[9,10] presented a detailed comparison of the equilibrium densities of a nematic liquid crystal (where x represents the nematic director) when $\lambda(t)$ is either a square-wave modulation or a dichotomous Markov process. Results for the equilibrium density of a cubic bistable electronic circuit under a variety of modulations have also been presented.[11] An important result of such studies is the observation of fluctuation-induced transitions which are absent in the white-noise limit. The aim of this Letter is to report a new fluctuation-induced transition. We show that there exist regions of negative differential resistance where the response amplitude of the system decreases as the amplitude of the external fluctuations increases. A detailed analysis of negative differential resistance in a cubic bistable system is presented and a criterion for the onset of this phenomenon in other overdamped nonlinear systems is obtained.

In order to illustrate negative differential resistance we first examine the response of an electronic circuit[12] designed to model the equation $dx/dt=-x^3+\lambda(t)x^2-Qx+R$, with $Q=3$ and $R=0.7$. The periodically fluctuating parameter $\lambda(t)=\lambda(t+\tau)$ is conveniently written in the form $\lambda(t)=\lambda_0+V_0h(t)$, where $h(t)$ is a normalized modulation with zero mean satisfying

$$\tau^{-1}\int_0^\tau dt\,|h(t)|=1,\quad \int_0^\tau dt\,h(t)=0.$$

This system is an example of the more general equation

$$dx/dt=f(x)+V_0h(t)g(x), \tag{1}$$

where, for the bistable circuit,

$$f(x)=-x^3+\lambda_0x^2-Qx+R \tag{2}$$

and

$$g(x)=x^2. \tag{3}$$

For fixed values of λ_0 and V_0, the response amplitude $A(\lambda_0,V_0)$ of a given equilibrium solution $x(t)$ is defined to be the difference between the maximum and minimum values x_+,x_- of $x(t)$. For a fixed period of $\tau=\frac{1}{6}$, the left-hand curves of Fig. 1 show the variation of A with drive amplitude V_0, obtained by our allowing the system to relax to equilibrium from a large positive value of $x(0)$. For $V_0=3.6\pm0.1$, the squares show results obtained from the electronic circuit[12] in the presence of a square-wave modulation of the form $h(t)=+1$ for $0\le t$

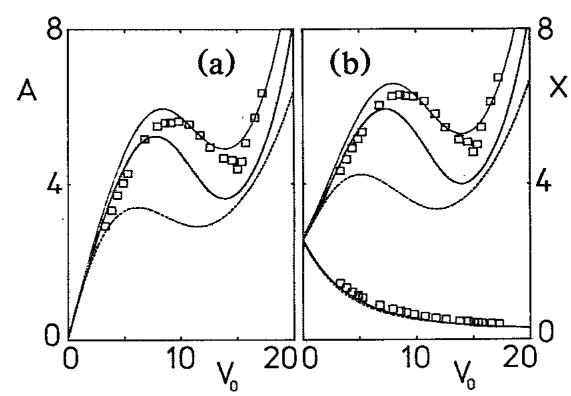

FIG. 1. (a) Response amplitude $A=|x_+-x_-|$ vs the drive amplitude V_0 for a variety of modulations; (b) the corresponding values of the limits $x_\pm$. Note the regions of negative slope which occur over a finite region of V_0.

(mod τ) < $\frac{1}{2}\tau$, and $h(t)=-1$ for $\frac{1}{2}\tau \leq t$ (mod τ) < τ. For $V_0=3.6$ and 3.65, respectively, the bold and thin solid lines show square-wave results obtained analytically by solution of the pair of simultaneous equations

$$\tfrac{1}{2}\tau=\int_{x_-}^{x_+} dx[f(x)+V_0 g(x)]^{-1}, \quad (4)$$

$$\tfrac{1}{2}\tau=\int_{x_+}^{x_-} dx[f(x)-V_0 g(x)]^{-1}. \quad (5)$$

The dashed line shows sine-wave results obtained by digital integration of Eq. (1) with $h(t)=\frac{1}{2}\pi\sin(2\pi t/\tau)$ and $V_0=3.6$. For comparison the right-hand figure shows the corresponding values of the limits $x_\pm$. These results clearly demonstrate the existence of a negative-differential-resistance region in which $dA/dV_0 < 0$.

To analyze this interesting behavior, we focus attention initially on the response of the system to a square-wave modulation. The analysis is simplified in the limit $V_0 \to \infty$, $\tau \to 0$, $\mu=\frac{1}{2}V_0\tau=$finite. In this limit the sum and difference of Eqs. (4) and (5) reduce to the form

$$\int_{x_-}^{x_+} dx\, f(x)/[g(x)]^2=0, \quad (6)$$

$$\int_{x_-}^{x_+} dx\,[g(x)]^{-1}=\mu. \quad (7)$$

For the electronic circuit, with $f(x)$ and $g(x)$ given by Eqs. (2) and (3), these yield

$$-\ln(x_+/x_-)+\lambda_0(x_-^{-1}-x_+^{-1}) - \tfrac{1}{2}Q(x_-^{-2}-x_+^{-2})+\tfrac{1}{3}R(x_-^{-3}-x_+^{-3})=0 \quad (8)$$

and

$$x_-^{-1}-x_+^{-1}=\mu. \quad (9)$$

Equations (8) and (9) demonstrate that for large V_0 the bounds $x_\pm$, and hence the amplitude $A=|x_+-x_-|$, approach universal functions of the parameter μ independent of τ and V_0 separately. In this limit, τ is simply a scaling factor for the drive amplitude V_0. For $\lambda_0=3.6$, the solid lines of Fig. 2 show plots of $x_\pm$ versus μ obtained from Eqs. (8) and (9). For comparison the dashed lines show plots obtained from solutions of the exact formulas (4) and (5) for a variety of V_0. These illustrate how the universal curves are approached with increasing V_0 and emphasize that negative resistance is a large-V_0 phenomenon. For clarity in Fig. 2 we have chosen to show values of x_+ only, for both positive and negative μ. Values of x_- for positive μ are obtained from the relation $x_-(\mu)=x_+(-\mu)$; i.e., by reflection about the vertical axis at $\mu=0$.

The curves of Fig. 2 cross the $\mu=0$ vertical axis at the three points x_i, $i=1$, 2, and 3. These are the three real roots of $f(x)$ which occur for this value of λ_0. x_1 and x_3 are stable roots of $f(x)$, and x_2 is unstable. Starting from any of the equilibrium values, x_i, of the unperturbed system, the effect of a nonzero μ is to produce an equilibrium solution $x(t)$ with bounds $x_\pm$ lying on either side of x_i. The periodic solution centered on x_2 is unstable and in practice will decay onto one of the other two periodic solutions. The unstable solution always forms part of a closed curve in the (μ,x_+) plane. This means that only one of the equilibrium solutions is stable for all μ, because with increasing μ the universal function will reach a point beyond which $x_+(\mu)$ becomes single valued. For the parameters of Fig. 2 this is the solution which evolves from the upper stable root of $f(x)$. The response amplitude $|x_+-x_-|$ for this solution is shown in Fig. 1(a).

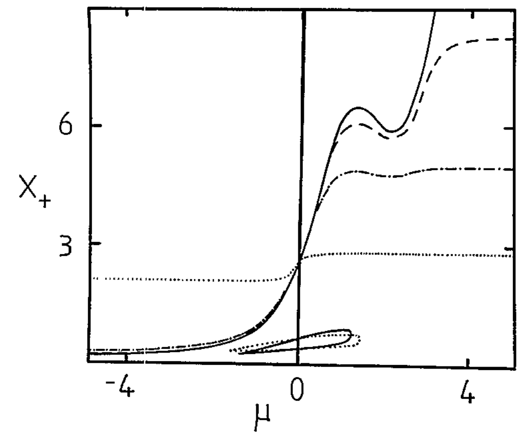

FIG. 2. Broken lines show $x_+ \to \mu$ for $\lambda_0=3.6$ obtained by solution of the pair of equations (4) and (5) for the following values of V_0: $V_0=0.2$ (dotted lines); $V_0=2$ (dot-dashed lines); $V_0=5$ (dashed lines). The solid curve is the solution of Eqs. (8) and (9), valid in the limit $V_0\to\infty$, $\tau\to 0$, $\mu=V_0\tau/2$ finite.

The extrema of the solid lines of Fig. 2 can be located by differentiation of Eqs. (6) and (7) with respect to μ and by combination of the results to yield

$$\frac{dx_+}{d\mu}=\frac{g(x_+)}{1-f(x_+)g(x_-)/f(x_-)g(x_+)}. \quad (10)$$

Hence an extremum occurs when the right-hand side of this equation vanishes. To simplify this result we note that, provided that the amplitude $|x_+-x_-|$ is nonzero and that $f(x)$ and $g(x)$ are analytic in the region $[x_-,x_+]$, $g(x)$ cannot vanish in this interval. To prove this one notes that if a point $x_0 \in [x_-,x_+]$ exists for which $g(x_0)$ vanishes, then the velocity at x_0 is finite. Consequently, if $g(x)$ is analytic, the distance moved from x_0 in any time interval of order τ vanishes as $\tau\to 0$, which contradicts the assertion that $|x_+-x_-|$ is nonzero. The combination of this result with Eq. (10) shows that the universal curves of x_+ versus μ possess extrema when the following condition is satisfied:

$$f(x_-)/f(x_+)=0. \quad (11)$$

Similarly curves of x_- versus μ possess extrema when $f(x_+)/f(x_-)=0$. Equation (11) is a general condition for the existence of extrema in the universal curve of Fig. 2 and locates the regions of negative slope. For a non-

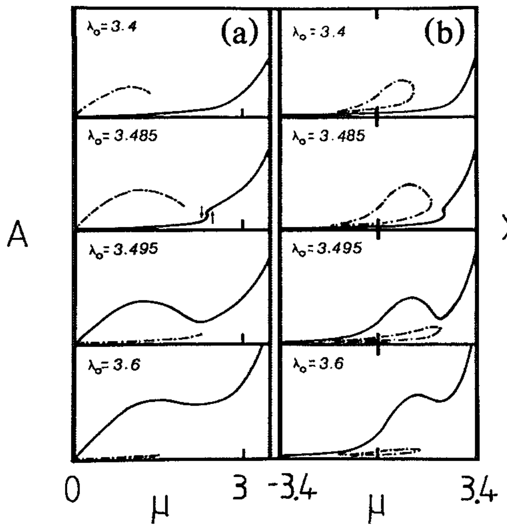

FIG. 3. Universal curves of (a) $A\rightarrow\mu$ and (b) $x_+\rightarrow\mu$ for a variety of λ_0.

linear system such as Eqs. (2) and (3), $f(x_-)$ and $f(x_+)$ cannot simultaneously vanish except perhaps at a small number of values of λ_0. Excluding this set of measure zero, Eq. (11) reduces to the condition

$$f(x_-)=0. \tag{12}$$

It is interesting to note that the potential of the fluctuation-free system is $U(x)=-\int dx f(x)$ and, despite the fact that the modulated system is never subject to this potential, Eq. (12) reveals that $U(x)$ determines the locations of the extrema. The right-hand column of Fig. 3 shows universal curves for a variety of different λ_0 and the left-hand column shows the corresponding response amplitudes. In each case the solid line represents the globally stable solution, and the dashed line shows the solution which is stable over only a finite range of μ.

The response of the system to a sine-wave modulation, as exemplified by the dashed line of Fig. 1, shows that the features we have been discussing are not simply an artifact of a square-wave modulation. To obtain an extension of the above analysis which encompasses a more general form of modulation, we consider a periodic fluctuation $h(t)=h(t+n\tau)$ which vanishes only at times $t=n\tau$ and $t=n\tau+t_+$, where n is an integer. In the limit $V_0\rightarrow\infty$, $\tau\rightarrow 0$, μ finite, Eq. (1) shows that $\tau dx/dt$ also changes sign at $n\tau$ and $n\tau+t_+$ only. To obtain expressions for $x_\pm$ consider an equilibrium solution $x(t)$ whose velocity dx/dt is positive over the half-cycle $n\tau\le t<n\tau+t_+$ and negative in the interval $n\tau+t_+\le t\le(n+1)\tau$. Over the positive (negative) half-cycle we denote the equilibrium solution by $x^+(t)$ [$x^-(t)$] and its inverse by $t^+(x)$ [$t^-(x)$]. The multiplication of Eq. (1) by an arbitrary function $F(x)$ then yields, in the limit $V_0\rightarrow\infty$, $\tau\rightarrow 0$, μ finite,

$$\int_{x_-}^{x_+}dx\,F(x)=2\mu\tau^{-1}\int_0^{t_+}dt\,h(t)F(x^+(t))g(x^+(t)),$$

$$\int_{x_+}^{x_-}dx\,F(x)=2\mu\tau^{-1}\int_{t_+}^{\tau}dt\,h(t)F(x^-(t))g(x^-(t)).$$

The subtraction of these equations yields the important result

$$\int_{x_-}^{x_+}dx\,F(x)=\mu\langle Fg\rangle, \tag{13}$$

where we have defined

$$\langle Fg\rangle=\tau^{-1}\int_0^{\tau}dt\,|h(t)|F(x(t))g(x(t)).$$

The choice of $F(x)=g^{-1}(x)$ shows that the square-wave equation (7) is valid whatever the shape of the parametric wave form $h(t)$. The choice of $F(x)=f(x)/g^2(x)$ shows that the general form of Eq. (6) is

$$\int_{x_-}^{x_+}dx\,f(x)/g^2(x)=\mu\epsilon, \tag{14}$$

where

$$\epsilon=\langle f/g\rangle. \tag{15}$$

The "coefficient of castellation" ϵ is identically zero for a square-wave modulation; it provides a measure of the deviation of $h(t)$ from such a wave form. The significance of a nonzero ϵ is mostly easily seen by writing $f(x)=f_0(x)+\lambda_0 g(x)$ [cf. Eqs. (2) and (3)], which yields for Eq. (14)

$$\int_{x_-}^{x_+}dx\,f_0(x)/g^2(x)=-\mu\lambda^*, \tag{16}$$

where $\lambda^*=\lambda_0-\epsilon$.

Equations (7) and (16) form rather a general pair of simultaneous equations for $x_\pm$. Differentiating these with respect to μ shows that the corresponding generalization of condition (12) for an extremum in x_+ is

$$f(x_-)=g(x_-)(\epsilon+\mu\,d\epsilon/d\mu). \tag{17}$$

For a square wave where $\epsilon=0$ and $\lambda^*=\lambda_0$ the values of $x_\pm$ for a given λ_0,μ can be read directly from the univer-

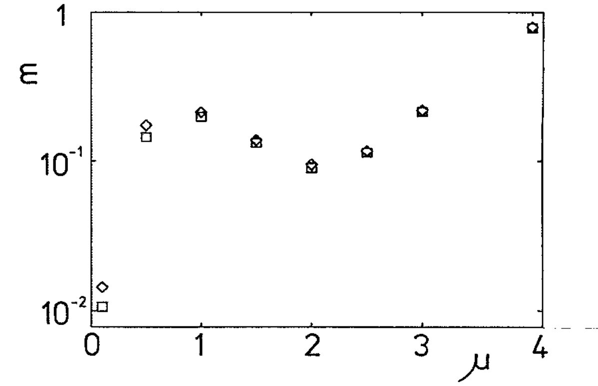

FIG. 4. Curves of the coefficient of castellation ϵ vs μ for a triangular wave (lozenges) and a sine wave (squares).

sal curves of Fig. 3. Remarkably Eq. (16) shows that whatever the form of $h(t)$, $x_{\pm}$ can be obtained from the same family of curves, provided λ_0 is replaced by λ^*. Thus the presence of a nonzero coefficient of castellation shifts the mean of $\lambda(t)$. For $\lambda_0=3.6$ Fig. 4 shows the variation of ϵ with μ for the bistable system (2) and (3) in the presence of sinusoidal and triangular wave forms. These results were obtained by numerical integration of Eq. (1). For each value of μ, f the system was allowed to relax to equilibrium from a large initial value of x and the solution $x(t)$ used to evaluate the integral on the right-hand side of Eq. (15). In the vicinity of the extrema of the $x_+(\mu)$ curves of Fig. 3, Fig. 4 shows that both ϵ and $d\epsilon/d\mu$ are small. Hence, for this example of a nonlinear system, the right-hand side of Eq. (17) remains close to zero and the positions of the extrema are given approximately by the simpler condition (12).

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