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Three-loop β Function(s) and Effective Potential in the Gross-Neveu Model

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ABSTRACT

We extend the perturbative analysis of the two dimensional Gross-Neveu model by computing within the MS scheme the three loop renormalization functions β, γ and γ_m in a two-parameter version of the model appropriate to the discussion of the RG properties of the effective potential, whose perturbative expansion is performed to the same order.

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Sec.1-Introduction

The evolution of our understanding of quantum field theories seems to be tightly related in most cases to a better and better knowledge of higher orders of perturbation theory. This may not exhaust the richness of a model, but it is certainly preliminary and complementary to any kind of non perturbative consideration .

Within a wide program of analysis of the perturbative and non perturbative features of two dimensional asymptotically free quantum field theories, and especially as a natural development of the results presented in Ref.[1], we were thus led to consider the third non trivial order of perturbation theory (three loops approximation) for the most general four fermion two dimensional model.

In this paper we shall present results concerning a special case of this general interaction, the so called Gross-Neveu (GN) model [2]. There are a few good reasons for this restriction and for a separate presentation: the GN model is essentially the only model in this class admitting a separate self-consistent treatment in the context of dimensional regularization and minimal subtraction renormalization; it provides a benchmark for future calculations, not only in the context of standard perturbation theory, but also within the $1/N$ expansion approach. Many non perturbative features of the model have been analytically determined, including the explicit factorized S-matrix. Last but not least we can take the chance to discuss and clarify a subtle question concerning the effective potential for composite operators and auxiliary fields, that has been till now understated or obscured in all discussions of this issue. To say it briefly, we shall give an answer to the question: "Which renormalization group equation (if any) is solved by the (renormalized) non-derivative part of the effective action of the model expressed as a function of the auxiliary bosonic fields introduced in order to eliminate the four fermion interaction?"

We stress that the complete three loops analysis of the most general model is such a cumbersome task that even a partial result like ours requires a considerable effort in concision in order to offer a self-contained but reasonably short and readable presentation.

This paper is organized as follows:
In sec.2 we define the model and very briefly comment about the regularization and renormalization procedures and notations we have adopted .
A cursory review of the one and two loops results is given in sec.3.
In sec.4 we classify the relevant three loops diagrams, discuss the main technical problems related to their computation in the schemes we have followed, and present the value of the divergent (pole) part of every topologically distinct group of diagrams entering the calculations.
In sec.5 we briefly discuss the relationships satisfied by the coefficients of the multiple poles and extract from the coefficients of the simple poles the renormalization group functions of the model, commenting about their behaviour for some special values of N .
In sec.6 we construct the abovementioned renormalized effective potential and discuss its properties from the point of view of the RG equations.

Sec.2-The Gross Neveu model in dimensional regularization

Let us consider the following Euclidean Lagrangian involving N Dirac fermions and enjoying a manifest $U(N)$ symmetry

$$L_1(g_o) = \bar{\psi}_o \partial\!\!\!/ \psi_o + \bar{\psi}_o \psi_o m_o - \frac{1}{2} g_o (\bar{\psi}_o \psi_o)^2 \tag{2.1}$$

In two dimensions this theory enjoys a hidden $O(2N)$ symmetry between the Majorana components of the fermion fields, and is therefore stable under renormalization against the appearance of different four fermion interactions [3]. In the limit $m_o \to 0$ the theory possesses a discrete γ_5 invariance $\psi \to \exp \frac{\pi}{2}\gamma_5\, \psi$, whose spontaneous breakdown leads to a non zero vacuum expectation value for the composite field $\bar{\psi}\psi$ and to a dynamical mass generation [2]. This model is known to be asympotically free in two dimensions and to possess (at $m_o = 0$) a factorized S matrix and infinitely many conservation laws. Moreover, the model is $1/N$ expandable, and many results are known about its large N limit and the first $1/N$ corrections [4].

Special properties are enjoyed by the models corresponding to several special values of N; among known properties we mention:
at $N = 1$ the quantum equivalence with the $U(1)$ Thirring model and the vanishing of the RG β function [5];
at $N = \frac{3}{2}$ the equivalence with the Supersymmetric Sine Gordon model at the critical coupling [6];
at $N = 2$ the decoupling into two distinct $SU(2)$ Thirring models [6];
at $N = 3$ the quantum equivalence with $SU(4)$ Thirring model [7].

We would like to add a very trivial, nevertheless very useful, observation: at $N = \frac{1}{2}$, due to the Majorana property, there is no interaction term, and therefore the model is just free field theory in a trivial realization. Keeping in mind this fact we shall be able to obtain highly non trivial checks of our perturbative results.

When we consider the dimensionally regularized version of the model, we must in principle worry about the possibility of losing multiplicative renormalizability, due to possible mixing with "evanescent operators" (i.e. operators whose tree level matrix elements are vanishing) induced by radiative corrections. As a matter of fact, no contributions from such operators appear in the perturbative expansion of the GN model up to three loops, and we suspect (although we have no proof of it) this to be a more general feature of the model, related to the $O(2N)$ invariance of its Majorana formulation.

Before delving into renormalization problems, let us consider a formal extension of the bare model to a two coupling constant system obtained by introducing an auxiliary field σ_o carrying the same quantum numbers as $\bar{\psi}_o\psi_o$:

$$L_2(g_o, h_o) = \bar{\psi}_o[\partial\!\!\!/ + m_o]\psi_o + \sigma_o\bar{\psi}_o\psi_o + \frac{1}{2}\frac{\sigma_o^2}{g_o} - \frac{1}{2}h_o(\bar{\psi}_o\psi_o)^2 \tag{2.2}$$

As long as we consider only fermionic Green's functions, by a trivial Gaussian integration we can show this Lagrangian to be related to the previous one by

$$L_2(g_o, h_o) \sim L_1(g_o + h_o) \tag{2.3}$$

We may therefore be tempted to start with the simpler bare Lagrangian $L_2(g_o, h_o = 0)$ and recover all interesting physical information via the equivalence eq.(2.3). However, we are sometimes interested in the correlation functions of the σ fields, as it is the case when considering the construction of an effective potential for the GN model, or when performing a $1/N$ expansion of it. In this case some dynamics for the σ field must be taken into account; albeit rather trivial, this dynamics will crucially affect the RG properties, and therefore the RG equations of the effective potential.

In order to clarify this point, we found it convenient to proceed to a comparison of the renormalized versions of the theory. By renormalizability, finite correlation functions can be obtained in the minimal subtraction scheme by performing all computations starting from the renormalized Lagrangians:

$$L_1(g) = Z(g)\bar{\psi}[\partial\!\!\!/ + Z_m(g)m]\psi - \frac{1}{2}Z_4(g)g(\bar{\psi}\psi)^2 \tag{2.4}$$

$$L_2(g,h) = Z(g,h)\bar{\psi}[\partial\!\!\!/ + Z_m(g,h)m]\psi + \frac{1}{2}Z_2(g,h)\frac{\sigma^2}{g} + Z_3(g,h)\sigma\bar{\psi}\psi - \frac{h}{2}Z_4(g,h)(\bar{\psi}\psi)^2 \tag{2.5}$$

where all functions Z, Z_m, Z_i depend on the renormalized couplings and are power series in the inverse powers of $\epsilon = d - 2$.

Again by renormalizability, we can assume this parametrization to be originated from renormalization of the bare coupling, mass and wavefunctions appearing in (2.1) or (2.2), via the relationships

$$\psi_o = Z^{\frac{1}{2}}(g)\psi \qquad m_o = Z_m(g)m \qquad \text{and} \qquad g_o = Z_g(g)g \tag{2.6}$$

implying

$$Z_4(g) = Z^2(g)Z_g(g) \tag{2.7}$$

or respectively

$$\psi_o = Z^{\frac{1}{2}}(g,h)\psi \qquad \sigma_o = Z_\sigma^{\frac{1}{2}}(g,h)\sigma \qquad m_o = Z_m(g,h)m \qquad \text{and}$$

$$g_o = Z_g(g,h)g \qquad h_o = Z_h(g,h)h \tag{2.8}$$

implying

$$Z_2(g,h) = \frac{Z_\sigma(g,h)}{Z_g(g,h)} \qquad Z_3(g,h) = Z_\sigma^{\frac{1}{2}}(g,h)Z(g,h) \qquad Z_4(g,h) = Z^2(g,h)Z_h(g,h) \tag{2.9}$$

We can now perform the Gaussian integration on the renormalized field σ and obtain

$$L_2(g,h) \sim Z(g,h)\bar{\psi}[\partial\!\!\!/ + Z_m(g,h)m]\psi - \frac{1}{2}\left(\frac{Z_3^2(g,h)}{Z_2(g,h)}g + Z_4(g,h)h\right)(\bar{\psi}\psi)^2 \tag{2.10}$$

By a trivial comparison with eq.(2.4) we obtain the relationships

$$Z(g,h) = Z(g+h) \tag{2.11a}$$

$$Z_m(g,h) = Z_m(g+h) \tag{2.11b}$$

$$Z_g(g,h)g + Z_h(g,h)h = Z_{g+h}(g+h)(g+h) \tag{2.11c}$$

Eqs.(2.11) sound very formal and void of physical content, until we come to consider the diagrammatic origin of the contributions to Z_3 and Z_4 respectively. We must then recognize that hZ_4 is generated by the 1PI contributions to the four fermion amplitude, and these can be non vanishing even if we set $h = 0$. This means we cannot avoid the dynamical generation of a bare effective four fermion vertex even if we pretended the corresponding renormalized coupling to be zero. The corresponding effective one-parameter Lagrangian $L_2(g, h = 0)$ can be reinterpreted as the renormalized version of $L_2(g_o, 0)$ only if we introduce the following unconventional relationship between bare and renormalized quantities [8]

$$\sigma_o = \tilde{Z}_\sigma^{\frac{1}{2}}(g)\sigma + \delta(g)gZ_g(g)Z(g)\bar{\psi}\psi \tag{2.12}$$

related to the mixing between the fields σ and $\bar{\psi}\psi$ carrying the same quantum numbers.

As a consequence, the following unusual relations hold:

$$Z_3(g,0) = \tilde{Z}_\sigma^{\frac{1}{2}}(g)Z(g)(1+\delta(g)) \tag{2.13a}$$

$$\tilde{Z}_4(g) \equiv \frac{hZ_4(g,0)}{g} = Z_g(g)Z^2(g)[1-(1+\delta(g))^2] \tag{2.13b}$$

and we can solve eqs.(2.13) to

$$\tilde{Z}_\sigma(g) = \frac{Z_3^2(g,0) + Z_2(g,0)\tilde{Z}_4(g)}{Z^2(g)} = Z_2(g,0)Z_g(g) \tag{2.14a}$$

$$1+\delta(g) = \left[1 + \frac{Z_2(g,0)\tilde{Z}_4(g)}{Z_3^2(g,0)}\right]^{-\frac{1}{2}} \tag{2.14b}$$

$\tilde{Z}_4(g)$ and $\delta(g)$ are higher order quantities both in standard perturbation theory and in the $1/N$ expansion, but they are certainly relevant to our computation, and by no means irrelevant in the discussion of the RG properties of the effective potential.

One of the main purposes of this work is the actual three-loop computation of all the Z functions defined in this section for arbitrary g and h. Postponing to sec.4 and 5 the discussion of some technicalities and checks of the computation, we want to introduce here some general observations and fix our notations.

Let's notice first that in the minimal subtraction scheme the coefficients of the inverse powers of ϵ appearing in the Z functions, being directly related to the renormalization group functions $\beta_g, \beta_h, \gamma_m, \gamma_\sigma, \gamma_\psi$, cannot depend on the low energy behaviour of the Feynman amplitudes. This means that we have at our disposal a number of alternative approaches to the computation of these coefficients, and, as far as these approaches are computationally viable, they provide independent determinations of the quantities we are going to compute, and their consistency is a formidable check of accuracy in the computation.

Let's discuss the three alternatives we have been to a different extent considering in our work:

(a) When starting from the model with vanishing bare mass $m = 0$ no mass counterterms are needed in dimensional regularization, and therefore all Feynman amplitudes depend only on the scale of the external momenta. Those kinematical configurations depending effectively on only one external momentum scale will then show a factorized dependence on this scale, fixed by naive power counting in d-dimensions. As a consequence, at least for these special configurations of momenta, all of the one- and two-loop diagrams and many of the three-loop diagrams can be computed in closed form in terms of special functions, and their pole contributions can be expressed in terms of rational numbers.

In the actual computation, we found it very convenient to absorb a bunch of numerical factors, whose $\epsilon \longrightarrow 0$ limit is $\frac{1}{\pi}$, in a redefinition of the couplings, by the substitutions

$$4\,\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\,\frac{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(d-1)}\,g \longrightarrow g \tag{2.15a}$$

$$4\,\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\,\frac{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(d-1)}\,h \longrightarrow h \tag{2.15b}$$

The independence of the RG functions from these rescalings (apart from suppression of a trivial dependence on π) is a well known result.

This is certainly the simplest approach to the computation of the fermion wavefunction renormalization constant; it also easily applies to most three-loop topologies, but it is useless in order to determine the mass renormalization constant and to compute the effective potential.

(b) Alternatively, we can start with a massive version of the model and compute all diagrams at zero external momenta. The mass provides the factorized scale whose power dependence is fixed by power counting; thanks to some computational tricks we shall discuss later every three-loop diagram can now be explicitly computed and its pole part can be shown to have coefficients that are all rational numbers.

(c) A slight modification of procedure (b), already discussed in Ref.[1], can simplify the computations or be used as a further check of their consistency and accuracy.

We start from the observation that the massive fermion propagator takes the form

$$S(p) = \frac{1}{i\not{p} + m} = \frac{-i\not{p} + m}{p^2 + m^2} \tag{2.16}$$

However, in a renormalizable infrared finite theory, like the one we are analyzing, we expect on general grounds to have no dependence of RG functions on the mass term appearing on the numerator of eq.(2.16). We can therefore perform all calculations using consistently everywhere the effective propagator

$$\tilde{S}(p) = \frac{-i\not{p}}{p^2 + m^2} \tag{2.17}$$

or alternatively introduce the mass term and verify that its contribution to divergent terms cancels between any diagram and its counterterms. We successfully performed this control for all independent topologies appearing in three-loop diagrams we had to compute, and this provided us further confidence in the accuracy of our results. Needless to say, procedure (c) does not apply to the computation where the mass parameter has an intrinsic meaning, like mass renormalization and contributions to the effective potential.

In applying procedures (b) and (c) we adopted the following notation, defining the simple pole by a rescaling of the couplings:

$$g \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m^2} \equiv gI \longrightarrow -\frac{gm^\epsilon}{2\epsilon} \tag{2.18a}$$

hence

$$I = \Gamma(1 - \frac{d}{2}) m^{d-2} \frac{1}{(4\pi)^{\frac{d}{2}}} \rightarrow -\frac{m^\epsilon}{2\epsilon} \qquad \hat{I} = -\frac{1}{2\epsilon} \tag{2.18b}$$

Before concluding this section, we must mention another feature of the model that proves relevant to the computations of Z functions.

Let's consider, in the contest of the procedure (b), the set of diagrams contributing to the inverse two-point fermionic function. Since we are working at zero external momentum, these diagrams are simply the contributions to the mass renormalization of the model. To make this identification we must however be careful in including the one σ particle reducible tadpole contributions; otherwise we would not be able to reproduce eq.(2.11b).

Let's now define the function

$$m_o(m, g, h) \equiv m Z_m(g + h) \equiv m[Z_I(g, h) Z_R(g, h)] \tag{2.19}$$

where we have factorized the contribution Z_R of the reducible tadpole diagrams.

Let's now take the derivative with respect to the mass parameter of the abovementioned set of diagrams; from the relationship

$$\frac{\partial}{\partial m} S(p, m) = -S^2(p, m) \tag{2.20}$$

it is immediate to recognize that this derivative corresponds graphically to the insertion of an external σ line carrying zero external momentum in all possible ways along internal fermions lines. Whenever the insertion occurs in a irreducible self-energy diagram or subdiagram it generates an irreducible vertex diagram or subdiagram. Whenever the insertion acts on a reducible tadpole diagram or subdiagram it generates the insertion of a dressed σ line in a vertex diagram.

If we consider only the divergent parts of the self-energy diagrams, we know we are simply computing the contributions to the function $m_o(m, g, h)$, whose dependence on m is all contained in the trivial prefactor m. The divergent parts of the (irreducible and reducible) vertex diagrams generated by the above insertions are in turn obviously related to the renormalization functions $Z_2(g, h)$ and $Z_3(g, h)$.

The factorization introduced in eq.(2.19) allows us to formalize this relationship by writing down the following "Ward identities":

$$Z_3(g, h) = Z(g, h) Z_I(g, h) \tag{2.21}$$

$$Z_2^{-1}(g, h) = Z_R(g, h) \tag{2.22}$$

and by recalling eq.(2.19) and (2.11a) we also find the following constraint:

$$Z_3(g, h) Z_2^{-1}(g, h) = Z\, Z_m\, (g + h) \tag{2.23}$$

As usual, we can take eqs.(2.21-2.23) either as a computational tool in order to reduce the harder task of computing Z_2 and Z_3 to the simpler problem of evaluating self-energy and tadpole contributions, or as a consistency condition to be verified after independent calculation of all the quantities involved. We shall draw all the consequences of these relationships in the dicussion of the RG functions presented in sec.5.

A last important observation concerns the evaluation of irreducible four-fermion diagrams contributing to Z_4. It's easy to get convinced, by working within scheme (a), that whenever the external fermion lines meet an even number of interaction vertices any resulting divergent contribution, being independent of external momenta, might only be proportional to $\gamma_\mu \otimes \gamma_\mu$. The renormalizability of the Gross Neveu model therefore insures us that the overall contribution of such classes of diagrams must be identically zero to any order of the loop expansion. We verified explicitly the cancellations implied by this statement, and we shall therefore drop these diagrams from our forthcoming discussion.

From now on, all the results we shall quote and discuss will be presented in the form obtained by applying procedure (b) of sec.2, unless otherwise stated. We stress that, besides all other consistency checks we shall discuss, in many cases independent determinations of the same results based on procedure (c) and (whenever feasible) procedure (a) have been obtained.

The order of presentation is the following: we shall first separately discuss the wave-function and mass renormalization, then extract the values of Z_2 and Z_3 from these results, and finally evaluate the irreducible contributions to Z_4. We shall drop out all finite contributions to single diagrams: finite parts will only be discussed in the contest of the effective potential calculation, where they become important and must be explicity computed.

Sec.3-One and two loops renormalization

The purpose of this section is twofold:
a) we want to review and extend previous computations by considering the jet untouched question of mass renormalization and introducing the two parameter formulation discussed in sec.2;
b) we want to fix the notation and to give a mathematical and graphical definition of the Lagrangian counterterms to be applied in the classification of counterterm insertions accompanying the three loop diagrams.

As a matter of presentation, we shall always denote by a number in square brakets the value of the diagram drawn in the corresponding figure, computed in scheme (b) if not otherwise stated, and irrespective of its dependence from the coupling constants. Establishing this dependence is in most cases a reasonably simple combinatorial exercise and in general no details will be offered, but for the graphical decomposition of each topological class into the subclasses obtained by performing all different contractions of color indices. Solid lines are fermion propagators. Dashed lines represent both four-fermion vertices (coupling h) and σ propagators (coupling g); only in the second case they are reducible.

At the one loop level there is no wavefunction renormalization: the only topologically distinct contribution to the two point function is drawn in fig.1 and is manifestly independent of the external momentum.

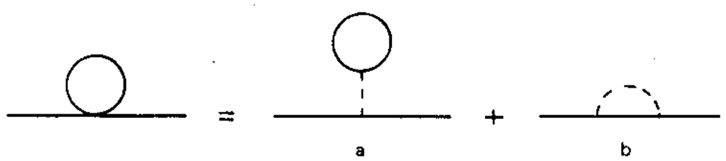


fig 1

The value of this contribution is:

$$[1] = \int \frac{d^d q}{(2\pi)^d} \left[-NTr \frac{1}{i\not{q} + m} + \frac{1}{i\not{q} + m} \right] = -2(N - \frac{1}{2}) Im \tag{3.1}$$

As a consequence, by recognizing that diagram [1a], when the dashed propagator is a σ line, is a reducible tadpole, we obtain the following one loop results:

$$Z^{(1)} = 0 \tag{3.2a}$$

$$Z_R^{(1)} = -2Ng\hat{I} = -Z_2^{(1)} \tag{3.2b}$$

$$Z_I^{(1)} = [g - 2(N - \frac{1}{2})h]\hat{I} = Z_3^{(1)} \tag{3.2c}$$

and by straightforward considerations

$$Z_4^{(1)} = [2g - 2(N-1)h]\hat{I} \tag{3.3}$$

Eqs (3.2) and (3.3) in turn may be used to derive the relationships:

$$Z_m^{(1)} = -2(N - \frac{1}{2})(g+h)\hat{I} \tag{3.4a}$$

$$Z_{g+h}^{(1)} = -2(N-1)(g+h)\hat{I} \tag{3.4b}$$

We are now ready to define the one loop counterterms, drawn in fig.2:

fig 2

Mass counterterm:

$$[2a] = 2(N - \frac{1}{2})(g+h)\hat{I}m \tag{3.5a}$$

σ wavefunction counterterm:

$$[2b] = -2Ng\hat{I} \tag{3.5b}$$

3-vertex counterterm:

$$[2c] = [g - 2(N - \frac{1}{2})h]\hat{I} \tag{3.5c}$$

4-vertex counterterm:

$$[2d] = [2g - 2(N-1)h]h\hat{I} \tag{3.5d}$$

For future purposes it is also convenient to define a total effective vertex counterterm, drawn in fig.3:

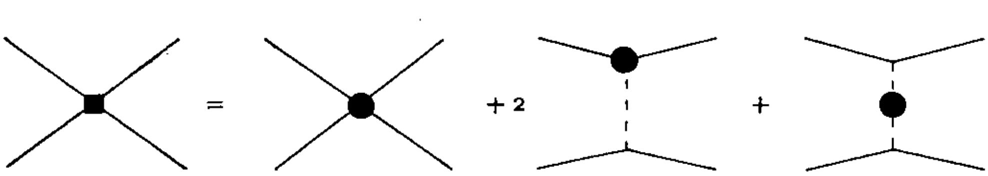


fig 3

$$[3] = -2(N-1)(g+h)^2\hat{I} \tag{3.6}$$

The two-loops contributions to the two point function belong to two different topologies, which are drawn in figs.4 and 5.

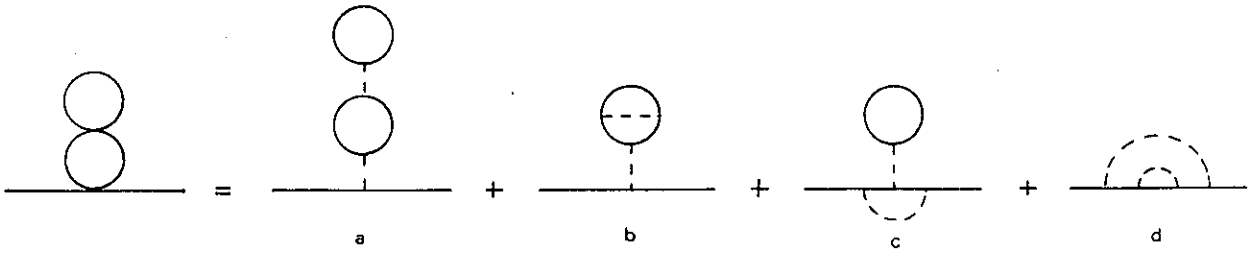


fig 4

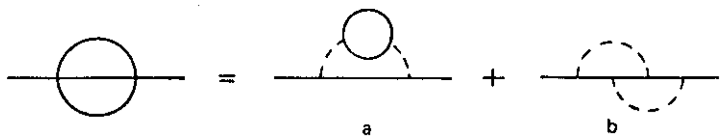


fig 5

Contributions corresponding to diagram [4] are independent of the external momentum and when the vertical dashed line is a σ propagator diagrams [4a],[4b] and [4c] are reducible tadpoles. The value of diagram [4] is

$$[4] = \int \frac{d^d q}{(2\pi)^d}\frac{d^d r}{(2\pi)^d}\left\{ N^2 Tr \frac{1}{i\not q + m} Tr \frac{1}{(i\not r + m)^2} - N Tr \frac{1}{i\not q + m}\frac{1}{(i\not r + m)^2} - \right.$$

$$\left. - N\left[Tr \frac{1}{i\not q + m}\right]\frac{1}{(i\not r + m)^2} + \frac{1}{i\not r + m}\frac{1}{i\not q + m}\frac{1}{i\not r + m}\right\} =$$

$$= 4(N - \frac{1}{2})^2 \int \frac{d^d q}{(2\pi)^d}\frac{1}{i\not q + m}\int \frac{d^d r}{(2\pi)^d}\frac{1}{(i\not r + m)^2} = -4(N - \frac{1}{2})^2(1+\epsilon)I^2 m \qquad (3.7)$$

The value of diagram [5] is obtained from

$$[5] = \int \frac{d^d l_1}{(2\pi)^d}\frac{d^d l_2}{(2\pi)^d}\left\{ -N\frac{1}{i\not l_3 + m} Tr \frac{1}{i\not l_1 + m}\frac{1}{-i\not l_2 + m} + \frac{1}{i\not l_1 + m}\frac{1}{-i\not l_2 + m}\frac{1}{i\not l_3 + m}\right\} \equiv$$

$$\equiv -2(N - \frac{1}{2})\int \frac{d^d l_1}{(2\pi)^d}\frac{d^d l_2}{(2\pi)^d}\frac{m^2 + l_1 \cdot l_2}{(l_1^2 + m^2)(l_2^2 + m^2)}\frac{1}{i\not l_3 + m} \qquad (3.8)$$

where $l_1 + l_2 + l_3 = k$, and the identity is easily demonstrated. Evaluating this expression we find:

$$[5] = (N - \frac{1}{2})[I^2(m + \frac{1}{2}i\not k\epsilon) - 3Jm] + \text{finite momentum dependent terms} \qquad (3.9)$$

where

$$J = \int \frac{d^d l_1}{(2\pi)^d} \frac{d^d l_2}{(2\pi)^d} \frac{1}{l_1^2 + m^2} \frac{1}{l_2^2 + m^2} \frac{1}{l_3^2 + m^2} m^2 \qquad l_1 + l_2 + l_3 = 0 \tag{3.10}$$

is a dimensionless finite quantity when $d = 2$.

Let's now draw the corresponding one loop diagrams with inserted one loop counterterms. The counterterms associated with diagrams [4] and [5] are drawn in figs.6 and 7.

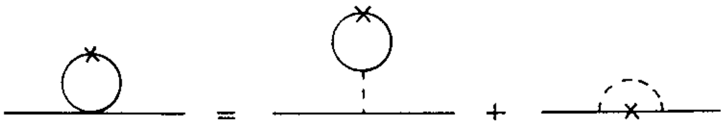


fig 6

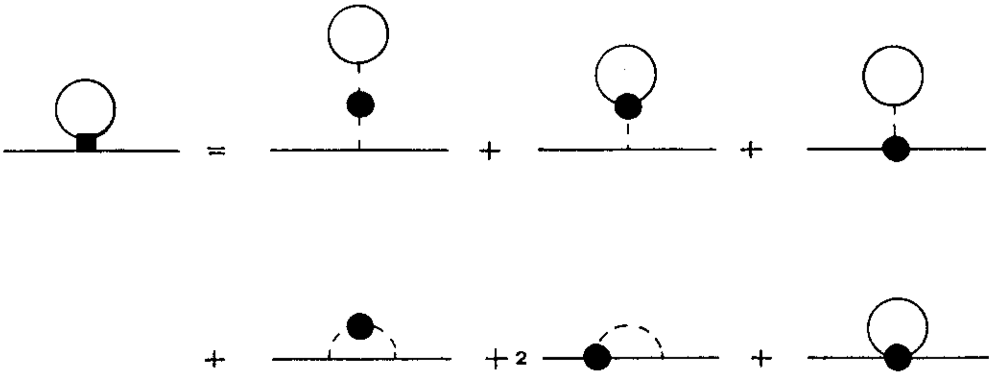


fig7

Each contribution can be shown to be proportional to $(g + h)^2$, and the respective weights are

$$[6] = 4(N - \frac{1}{2})^2(1 + \epsilon)I\hat{I}m \tag{3.11a}$$

$$[7] = 4(N - 1)(N - \frac{1}{2})I\hat{I}m \tag{3.11b}$$

Computations are straightforward and the results can be summarized as follows:

$$Z^{(2)} = -\frac{1}{4}(N - \frac{1}{2})(g + h)^2 \hat{I} \tag{3.12a}$$

$$Z_R^{(2)} = 4N(N - \frac{1}{2})g(g + h)\hat{I}^2 \tag{3.12b}$$

$$Z_I^{(2)} = \frac{1}{4}(N - \frac{1}{2})(g + h)^2 \hat{I} + [4(N - \frac{1}{2})(N - \frac{3}{4})h^2 - 6(N - \frac{1}{2})gh - (N - \frac{3}{2})g^2]\hat{I}^2 \tag{3.12c}$$

Eqs.(3.12) immediately imply

$$Z_2^{(2)} = [2Ng^2 - 4N(N - \frac{1}{2})gh]\hat{I}^2 \tag{3.13a}$$

$$Z_3^{(2)} = [4(N-\frac{1}{2})(N-\frac{3}{4})h^2 - 6(N-\frac{1}{2})gh - (N-\frac{3}{2})g^2]\hat{I}^2 \tag{3.13b}$$

and

$$Z_m^{(2)} = \frac{1}{4}(N-\frac{1}{2})(g+h)^2\hat{I} + 4(N-\frac{1}{2})(N-\frac{3}{4})(g+h)^2\hat{I}^2 \tag{3.14}$$

This is not however the end of the story, since at the two-loop level an irreducible, h independent contribution to the four-point function comes from the diagrams in fig.8, when the dashed lines are σ propagators .

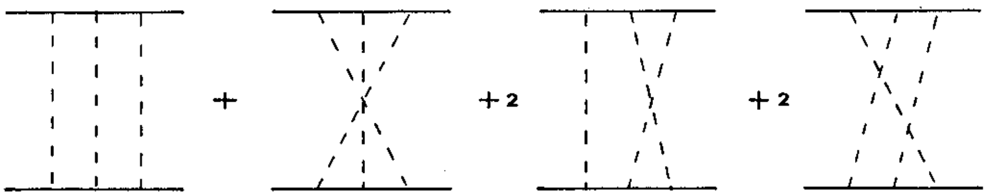


fig 8

Without delving into the details of the computation, we just quote the result (computed in scheme (b)):

$$[8] = -\frac{1}{2}\epsilon(1-\epsilon)I^2 - 3(1+\epsilon)J \to \frac{1}{4}\hat{I} + \text{finite terms} \tag{3.15}$$

Therefore

$$hZ_4^{(2)} = [4N^2h^3 - 6Nh^2(g+h) + (4-2N)h(g+h)^2]\hat{I}^2 - \frac{1}{4}(g+h)^3\hat{I} \tag{3.16}$$

and finally

$$Z_{g+h}^{(2)} = 4(N-1)^2(g+h)^2\hat{I}^2 + \frac{1}{2}(N-1)(g+h)^2\hat{I} \tag{3.17}$$

Eqs.(3.14) and (3.17) reflect the general renormalization properties of the model expressed in eqs.(2.11). We stress that we have not imposed this result; rather we computed every single contribution and obtained the quoted values satisfying this consistency check.

We can now finally define the two-loop counterterms, drawn in fig.9.

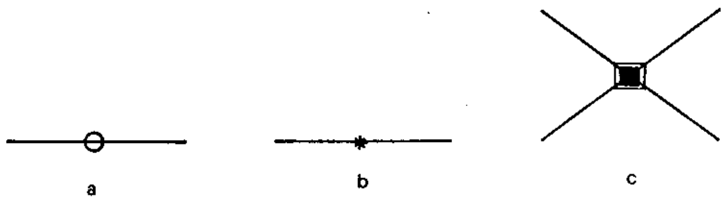


fig 9

Wavefunction counterterm:

$$[9a] = i\not{k}\frac{1}{4}(N - \frac{1}{2})(g+h)^2\hat{I} = -Z^{(2)}i\not{k} \tag{3.18a}$$

Mass counterterm:

$$[9b] = -4(N - \frac{1}{2})(N - \frac{3}{4})\hat{I}^2(g+h)^2 m = -[ZZ_m]^{(2)}m \tag{3.18b}$$

Effective vertex countertem:

$$[9c] = [4(N-1)^2\hat{I}^2 - \frac{1}{4}\hat{I}](g+h)^3 \tag{3.18c}$$

Sec.4-The three-loop diagrams

Following the pattern of presentation introduced in sec.3 we shall now first discuss the different classes of three-loop contributions to the two-point function, classified according to their topological structure.

We recognize the existence of five topologically distinct classes of diagrams, drawn in figs.10-13.

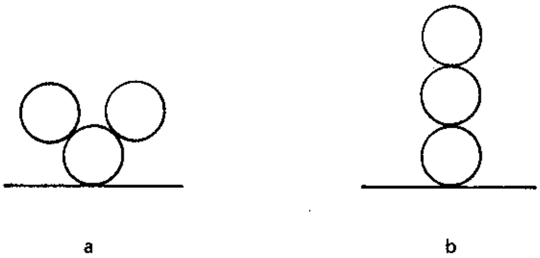


fig 10

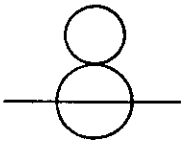


fig 11

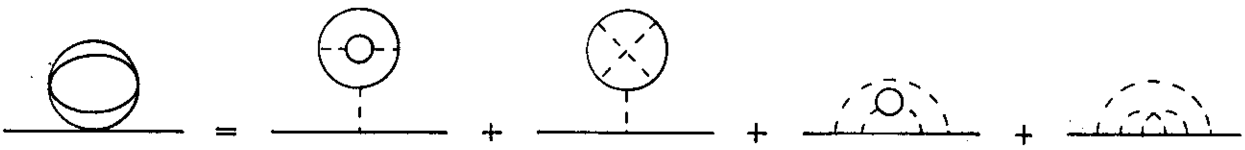


fig 12

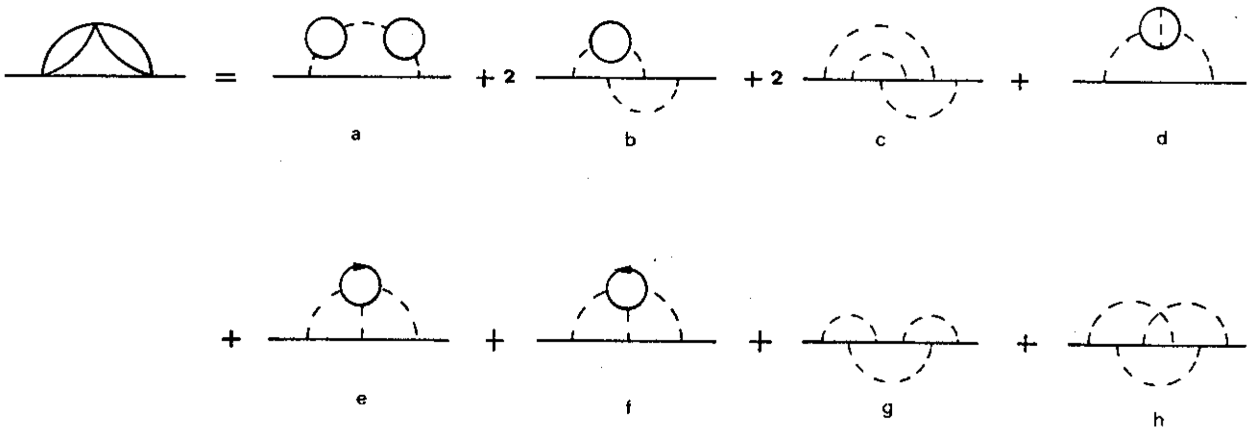


fig 13

Contributions corresponding to the topologies [10] are independent of the external momentum. Moreover, diagrams [10] and [11] simply amount to factorized tadpole insertions into lower order diagrams, which implies they can only contribute to mass renormalization, and allow us to perform a zero external momentum calculation of diagram [11]. The only truly original calculations will be those requested by the momentum-independent diagrams [12] and especially by [13].

Let's start our analysis from the topology drawn in fig.10a. Without entering any (rather trivial) detail we shall only quote the final result:

$$[10a] = -4(N - \frac{1}{2})^3 \epsilon(1+\epsilon) I^3 m \tag{4.1}$$

When including the counterterm contributions coming from the diagrams drawn in fig.14 one easily finds out that the total contribution is convergent, as expected on general grounds.

$$[10a] + [14] = -4(N - \frac{1}{2})^3 (1+\epsilon)(\epsilon I)(I - \hat{I})^2 m \tag{4.2}$$

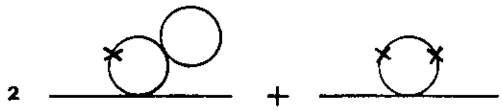


fig 14

The analysis of diagram [10b] is also rather trivial. The resulting contribution is:

$$[10b] = -8(N - \frac{1}{2})^3 (1+\epsilon)^2 I^3 m \tag{4.3}$$

As a check of these calculations, we may verify the simple relationship:

$$2[10a] + [10b] = 2(N - \frac{1}{2}) I m \frac{\partial}{\partial m}[4] \tag{4.4}$$

Similarly, one can easily recognize that the diagram in fig.11 leads to the expression:

$$[11] = 4(N - \frac{1}{2})^2 I m \int \frac{d^d l_1}{(2\pi)^d} \frac{d^d l_2}{(2\pi)^d} \frac{m^2 + l_1 \cdot l_2}{(l_1^2 + m^2)(l_2^2 + m^2)} \frac{m^2 - l_3^2}{(l_3^2 + m^2)^2} \tag{4.5}$$

where $l_1 + l_2 + l_3 = k$, and one can prove the relationship:

$$[11] = 2(N - \frac{1}{2}) I m \frac{\partial}{\partial m}[5] \underset{k \to 0}{\longrightarrow} 2(N - \frac{1}{2})^2 (1 + 2\epsilon)[I^3 - 3IJ] m \tag{4.6}$$

Let's now consider the topology in fig.12. The overall contribution of this set of diagrams is obtained after some manipulation of the integrands in the corresponding Feynman integrals, and can be cast into the form

$$[12] = -(N-\frac{1}{2})^2\frac{\partial}{\partial m}\int \frac{d^dl_1}{(2\pi)^d}\frac{d^dl_2}{(2\pi)^d}\frac{d^dl_3}{(2\pi)^d}\frac{m^2+l_1\cdot l_2}{(l_1^2+m^2)(l_2^2+m^2)}\frac{m^2+l_3\cdot l_4}{(l_3^2+m^2)(l_4^2+m^2)} =$$

$$= 4(N-\frac{1}{2})^2 m\int \frac{d^dl_1}{(2\pi)^d}\frac{d^dl_2}{(2\pi)^d}\frac{d^dl_3}{(2\pi)^d}\left[2\frac{m^2+l_1\cdot l_2}{(l_1^2+m^2)^2}-\frac{1}{l_1^2+m^2}\right]\frac{1}{l_2^2+m^2}\frac{m^2+l_3\cdot l_4}{(l_3^2+m^2)(l_4^2+m^2)} \tag{4.7}$$

where $l_1+l_2+l_3+l_4=0$. Let's briefly describe the computational tricks involved in the evaluation of this integral. First we define the typical three-loop integrals:

$$A_n = \int \frac{1}{(l_1^2+m^2)^n}\frac{1}{l_2^2+m^2}\frac{1}{l_3^2+m^2}\frac{1}{l_4^2+m^2}m^{2n} \tag{4.8}$$

$$B_n = \int \frac{l_1\cdot l_2}{(l_1^2+m^2)^n}\frac{1}{l_2^2+m^2}\frac{1}{l_3^2+m^2}\frac{1}{l_4^2+m^2}m^{2n-2} \tag{4.9a}$$

$$B'_n = \int \frac{l_3\cdot l_4}{(l_1^2+m^2)^n}\frac{1}{l_2^2+m^2}\frac{1}{l_3^2+m^2}\frac{1}{l_4^2+m^2}m^{2n-2} \tag{4.9b}$$

$$C_n = \int \frac{l_1\cdot l_2}{(l_1^2+m^2)^n}\frac{l_3\cdot l_4}{l_2^2+m^2}\frac{1}{l_3^2+m^2}\frac{1}{l_4^2+m^2}m^{2n-4} \tag{4.10}$$

and notice that A_n integrals are finite in the neighborhood of $d=2$, and that the following relation holds:

$$B_1 = \frac{1}{3}(A_1 - I^3) \tag{4.11}$$

In general, all type B, B' and C integrals may be computed in terms of I and type-A integrals, but we shall not belabor on this point . In the case at hand we may use the identity:

$$0 \equiv \int \frac{\partial}{\partial l_1^\alpha}\left[\frac{1}{l_1^2+m^2}\frac{1}{l_2^2+m^2}\frac{1}{l_3^2+m^2}\frac{1}{l_4^2+m^2}(-m^2l_2^\alpha + l_3\cdot l_4 l_1^\alpha)\right] \tag{4.12}$$

in order to prove the following relationship:

$$2A_2 + 2C_2 + 2(B_2+B'_2) + \epsilon(A_1+B_1) = 0 \tag{4.13}$$

By straightforward substitutions in eq.(4.7) we can find

$$[12] = 4(N-\frac{1}{2})^2[2A_2+2C_2+2(B_2+B'_2)-(A_1+B_1)]m =$$

$$= \frac{4}{3}(N-\frac{1}{2})^2(1+\epsilon)(I^3-4A_1)m \tag{4.14}$$

where we made use of eqs.(4.11) and (4.13). For further reference we shall also quote the following result:

$$\lim_{d\to 2} A_1 = \frac{7}{64\pi^3}\zeta(3) \tag{4.15}$$

Let's finally discuss the topology drawn in fig.13. These are the only three-loop diagrams carrying a contribution to wavefunction renormalization , besides their mass renormalization effect. In order to simplify the computational task, we found it convenient to calculate the wavefunction renormalization effects in scheme a), thus dramatically reducing the algebraic manipulations. The relevant diagrams and counterterms are drawn in fig.15.

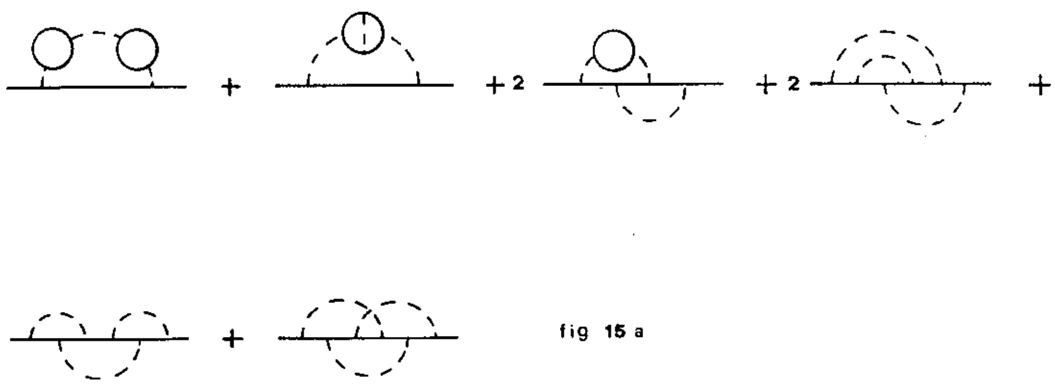


fig 15 a

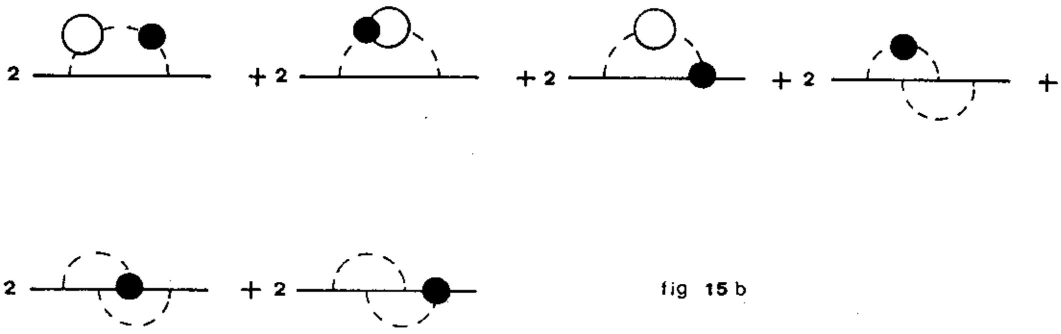


fig 15 b

The value of these diagrams in scheme a) can be obtained by first proving that

$$[15a] = 4(N-\frac{1}{2})(N-1)\int \frac{d^dl_1}{(2\pi)^d}\frac{d^dl_2}{(2\pi)^d}\frac{d^dl_3}{(2\pi)^d}\frac{\not l_1}{l_1^2}\frac{l_2\cdot(k-l_1-l_2)}{l_2^2(k-l_1-l_2)^2}\frac{l_3\cdot(k-l_1-l_3)}{l_3^2(k-l_1-l_3)^2} \tag{4.16a}$$

$$[15b] = -16(N-\frac{1}{2})(N-1)\frac{1}{\epsilon}\int \frac{d^dl_1}{(2\pi)^d}\frac{d^dl_2}{(2\pi)^d}\frac{d^dl_3}{(2\pi)^d}\frac{\not l_1}{l_1^2}\frac{l_2\cdot(k-l_1-l_2)}{l_2^2(k-l_1-l_2)^2} \tag{4.16b}$$

It is now an exercise in the calculation of massless dimensionally regularized integrals to show that

$$[15] = \frac{1}{4}(N - \frac{1}{2})(N - 1)\left[\frac{1}{3\epsilon^2} - \frac{1}{6\epsilon}\right] i\not{k} + \text{finite terms} \tag{4.17}$$

Mass renormalization effects can now be computed by considering simply the zero-momentum contribution of diagrams [13]. After some algebraic manipulations, one can show that this contribution can be cast into the form

$$[13] = 4(N-\frac{1}{2})(N-1)\int \frac{d^d l_1}{(2\pi)^d}\frac{d^d l_3}{(2\pi)^d}\frac{d^d l_5}{(2\pi)^d}\frac{m^2 + l_1 \cdot l_2}{(l_1^2 + m^2)(l_2^2 + m^2)}\frac{m^2 + l_3 \cdot l_4}{(l_3^2 + m^2)(l_4^2 + m^2)}\frac{m}{l_5^2 + m^2}$$

$$-(N - \frac{1}{2})\int \frac{d^d l_1}{(2\pi)^d}\frac{d^d l_3}{(2\pi)^d}\frac{d^d l_5}{(2\pi)^d} Tr\left(\frac{1}{i\not{l}_1 + m}\frac{1}{i\not{l}_3 + m}\frac{1}{i\not{l}_5 + m} + \right.$$

$$\left. + \frac{1}{i\not{l}_5 + m}\frac{1}{i\not{l}_3 + m}\frac{1}{i\not{l}_1 + m}\right)\frac{1}{-i\not{l}_4 + m}\frac{1}{-i\not{l}_2 + m} \tag{4.18}$$

where $l_1 + l_2 = l_3 + l_4 = l_5$. We shall not give a detailed proof of eq.(4.18); let's only mention that its derivation requires the use of the following subtle diagrammatic identity:

$$[13e] + [13f] = -2N\{[13g] + [13h]\} \tag{4.19}$$

A discussion of the algebraic techniques needed in order to reduce the integrals appearing in eq.(4.18) to functions of I and of known finite quantities is beyond the scope of the present work; suffice it to say that these techniques may be systematically applied, and the final result involves, besides I, J and A_1, only the quantity:

$$K = \int \frac{d^d l_1}{(2\pi)^d}\frac{d^d l_3}{(2\pi)^d}\frac{d^d l_5}{(2\pi)^d}\frac{1}{l_1^2 + m^2}\frac{1}{l_2^2 + m^2}\frac{1}{l_3^2 + m^2}\frac{1}{l_4^2 + m^2}\frac{1}{l_5^2 + m^2} m^4 \tag{4.20}$$

finite and dimensionless when $d = 2$

We quote the final result, including for completeness the finite parts, in the zero-momentum limit:

$$[13] = (N - \frac{1}{2})(N - 1)m[\frac{4}{3}I^3 - 12IJ + \frac{17}{3}A_1 + 9K] +$$

$$+(N - \frac{1}{2})m[\frac{1}{3}\frac{\epsilon}{1+\epsilon}I^3 + 4A_1 - 8K] \tag{4.21}$$

The next step in our calculations is the evaluation of the one and two loop diagrams with counterterm insertions that are relevant for the three loop renormalization of the model. These diagrams are drawn in figs.16-19.

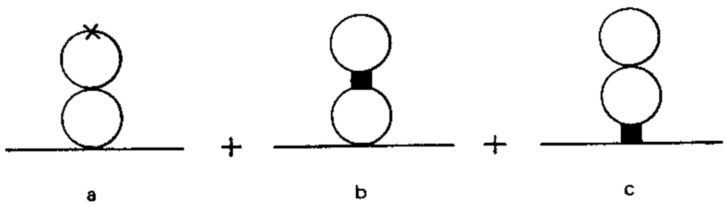


fig 16

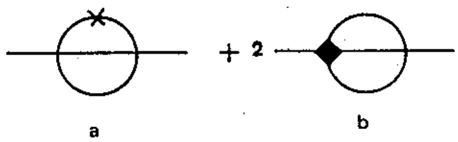


fig 17

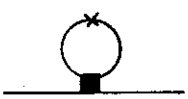


fig 18

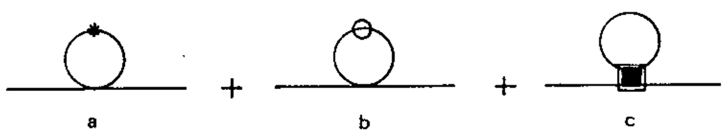


fig 19

In figs.16 and 17 are the two loop-diagrams involving a one-loop counterterm. In fig.18 are drawn the one-loop diagrams with two one-loop counterterms. In fig.19 are the one-loop diagrams with a two-loop counterterm. Evaluations are straightforward, and we shall only quote the final results:

$$[16a] = 8(N - \frac{1}{2})^3(1+\epsilon)^2 \hat{I} I^2 m \tag{4.22a}$$

$$[16b] + [16c] = 16(N-1)(N - \frac{1}{2})^2(1+\epsilon)\hat{I} I^2 m \tag{4.22b}$$

$$[17a] = -2(N - \frac{1}{2})^2(1+2\epsilon)\hat{I}[I^2 - 3J]m \tag{4.23a}$$

$$[17b] = -4(N - \frac{1}{2})(N-1)\hat{I}[I^2 - 3J]m \tag{4.23b}$$

$$[18] = -8(N-\frac{1}{2})^2(N-1)(1+\epsilon)\hat{I}^2 Im \tag{4.24}$$

$$[19a] = -8(N-\frac{1}{2})^2(N-\frac{3}{4})(1+\epsilon)\hat{I}^2 Im \tag{4.25a}$$

$$[19b] = -(N-\frac{1}{2})^2(1+\frac{1}{2}\epsilon)\hat{I}Im \tag{4.25b}$$

$$[19c] = -2(N-\frac{1}{2})[4(N-1)^2\hat{I}^2 - \frac{1}{4}\hat{I}]Im \tag{4.25c}$$

We can now collect all the results presented in this section, remove all the finite terms and find the following cumulative three-loop contributions:

$$Z^{(3)} = \frac{1}{3}(N-\frac{1}{2})(N-1)(g+h)^3(\hat{I}^2 + \frac{1}{4}\hat{I}) \tag{4.26}$$

$$[ZZ_m]^{(3)} = (N-\frac{1}{2})(g+h)^3[-8(N-\frac{3}{4})(N-\frac{5}{6})\hat{I}^3 - \frac{2}{3}(N-1)\hat{I}^2 + \frac{1}{4}(N-\frac{5}{6})\hat{I}] \tag{4.27}$$

implying immediatelly

$$Z_m^{(3)} = (N-\frac{1}{2})(g+h)^3[-8(N-\frac{3}{4})(N-\frac{5}{6})\hat{I}^3 - \frac{3}{2}(N-\frac{5}{6})\hat{I}^2 + \frac{1}{6}(N-\frac{3}{4})\hat{I}] \tag{4.28}$$

By carefully distinguishing the reducible and irreducible contributions to mass renormalization we may check the decomposition eq.(2.19) and find

$$Z_R^{(3)} = -2N(N-\frac{1}{2})g(g+h)^2[4(N-\frac{2}{3})\hat{I}^3 + \frac{1}{3}\hat{I}^2 - \frac{1}{8}\hat{I}] \tag{4.29}$$

$$Z_I^{(3)} = [\frac{4N-5}{3}(N-\frac{3}{2})g^3 + (\frac{22}{3}N - 15)(N-\frac{1}{2})g^2h + (\frac{68}{3}N-15)(N-\frac{1}{2})gh^2 -$$
$$-8(N-\frac{1}{2})(N-\frac{3}{4})(N-\frac{5}{6})h^3]\hat{I}^3 + (N-\frac{1}{2})(g+h)^2[-\frac{3}{2}(N-\frac{5}{6})h + (-\frac{1}{3}N+\frac{5}{4})g]\hat{I}^2 +$$
$$+\frac{1}{6}(N-\frac{1}{2})(g+h)^2[(N-\frac{3}{4})h - \frac{1}{2}(N+\frac{3}{2})g]\hat{I} \tag{4.30}$$

As a consequence, we can immediately compute

$$Z_2^{(3)} = [8N(N-\frac{1}{2})(N-\frac{2}{3})gh^2 - \frac{32}{3}N(N-\frac{1}{2})g^2h - \frac{4}{3}N(N-2)g^3]\hat{I}^3 +$$
$$+2N(N-\frac{1}{2})g(g+h)^2(\frac{1}{3}\hat{I}^2 - \frac{1}{8}\hat{I}) \tag{4.31}$$

$$Z_3^{(3)} = [\frac{4N-5}{3}(N-\frac{3}{2})g^3 + (\frac{22}{3}N-15)(N-\frac{1}{2})g^2h + (\frac{68}{3}N-15)(N-\frac{1}{2})gh^2 -$$
$$-8(N-\frac{1}{2})(N-\frac{3}{4})(N-\frac{5}{6})h^3]\hat{I}^3 + \frac{2}{3}(N-\frac{1}{2})(g+h)^2[g-(N-1)h]\hat{I}^2 +$$

$$+\frac{1}{4}(N-\frac{1}{2})(g+h)^2[(N-\frac{5}{6})h-\frac{5}{6}g]\hat{I} \tag{4.32}$$

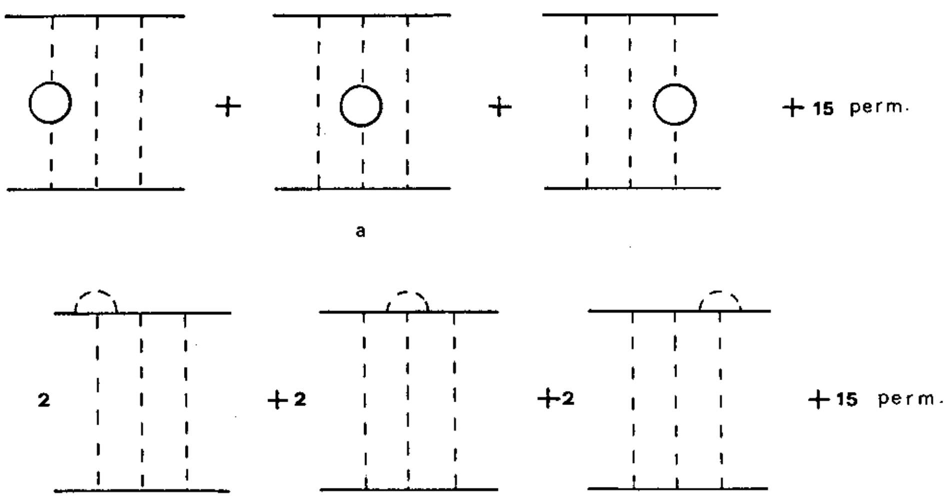
As already discussed, most three-loop contributions to the pole part of the four-point function can be extracted directly from our two-point function calculations.

We are left with the task of computing a limited number of "irreducible" contributions, belonging to three different classes, that are drawn in figs. 20-22. In order to identify the divergent part of these integrals we may as usual evaluate them in scheme (b) at zero external momentum. The result of this computation is

$$[20] = -2(N-1)[(1-\frac{5}{3}\epsilon)\epsilon I^3 + 9IJ] + \text{finite terms} \tag{4.33}$$

$$[21] = \frac{1}{3}(1+3\epsilon)\frac{\epsilon}{1+\epsilon}I^3 + \text{finite terms} \tag{4.34}$$

$$[22] = 0 \tag{4.35}$$



a

b

fig 20

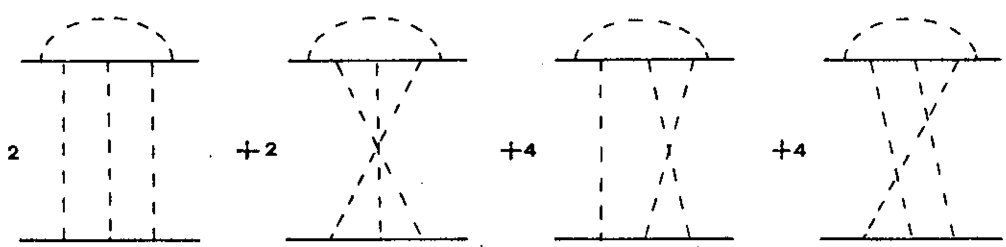


fig 21

$$[18] = -8(N-\frac{1}{2})^2(N-1)(1+\epsilon)\hat{I}^2 Im \tag{4.24}$$

$$[19a] = -8(N-\frac{1}{2})^2(N-\frac{3}{4})(1+\epsilon)\hat{I}^2 Im \tag{4.25a}$$

$$[19b] = -(N-\frac{1}{2})^2(1+\frac{1}{2}\epsilon)\hat{I} Im \tag{4.25b}$$

$$[19c] = -2(N-\frac{1}{2})[4(N-1)^2\hat{I}^2 - \frac{1}{4}\hat{I}]Im \tag{4.25c}$$

We can now collect all the results presented in this section, remove all the finite terms and find the following cumulative three-loop contributions:

$$Z^{(3)} = \frac{1}{3}(N-\frac{1}{2})(N-1)(g+h)^3(\hat{I}^2+\frac{1}{4}\hat{I}) \tag{4.26}$$

$$[ZZ_m]^{(3)} = (N-\frac{1}{2})(g+h)^3[-8(N-\frac{3}{4})(N-\frac{5}{6})\hat{I}^3 - \frac{2}{3}(N-1)\hat{I}^2 + \frac{1}{4}(N-\frac{5}{6})\hat{I}] \tag{4.27}$$

implying immediatelly

$$Z_m^{(3)} = (N-\frac{1}{2})(g+h)^3[-8(N-\frac{3}{4})(N-\frac{5}{6})\hat{I}^3 - \frac{3}{2}(N-\frac{5}{6})\hat{I}^2 + \frac{1}{6}(N-\frac{3}{4})\hat{I}] \tag{4.28}$$

By carefully distinguishing the reducible and irreducible contributions to mass renormalization we may check the decomposition eq.(2.19) and find

$$Z_R^{(3)} = -2N(N-\frac{1}{2})g(g+h)^2[4(N-\frac{2}{3})\hat{I}^3 + \frac{1}{3}\hat{I}^2 - \frac{1}{8}\hat{I}] \tag{4.29}$$

$$\begin{aligned}Z_I^{(3)} = &[\frac{4N-5}{3}(N-\frac{3}{2})g^3 + (\frac{22}{3}N-15)(N-\frac{1}{2})g^2h + (\frac{68}{3}N-15)(N-\frac{1}{2})gh^2 -\\ &-8(N-\frac{1}{2})(N-\frac{3}{4})(N-\frac{5}{6})h^3]\hat{I}^3 + (N-\frac{1}{2})(g+h)^2[-\frac{3}{2}(N-\frac{5}{6})h + (-\frac{1}{3}N+\frac{5}{4})g]\hat{I}^2+\\ &+\frac{1}{6}(N-\frac{1}{2})(g+h)^2[(N-\frac{3}{4})h - \frac{1}{2}(N+\frac{3}{2})g]\hat{I}\end{aligned} \tag{4.30}$$

As a consequence, we can immediately compute

$$\begin{aligned}Z_2^{(3)} = &[8N(N-\frac{1}{2})(N-\frac{2}{3})gh^2 - \frac{32}{3}N(N-\frac{1}{2})g^2h - \frac{4}{3}N(N-2)g^3]\hat{I}^3+\\ &+2N(N-\frac{1}{2})g(g+h)^2(\frac{1}{3}\hat{I}^2 - \frac{1}{8}\hat{I})\end{aligned} \tag{4.31}$$

$$\begin{aligned}Z_3^{(3)} = &[\frac{4N-5}{3}(N-\frac{3}{2})g^3 + (\frac{22}{3}N-15)(N-\frac{1}{2})g^2h + (\frac{68}{3}N-15)(N-\frac{1}{2})gh^2 -\\ &-8(N-\frac{1}{2})(N-\frac{3}{4})(N-\frac{5}{6})h^3]\hat{I}^3 + \frac{2}{3}(N-\frac{1}{2})(g+h)^2[g-(N-1)h]\hat{I}^2+\end{aligned}$$

$$+\frac{1}{4}(N-\frac{1}{2})(g+h)^2[(N-\frac{5}{6})h-\frac{5}{6}g]\hat{I} \qquad (4.32)$$

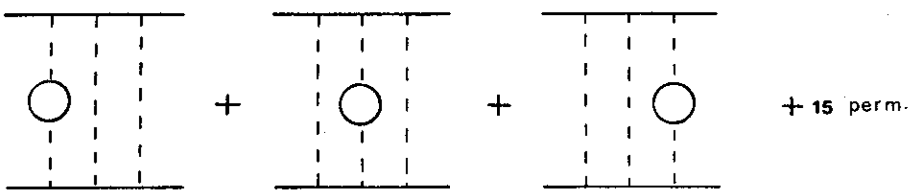
As already discussed, most three-loop contributions to the pole part of the four-point function can be extracted directly from our two-point function calculations.

We are left with the task of computing a limited number of "irreducible" contributions, belonging to three different classes, that are drawn in figs. 20-22. In order to identify the divergent part of these integrals we may as usual evaluate them in scheme (b) at zero external momentum. The result of this computation is

$$[20]=-2(N-1)[(1-\frac{5}{3}\epsilon)\epsilon I^3+9IJ]+\text{finite terms} \qquad (4.33)$$

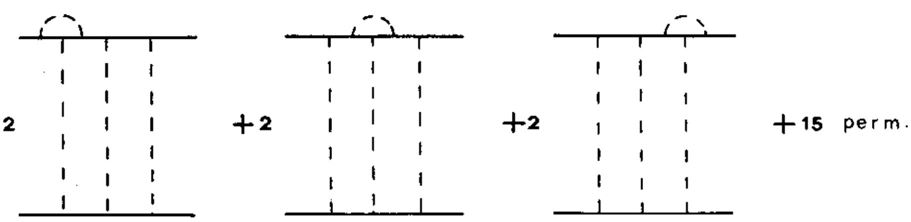
$$[21]=\frac{1}{3}(1+3\epsilon)\frac{\epsilon}{1+\epsilon}I^3+\text{finite terms} \qquad (4.34)$$

$$[22]=0 \qquad (4.35)$$



a

fig 20



b

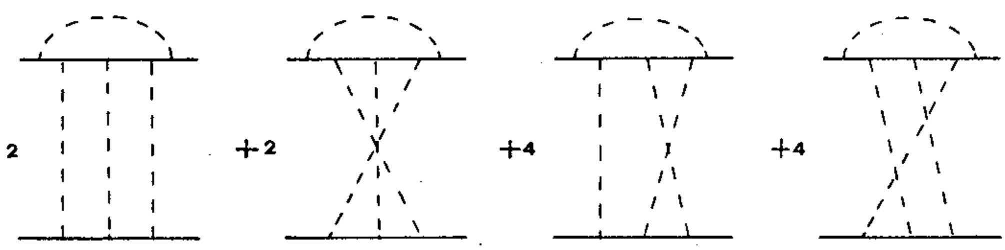
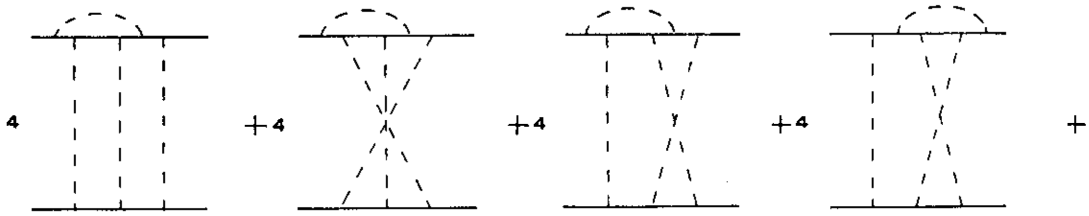


fig 21



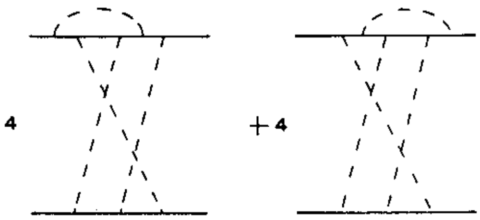


fig 22

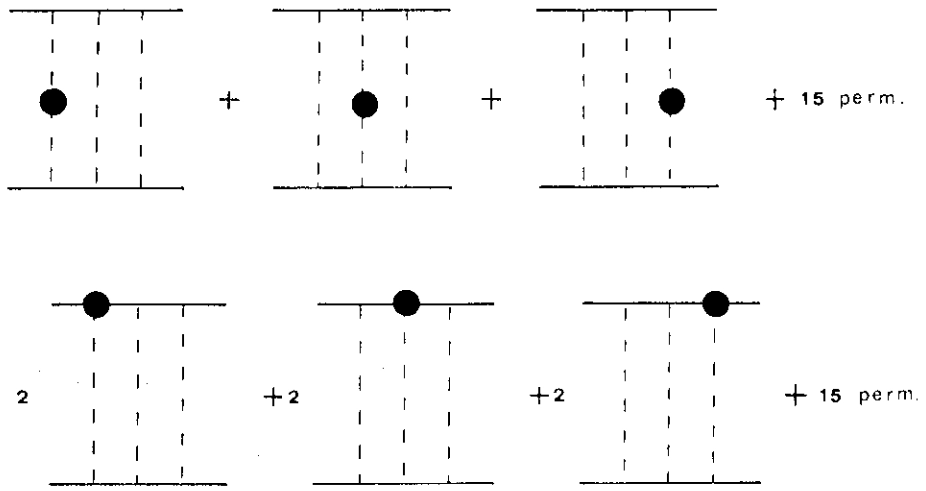


fig 23

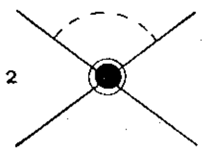


fig 24

Contributions coming from counterterm insertions are drawn in figs.23 and 24.

Evaluation of these diagrams leads to

$$[23] = 6(N-1)[\frac{1}{2}\epsilon(1-\epsilon)I^2 + 3(1+\epsilon)J]\hat{I} \quad (4.36)$$

$$[24] = \frac{1}{2}(1+\epsilon)I\hat{I} \quad (4.37)$$

We therefore obtain the renormalized contributions

$$[20] + [23] = -(N-1)(\frac{1}{2}\hat{I}^2 - \frac{1}{12}\hat{I}) \quad (4.38)$$

$$[21] + [24] = \frac{1}{3}\hat{I}^2 - \frac{1}{12}\hat{I} \quad (4.39)$$

Collecting these results and all previous diagram computations we can now evaluate the three-loop contribution to Z_4:

$$hZ_4^{(3)}(g,h) = [-8N^3h^4 + 16N^2h^3(g+h) + (\frac{16}{3}N^2 - \frac{44}{3}N)h^2(g+h)^2 + \frac{8}{3}(N-\frac{3}{2})(N-2)$$

$$h(g+h)^3]\hat{I}^3 + [-\frac{2}{3}N(N-\frac{1}{2})h^2(g+h)^2 + \frac{4}{3}(N-\frac{1}{2})h(g+h)^3 + (\frac{1}{2}N-\frac{5}{6})(g+h)^4]\hat{I}^2 +$$

$$+[\frac{1}{4}N(N-\frac{1}{2})h^2(g+h)^2 - \frac{5}{12}(N-\frac{1}{2})h(g+h)^3 - \frac{1}{12}(N-2)(g+h)^4]\hat{I} \quad (4.40)$$

Finally we are able to compute

$$g\,\frac{Z_3^2(g,h)}{Z_2(g,h)} + hZ_4(g,h) \cong (g+h) - 2(N-1)(g+h)^2\hat{I} + [4(N-1)^2(g+h)^3\hat{I}^2 - \frac{1}{4}(g+h)^3\hat{I}] +$$

$$+\frac{1}{4}(N-1)(N-\frac{3}{2})(g+h)^4\hat{I} + \frac{1}{6}(N-1)(9-4N)(g+h)^4\hat{I}^2 - 8(N-1)^3(g+h)^4\hat{I}^3 \quad (4.41)$$

Due to our procedure, the fact that eq.(4.41) turns out to be a function of the single variable $g+h$ is a consistency check to our computation.

Eqs.(4.26),(4.28) and (4.41) are the most relevant results presented in this section. We shall not belabor on the number of different cross-checks we performed in our computations. Suffice it to say that the quantities $Z_2(g,h)$ and $Z_3(g,h)$ have been independently computed starting from their own definitions, and found in agreement with those obtained from the renormalization of the two point function.

Two more considerations are worth mentioning: our results satisfy the condition that $Z(g+h) = Z_m(g+h) = 1$ when $N = \frac{1}{2}$ (free Majorana fermion) and also the subtler request that $g\,Z_3^2/Z_2 + hZ_4 = (g+h)Z^2$ when $N = 1$, related to the conformal invariance (vanishing β function) of the Abelian Thirring model.

A last crucial consistency check follows from the analysis of the coefficients of the multiple poles, related to those of the simple poles by the renormalization group equations. Next section will be devoted to this topic.

Sec.5-Renormalization group properties of the Z functions

The renormalizability of the Lagrangian $L_2(g,h)$ defined in eq.(2.5) insures us that we can write renormalization group equations satisfied by the renormalized Green's functions. In particular it must be possible to define the functions

$$\beta_g(g,h,\epsilon) \equiv \epsilon g + \hat{\beta}_g(g,h) = \mu\frac{\partial}{\partial\mu}g \tag{5.1a}$$

$$\beta_h(g,h,\epsilon) \equiv \epsilon h + \hat{\beta}_h(g,h) = \mu\frac{\partial}{\partial\mu}h \tag{5.1b}$$

$$\gamma_\psi(g,h) = \mu\frac{\partial}{\partial\mu}lnZ(g,h) \tag{5.1c}$$

$$\gamma_m(g,h) = \mu\frac{\partial}{\partial\mu}lnZ_m(g,h) \tag{5.1d}$$

$$\gamma_2(g,h) = \mu\frac{\partial}{\partial\mu}lnZ_2(g,h) \tag{5.1e}$$

$$\gamma_3(g,h) = \mu\frac{\partial}{\partial\mu}lnZ_3(g,h) \tag{5.1f}$$

where the $\hat{\beta}$'s and the γ's are independent of ϵ and of the subtraction point μ, and acting on functions of g,h the following relationship holds:

$$\mu\frac{\partial}{\partial\mu} = \beta_g\frac{\partial}{\partial g} + \beta_h\frac{\partial}{\partial h} \tag{5.2}$$

Eqs.(5.1) can be reinterpreted as recursion relations between the coefficients of the multiple poles in the loop expansion of the functions Z. We did not take this point of view, but verified instead that these equations are consistently satisfied by the choice

$$\hat{\beta}_g(g,h) = -(N-1)g^2 - 2(N-\frac{1}{2})gh + \frac{1}{2}(N-\frac{1}{2})g(g+h)^2 +$$

$$+\frac{1}{2}(N-\frac{1}{2})g(g+h)^2[\frac{N-3}{4}g + (N-\frac{3}{4})h] \tag{5.3a}$$

$$\hat{\beta}_h(g,h) = -(N-1)h^2 + gh + \frac{1}{2}(N-\frac{1}{2})h(g+h)^2 - \frac{1}{4}(g+h)^3 +$$

$$+\frac{1}{2}(N-\frac{1}{2})h(g+h)^2[\frac{N-3}{4}h - \frac{1}{2}(N+\frac{3}{2})g] - \frac{1}{8}(N-2)(g+h)^4 \tag{5.3b}$$

and

$$\gamma_\psi(g,h) = \frac{1}{4}(N-\frac{1}{2})(g+h)^2 - \frac{1}{8}(N-\frac{1}{2})(N-1)(g+h)^3 \tag{5.4a}$$

$$\gamma_m(g,h) = \gamma_I(g,h) + \gamma_R(g,h) \tag{5.4b}$$

where

$$\gamma_I(g,h) = -\frac{1}{2}g + (N-\frac{1}{2})h - \frac{1}{4}(N-\frac{1}{2})(g+h)^2 -$$

$$-\frac{1}{4}(N-\frac{1}{2})(g+h)^2[(N-\frac{3}{4})h - \frac{1}{2}(N+\frac{3}{2})g] \quad (5.5a)$$

$$\gamma_R(g,h) = Ng - \frac{3}{8}N(N-\frac{1}{2})g(g+h)^2 \quad (5.5b)$$

$$\gamma_2(g,h) = -Ng + \frac{3}{8}N(N-\frac{1}{2})g(g+h)^2 = -\gamma_R(g,h) \quad (5.5c)$$

$$\gamma_3(g,h) = -\frac{1}{2}g + (N-\frac{1}{2})h - \frac{3}{8}(N-\frac{1}{2})(g+h)^2[(N-\frac{5}{6})h - \frac{5}{6}g] = \gamma_I(g,h) + \gamma_\psi(g,h) \quad (5.5d)$$

Moreover, the following relationships hold:

$$\hat{\beta}_g(g,h) + \hat{\beta}_h(g,h) = -(N-1)(g+h)^2 + \frac{1}{2}(N-1)(g+h)^3 + \frac{1}{8}(N-1)(N-\frac{7}{2})(g+h)^4 = \hat{\beta}(g+h) \quad (5.6a)$$

$$\gamma_m(g,h) = (N-\frac{1}{2})(g+h) - \frac{1}{4}(N-\frac{1}{2})(g+h)^2 - \frac{1}{4}(N-\frac{1}{2})(N-\frac{3}{4})(g+h)^3 = \gamma_m(g+h) \quad (5.6b)$$

$$\gamma_\psi(g,h) = \gamma(g+h) \quad (5.6c)$$

and we can check that

$$-\frac{\hat{\beta}_g}{g} = 2\gamma_3 - \gamma_2 - 2\gamma_\psi = 2\gamma_I + \gamma_R \quad (5.7)$$

consistently with our definitions and with the relationships

$$\mu \frac{\partial}{\partial \mu}\left[g\, \frac{Z_3^2}{Z_2 Z^2}\right] = \epsilon \left[g\, \frac{Z_3^2}{Z_2 Z^2}\right] \quad (5.8a)$$

$$\mu \frac{\partial}{\partial \mu}\left[h Z_4\right] = \epsilon \left[h Z_4\right] \quad (5.8b)$$

Eqs.(5.3),(5.4),(5.5) are obviously valid within the minimal subtraction (MS) scheme, and they are consistent with the standard procedure of reconstructing the RG functions from the coefficients of the simple poles. The functions

$$\beta(g) = \epsilon g - (N-1)g^2 + \frac{1}{2}(N-1)g^3 + \frac{1}{8}(N-1)(N-\frac{7}{2})g^4 + \cdots \quad (5.9)$$

$$\gamma(g) = \frac{1}{4}(N-\frac{1}{2})g^2 - \frac{1}{8}(N-\frac{1}{2})(N-1)g^3 + \cdots \quad (5.10)$$

$$\gamma_m(g) = (N-\frac{1}{2})g - \frac{1}{4}(N-\frac{1}{2})g^2 - \frac{1}{4}(N-\frac{1}{2})(N-\frac{3}{4})g^3 + \cdots \quad (5.11)$$

are the RG functions of the Gross Neveu model, computed to three loops in the MS scheme. They seem to satisfy all our prejudices about the quantum structure of the model; in particular $\beta(g) - \epsilon g$ vanishes when $N = 1$, while $\gamma(g)$ and $\gamma_m(g)$ vanish at $N = \frac{1}{2}$, as expected. It's also worth mentioning that the identities

$$\beta_h(0,h) = \beta(h) \tag{5.12}$$

$$\gamma_I(0,h) = \gamma_m(h) \tag{5.13}$$

are satisfied, as they should.

The only possible comparison of our results with existing literature involves Gracey's paper [9]. Converting his results into our notation, one obtains the predictions

$$\beta(g) = \epsilon g - (N-1)g^2 + \frac{1}{2}(N-1)g^3 + \frac{1}{4}(N-1)(N-1+\frac{a}{2})g^4 + \cdots \tag{5.14}$$

$$\gamma(g) = \frac{1}{4}(N-\frac{1}{2})g^2 - \frac{1}{8}(N-\frac{1}{2})(N-1)g^3 + \frac{1}{8}(N-\frac{1}{2})(N-1)^2 g^4 + \cdots \tag{5.15}$$

and the author proposes the value $a = 4$ assuming the amalgam nature of the supersimmetric σ model to persist to $O(1/N^2)$. We find the value $a = -5$, and therefore come to the conclusion that this amalgam structure is broken $O(1/N^2)$, a result which could hardly be qualified as surprising.

Sec.6-The effective potential and its RG equation

In sec.2 we discussed at length the renormalization procedure one must apply in order to get finite results in the theory when formulated in terms of the two parameter (massless) Lagrangian

$$L_2(g,h) = Z(g,h)\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}Z_2(g,h)\frac{\sigma^2}{g} + Z_3(g,h)\sigma\bar{\psi}\psi - \frac{h}{2}Z_4(g,h)(\bar{\psi}\psi)^2 \qquad (6.1)$$

As we shall see, this is exactly the formulation of the theory we need in order to compute the renormalized effective potential. The computation proceeds in the following way: let's first perform the substitution

$$\sigma = \sigma_c + \sigma_q \qquad (6.2)$$

in eq.(6.1); σ_c is the "classical" value of the renormalized σ field appearing as a parameter in the effective potential. The resulting Lagrangian is:

$$L = Z(g,h)\bar{\psi}[\partial\!\!\!/ + Z_I(g,h)\sigma_c]\psi + \frac{1}{2}Z_2(g,h)\frac{\sigma_q^2}{g} + Z_3(g,h)\sigma_q\bar{\psi}\psi -$$
$$-\frac{h}{2}Z_4(g,h)(\bar{\psi}\psi)^2 + Z_2(g,h)\frac{\sigma_q\sigma_c}{g} + \frac{1}{2}Z_2(g,h)\frac{\sigma_c^2}{g} \qquad (6.3)$$

where we made use of eq.(2.21). By exploiting the well known relationship between the linear term in the Lagrangian and reducible tadpole insertions, we come to the conclusion that the effective potential is simply

$$V(\sigma_c) = \frac{1}{2}Z_2(g,h)\frac{\sigma_c^2}{g} + F(\sigma_c) \qquad (6.4)$$

where $F(m)$ is the sum of the irreducible vacuum diagrams in the theory whose Lagrangian is

$$L = Z(g,h)\bar{\psi}[\partial\!\!\!/ + Z_I(g,h)m]\psi + \frac{1}{2}Z_2(g,h)\frac{\sigma^2}{g} + Z_3(g,h)\sigma\bar{\psi}\psi - \frac{h}{2}Z_4(g,h)(\bar{\psi}\psi)^2 \qquad (6.5)$$

i.e. nothing but the original massive Lagrangian associated with the prescription of removing the reducible tadpoles. Renormalizability of L and the appearence of Z_I instead of Z_m in front of the mass term insures us that F defined as above will be exactly renormalized by the counterterms generated by

$$\frac{1}{2}Z_2(g,h)\frac{\sigma_c^2}{g} \qquad (6.6)$$

We draw the relevant topologies of vacuum diagrams up to four loops in fig. 25. One and two loop computations are straightforward. At the one loop level

$$NTrln\frac{1}{ip\!\!\!/ + m} = -N\frac{1}{1+\epsilon/2}m^2 I \qquad (6.7)$$

and including the terms originated by (6.6)

$$m^2 N(\hat{I} - I\frac{1}{1+\epsilon/2}) \underset{\epsilon\to 0}{\longrightarrow} m^2 \frac{N}{4}(ln\frac{m^2}{\mu^2} - 1) \qquad (6.8)$$

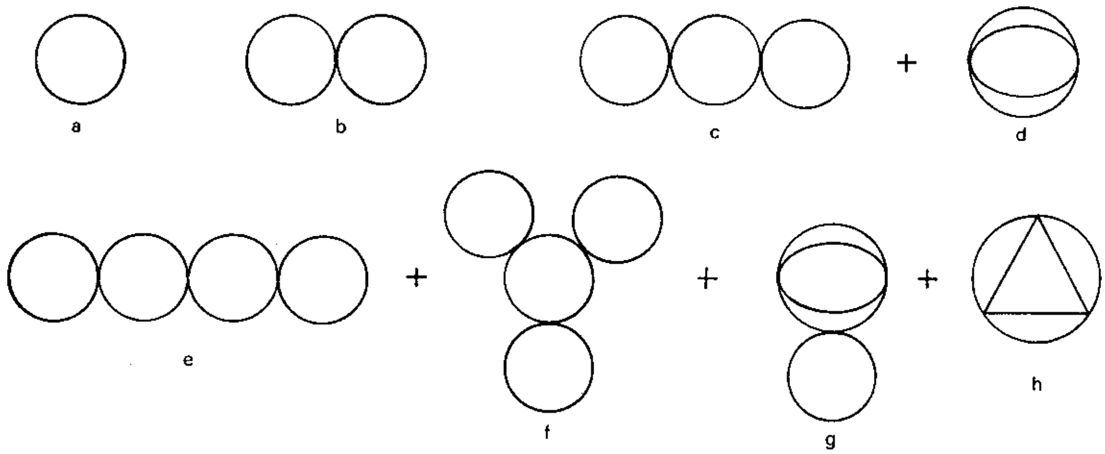


fig 25

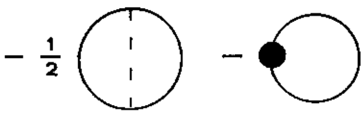


fig 26

At the two loop level the irreducible diagram dependent on g and its counterterm are depicted in fig.26

$$\frac{1}{2}NgTr\left[\int \frac{d^dp}{(2\pi)^d}\frac{1}{i\not{p}+m}\right]^2 - Ng\hat{I}mTr\int\frac{d^dp}{(2\pi)^d}\frac{1}{i\not{p}+m} =$$

$$= Ngm^2I^2 - 2Ngm^2I\hat{I} \qquad (6.9)$$

and including the term from (6.6)

$$Ngm^2(I-\hat{I})^2 \underset{\epsilon\to 0}{\longrightarrow} m^2 Ng(\frac{1}{4}ln\frac{m^2}{\mu^2})^2 \qquad (6.10)$$

An analogous computation of the terms depending on h leads to a renormalized contribution

$$-2N(N-\frac{1}{2})hm^2(\frac{1}{4}ln\frac{m^2}{\mu^2})^2 \qquad (6.11)$$

At the three-loop level, topology [25c] leads to the contributions

$$[25c] = m^2 I^3 N[-(g+h)^2 + 4Nh(g+h) - 4N^2h^2](1+\epsilon) \tag{6.12}$$

while topology [25d] leads to

$$[25d] = -N(N-\frac{1}{2})(g+h)^2 m^2[\frac{8}{3}A_1 + I^3(-\frac{2}{3}+\frac{\epsilon}{3}\frac{1}{1+3\epsilon/2})] \tag{6.13}$$

Including all counterterm insertions in the one and two loop diagrams and considering the contribution from eq.(6.6), we finally obtain the finite result

$$\begin{aligned} &m^2(g+h)^2N(N-\frac{1}{2})[\frac{2}{3}(I-\hat{I})^3+\frac{1}{2}(I-\hat{I})^2+\frac{3}{8}(I-\hat{I})+\frac{3}{32}-\frac{8}{3}A_1]+ \\ &+m^2[-4N^3h^2+4N^2h(g+h)-N(g+h)^2][(I-\hat{I})^3-\frac{1}{2}(I-\hat{I})^2] \underset{\epsilon\to 0}{\longrightarrow} \\ &-m^2(g+h)^2N(N-\frac{1}{2})[\frac{2}{3}(\frac{1}{4}ln\frac{m^2}{\mu^2})^3-\frac{1}{2}(\frac{1}{4}ln\frac{m^2}{\mu^2})^2+\frac{3}{32}ln\frac{m^2}{\mu^2}-\frac{3}{32}+\frac{7}{24}\zeta(3)]- \\ &-m^2[-4N^3h^2+4N^2h(g+h)-N(g+h)^2][(\frac{1}{4}ln\frac{m^2}{\mu^2})^3+\frac{1}{2}(\frac{1}{4}ln\frac{m^2}{\mu^2})^2] \end{aligned} \tag{6.14}$$

In conclusion, the renormalized effective potential computed up to three loops in the MS scheme is

$$V(\sigma_c, g, h, \mu) = \frac{1}{2g}\sigma_c^2\, v(t,g,h) \tag{6.15a}$$

where $t \equiv ln(\sigma_c/\mu)$ and

$$\begin{aligned} v(t,g,h) = {} & 1 + Ng(t-\frac{1}{2}) + \frac{1}{2}Ng^2t^2 - N(N-\frac{1}{2})ght^2+ \\ &+Ng[(g+h)^2-4Nh(g+h)+4N^2h^2][\frac{1}{4}t^3+\frac{1}{4}t^2]- \\ &-g(g+h)^2N(N-\frac{1}{2})[\frac{1}{6}t^3-\frac{1}{4}t^2+\frac{3}{8}t-\frac{3}{16}+\frac{7}{12}\zeta(3)] \end{aligned} \tag{6.15b}$$

The procedure we adopted in the definition and construction of V is completely canonical, because σ has been treated as a dynamical field, as opposed to the standard approach where σ is an auxiliary variable. Therefore we can write down a renormalization group equation for V in the standard form:

$$\left[\mu\frac{\partial}{\partial\mu} + \hat{\beta}_g(g,h)\frac{\partial}{\partial g} + \hat{\beta}_h(g,h)\frac{\partial}{\partial h} - \gamma_\sigma(g,h)\sigma_c\frac{\partial}{\partial\sigma_c}\right] V(\sigma_c,g,h,\mu) = 0 \tag{6.16}$$

where eq.(2.21) implies

$$\gamma_\sigma(g,h) \equiv \gamma_I(g,h) \tag{6.17}$$

By straightforward manipulations, we can rephrase eq.(6.16) in the form

$$\left[-(1+\gamma_I)\frac{\partial}{\partial t}+\hat{\beta}_g\frac{\partial}{\partial g}+\hat{\beta}_h\frac{\partial}{\partial h}+\gamma_R\right]v(t,g,h)=0 \qquad (6.18)$$

where we made use of eq.(5.7). One can easily check that eq.(6.18) is indeed satisfied by eq.(6.15), assuming the results of sec.5. However, if we try to take the limit $h\to 0$ in eq.(6.16), the resulting RG equation is no longer homogeneous; it can be recast in the form:

$$\left[\mu\frac{\partial}{\partial\mu}+\tilde{\beta}(g)\frac{\partial}{\partial g}-\tilde{\gamma}_\sigma(g)\sigma_c\frac{\partial}{\partial\sigma_c}\right]\tilde{V}(\sigma_c,g,\mu)=\tilde{\beta}_h(g)\tilde{V}_h(\sigma_c,g,\mu) \qquad (6.19)$$

where

$$\tilde{\beta}(g)\equiv\hat{\beta}_g(g,0)=-(N-1)g^2+\frac{1}{2}(N-\frac{1}{2})g^3+\frac{1}{8}(N-3)(N-\frac{1}{2})g^4+\cdots \qquad (6.20a)$$

$$\tilde{\gamma}_\sigma(g)\equiv\gamma_I(g,0)=-\frac{1}{2}g-\frac{1}{4}(N-\frac{1}{2})g^2+\frac{1}{8}(N+\frac{3}{2})(N-\frac{1}{2})g^3+\cdots \qquad (6.20b)$$

$$\tilde{\beta}_h(g)\equiv\hat{\beta}_h(g,0)=-\frac{1}{4}g^3-\frac{1}{8}(N-2)g^4+\cdots \qquad (6.20c)$$

and

$$\tilde{V}(\sigma_c,g,\mu)=\frac{\sigma_c^2}{2g}[1+Ng(t-\frac{1}{2})+Ng^2\frac{t^2}{2}+Ng^3(\frac{t^3}{4}+\frac{t^2}{4})-$$

$$-g^3N(N-\frac{1}{2})(\frac{t^3}{6}-\frac{t^2}{4}+\frac{3}{8}t-\frac{3}{16}+\frac{7}{12}\zeta(3))] \qquad (6.21a)$$

$$\tilde{V}_h(\sigma_c,g,\mu)=\frac{\sigma_c^2}{2}N(N-\frac{1}{2})[t^2+g(\frac{4}{3}t^3+\frac{1}{2}t^2+\frac{3}{4}t-\frac{3}{8}+\frac{7}{6}\zeta(3))] \qquad (6.21b)$$

The source of inhomogeneity is easily traced to the already discussed phenomenon of regeneration of the bare effective four-fermion vertex in the Lagrangian $L_2(g,h=0)$: this is the origin of a nonvanishing $\tilde{\beta}_h(g)$.

We must however mention the fact that the r.h.s. of eq.(6.19) is a quantity depressed by four powers of g and two powers of $1/N$ with respect to the leading term appearing in the l.h.s. of the same equation. As a consequence , the first three orders in the loop expansion and the first non trivial order in the $1/N$ expansion aren't affected by this phenomenon, and the corresponding truncated effective potential seems to satisfy an homogeneous RG equation [2,4]. It's however worth observing that, even in this case, the functions $\tilde{\beta}$ and $\tilde{\gamma}_\sigma$ do not agree with the RG functions β and γ_m; the discrepancy already occurs at the level of universal coefficients, thus indicating the absence of a direct physical interpretation for this equation.

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