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Dimensional Regularization in the $1/N$ Expansion

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ABSTRACT

Two classes of renormalizable $1/N$ expandable two dimensional models are analyzed to $O(1/N)$ and the asymptotic behavior of the renormalized two-point functions is nonperturbatively evaluated.

These results are taken as a benchmark to study the applicability of dimensional regularization and perturbative minimal subtraction renormalization to the context of the $1/N$ expansion. Perturbation theory is applied to $O(1/N)$ diagrams to all orders in the weak coupling constant, and after resummation the same finite renormalization group invariant asymptotic amplitudes are obtained.

As a byproduct, the $O(1/N)$ contributions to renormalization group Z functions in the minimal subtraction scheme are extracted and the critical index η is evaluated and compared to previous nonperturbative results, finding complete agreement.

An Appendix is devoted to the extension of these results to a supersymmetric version of the models.

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Finite size scaling in the $1/N$ expansion

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We develop a parametrization of finite size scaling appropriate to the description of N-component systems in the context of the $1/N$ expansion. We apply this formalism to the case of two-dimensional O(N) models and obtain the finite size scaling function of the susceptibility both numerically in the whole scaling region and analytically in the perturbative regime.

In recent years finite size scaling has become an increasingly important tool in the theoretical and numerical analysis of critical systems and lattice field theories [1]. To the best of our knowledge, however, despite the impressive theoretical development in this field, no explicit extension (and therefore no concrete application) of this formalism has been made in the context of $1/N$ expandable models.

These models are a typical testing ground of field-theoretical ideas, and especially of new numerical simulation algorithms, which must unavoidably face the problem of working with finite size systems and understanding finite size effects. Having recently made some progress in the evaluation of $1/N$ corrections for a two-dimensional spin model (O(N) sigma model), we decided to explore its finite lattice properties and were eventually led to introducing the formalism we are going to explain in this letter.

The comparison between theoretical analysis and numerical computation appears to be quite satisfactory and leads to the conclusion that $1/N$ expanded finite size scaling should prove to be a rather useful tool in the numerical analysis of lattice spin systems.

We shall explicitly discuss the finite size scaling relations from the point of view of the $1/N$ expansion, and obtain the $1/N$ expanded finite size relationships whose range of numerical applicability we want to explore.

Let us try to be rather general as far as possible. Any coordinate-independent physical quantity (like masses, susceptibilities, vacuum expectation values of composite operators) defined in the context of a $1/N$ expandable finite lattice model will in general depend on four different parameters:

$$Q \equiv Q(T, a, L, N) , \tag{1}$$

where T is the temperature, L^d is the physical volume in d dimensions and a is the lattice spacing (i.e. the number of lattice sites is $(L/a)^d$). In the infinite volume limit, in the presence of a critical point we can define a critical region, in which all separate dependence on T and a can be made to disappear by parametrizing everything in terms of the physical correlation length (inverse mass gap)

$$\xi \propto a \exp\left(\int^T \frac{dt}{\beta(t)} \right), \tag{2}$$

where $\beta(t)$ is the renormalization group beta function of the model and $\xi/a \to \infty$ when $T \to T_c$.

The finite size scaling relation stems from the observation that the infinite volume limit $(L/a \to \infty)$ can be reached by simultaneously moving towards the infinite correlation limit and keeping constant the ratio L/ξ. This corresponds to considering the physical system at criticality enclosed in a finite physical volume. By purely dimensional considerations, L and ξ being both renormalization group invariant quantities, we should get

$$\frac{Q(T,a,L,N)}{Q(T,a,\infty,N)} \xrightarrow[L/\xi=\text{const.}]{T\to T_c} f^{(Q)}(L^2/\xi^2, N)\,, \tag{3}$$

where $f^{(Q)}$ is the finite size scaling function of the quantity Q, and $f(Q)\to 1$ when $L/\xi\to\infty$.

Consider now the implications of the $1/N$ expandability on eq. (3). Since we shall only focus on the first non-trivial order of the expansion, we can reexpress eq. (3) in the form

$$\frac{Q_0(NT,a,L)+(1/N)Q_1(NT,a,L)+\dots}{Q_0(NT,a,\infty)+(1/N)Q_1(NT,a,\infty)+\dots} \xrightarrow[L/\xi=\text{const.}]{T\to T_c} f_0^{(Q)}(L^2/\xi^2)+(1/N)f_1^{(Q)}(L^2/\xi^2)+\dots\,, \tag{4}$$

where, with obvious notation, we have $1/N$ expanded both the physical quantity and its finite size scaling function.

However we must keep in mind that $L^2/\xi^2\equiv L^2m^2$ itself is a $1/N$ expandable physical quantity:

$$m^2=m_0^2(NT)+(1/N)m_1^2(NT)+\dots\,, \tag{5}$$

and by substituting eq. (5) in the expansion of f we get

$$\frac{Q_0(NT,a,L)}{Q_0(NT,a,\infty)}+\frac{1}{N}\left(\frac{Q_1(NT,a,L)}{Q_0(NT,a,\infty)}-\frac{Q_0(NT,a,L)Q_1(NT,a,\infty)}{[Q_0(NT,a,\infty)]^2}\right)+\mathrm{O}(1/N^2)$$

$$\to f_0^{(Q)}(m_0^2(NT)L^2)+(1/N)f_0'^{(Q)}(m_0^2(NT)L^2)m_1^2(NT)L^2+(1/N)f_1^{(Q)}(m_0^2(NT)L^2)+\mathrm{O}(1/N^2) \tag{6}$$

and by comparison

$$\frac{Q_0(NT,a,L)}{Q_0(NT,a,\infty)}\to f_0^{(Q)}(m_0^2(NT)L^2)\,, \tag{7a}$$

$$\frac{Q_1(NT,a,L)}{Q_0(NT,a,L)}-\frac{Q_1(NT,a,\infty)}{Q_0(NT,a,\infty)}\to\frac{f_0'^{(Q)}(m_0^2(NT)L^2)}{f_0^{(Q)}(m_0^2(NT)L^2)}m_1^2(NT)L^2+\frac{f_1^{(Q)}(m_0^2(NT)L^2)}{f_0^{(Q)}(m_0^2(NT)L^2)}\,. \tag{7b}$$

Eq. (7b) is obtained by making use of eq. (7a) in eq. (6) and is the most general form of the $1/N$ expanded finite size scaling relation.

In order to further develop our formalism, we shall restrict our attention on the two-dimensional O(N) non-linear sigma model (see ref. [2] and references therein), defined by the action

$$S=\frac{1}{T}\sum_{x,\mu}(1-\boldsymbol{S}_x\cdot\boldsymbol{S}_{x+\mu})\,,\quad \boldsymbol{S}_x\cdot\boldsymbol{S}_x=1\,. \tag{8}$$

Preliminary to all developments is a thorough investigation of the large-N finite size scaling function for the mass gap. In order to extract this function, it si convenient to analyze the finite and infinite lattice representation of the gap equation. In the infinite-volume limit and close to the critical point $T_c=0$ we have

$$\frac{1}{NT}=\int\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{1}{\hat{p}^2+m_0^2}\xrightarrow[T\to 0]{}\frac{1}{4\pi}\ln\frac{32}{a^2m_0^2}\,, \tag{9}$$

where $\hat{p}^2=(4/a^2)\sum_\mu\sin^2(ap_\mu/2)$, implying

$$m_0^2\equiv m_0^2(NT,\infty)=\frac{32}{a^2}\exp\left(-\frac{4\pi}{NT}\right)\,. \tag{10}$$

We can however introduce the finite-volume mass

$$m_L^2\equiv m_0^2(NT,L) \tag{11}$$

defined by the discrete sum

$$\frac{1}{NT}=\frac{1}{L^2}\sum_p \frac{1}{\hat{p}^2+m_L^2}, \tag{12}$$

where the sum over p runs on all the modes of the reciprocal lattice.

This representation is suitable for a finite volume weak coupling (small mass) expansion à la Flyvbjerg [3]:

$$\frac{1}{NT}=\frac{1}{m_L^2L^2}\left[1-\sum_{n=1}^{\infty}\sum_{p\neq 0}(-1)^n\left(\frac{m_L^2}{\hat{p}^2}\right)^n\right]. \tag{13}$$

It is not too difficult to prove the following asymptotic behaviour:

$$\sum_{p\neq 0}\frac{1}{\hat{p}^2}\xrightarrow[L\to\infty]{}\frac{L^2}{4\pi}\left[\ln\frac{L^2}{a^2}-\ln\frac{z_c^2}{32}+O\left(\frac{1}{L^2}\right)\right], \tag{14}$$

where $z_c=4.163948...$ and

$$\sum_{p\neq 0}\frac{1}{(\hat{p}^2)^n}\xrightarrow[L\to\infty]{}d_nL^{2n}\left[1+\frac{e_na^2}{L^2}+O\left(\frac{1}{L^4}\right)\right],\quad d_n=\frac{1}{(2\pi)^{2n}}\sum_{\substack{m_1,m_2=-\infty\\(m_1,m_2)\neq(0,0)}}^{\infty}\frac{1}{(m_1^2+m_2^2)^n}. \tag{15, 16}$$

In the $n\to\infty$ limit we can evaluate

$$d_n\approx\frac{4}{(2\pi)^{2n}}\left(1+\frac{1}{2^n}\right),\quad e_n\approx\frac{n\pi^2}{3}. \tag{17}$$

We would like to notice that the quantity z_c defined in eq. (14) is directly related to the constant S_c defined in ref. [4] through $z_c=2S_c$. More generally, all the coefficients d_n do not depend on the detailed structure of the lattice action and therefore lead to a universal definition of the scaling function, related only to the topology and the number of finite space dimensions in the physical system.

A finite size renormalization group invariant variable can now be defined by $z=m_LL$ and the large-N finite size scaling equation can be expressed in the form

$$-\ln\frac{(m_0L)^2}{32}\xrightarrow[T\to 0]{}\frac{4\pi}{NT}-\ln\frac{L^2}{a^2}\xrightarrow[L\to\infty]{}-\ln\frac{z_c^2}{32}+\omega(z^2),\quad \omega(z^2)=\frac{4\pi}{z^2}+4\pi\sum_{n=1}^{\infty}(-1)^n(z^2)^nd_{n+1}. \tag{18}$$

Comparing with eq. (7a) we can now identify the finite size scaling function

$$\frac{m_L^2}{m_0^2}\to f_0^{(m)}(m_0^2L^2)\equiv\frac{z^2}{z_c^2}\exp[\omega(z^2)], \tag{19}$$

and eq. (18) shows that any function of z can be reexpressed as a function of $m_0^2L^2$, as expected.

In the following we shall also be interested in the logarithmic derivative of $f_0^{(m)}$ with respect to $m_0^2L^2$. This can be achieved by observing that

$$m_0^2L^2\frac{\partial}{\partial(m_0^2L^2)}\ln f_0^{(m)}(m_0^2L^2)=m_0^2L^2\frac{\partial z^2}{\partial(m_0^2L^2)}\frac{\partial}{\partial z^2}\ln f_0^{(m)}(z)=\frac{1/z^2+\partial\omega(z^2)/\partial z^2}{\partial\ln(m_0^2L^2)/\partial z^2}$$

$$=-1-\frac{1}{z^2\partial\omega(z^2)/\partial z^2}. \tag{20}$$

From the definitions it follows that

$$z^2\frac{\partial\omega(z^2)}{\partial z^2}=m_L^2\frac{\partial}{\partial m_L^2}\frac{4\pi}{NT}=-\frac{4\pi}{L^2}\sum_p\frac{m_L^2}{(\hat{p}^2+m_L^2)^2}. \tag{21}$$

In fig. 1 we show the numerical evaluation of the function $\omega(z^2)+\ln(z^2/z_c^2)$ as a function of z. Its large-z behaviour can be proven proportional to $\exp(-z)$, as expected from the observation that $z\to\infty$ corresponds to the physical infinite-volume limit in presence of a nonvanishing mass gap, and in this limit we expect exponentially damped finite size deviations from the infinite-volume results. From eq. (17) it follows that the series appearing in eq. (18) and defining the function $\omega(z^2)$ has a radius of convergence in z equal to 2π.

There is a deep correspondence between our results and Lüscher's computation of the spectrum of low-lying states in the continuum finite-volume hamiltonian formulation of the model [5]. By recalling that in the large-N limit

$$m_0=\Lambda_{\overline{\rm MS}}=\sqrt{32}\Lambda_L\,, \tag{22}$$

we can rephrase our results in the form

$$\ln\frac{m_L}{\Lambda_{\overline{\rm MS}}}=\ln\frac{z}{z_c}+\tfrac{1}{2}\omega(z^2)\,, \tag{23}$$

and this is the same as Lüscher's result in an euclidean lattice formulation of the problem; the function $\omega(z)$ has even the same convergence radius. We refer to Lüscher's papers for any further consideration about his approach to finite size effects.

The main purposes of the $O(1/N)$ computation are the determination of the $1/N$ correction to the finite size scaling function, i.e. the function $f_1^{(Q)}(m_0^2(NT)L^2)$, and a comparison with numerical computations on finite lattices in order to have a quantitative estimate of the range of validity of the finite size scaling parametrization.

Our starting point will be eq. (7b), specialized to the case when the physical quantity under observation can be extracted from the dressed inverse two-point function at some special value of the momentum. We recall from ref. [2] that the inverse two-point function can be parametrized by

$$G^{-1}(p^2,T,a,L,N)=\hat{p}^2+m_0^2(NT,a,L)+(1/N)\Sigma_1(p^2,NT,a,L)+O(1/N^2)\,. \tag{24}$$

At first sight, the most interesting quantity to be taken into consideration is the zero of this function, corresponding in the infinite lattice limit to the value of the physical mass. However there is no simple interpretation of this zero value on a finite lattice. A more natural finite lattice operative definition of the mass scale might be the second moment of the correlation function (the "renormalized" mass of ref. [6]). However even in this case there is no unique definition on the finite lattice, and the study of the infinite lattice behaviour is slightly more involved.

Let us therefore focus on the inverse susceptibility

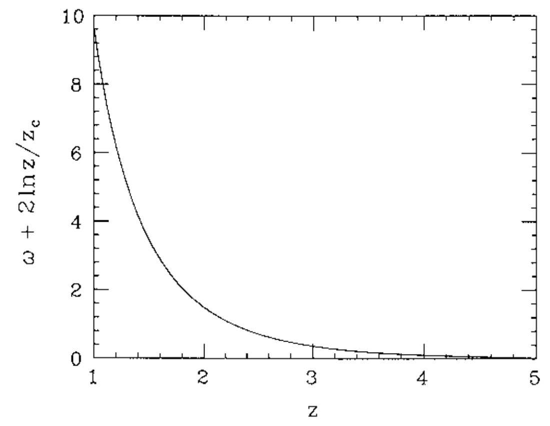

Fig. 1. The logarithm of the large-N finite size scaling function $\omega(z^2)+\ln(z^2/z_c^2)$ as a function of z.

$$G^{-1}(0,T,a,L,N)=\frac{NT}{\chi}\,,\quad \chi=a^2\sum_x\langle \boldsymbol{S}(0)\cdot\boldsymbol{S}(x)\rangle=\chi_0+(1/N)\chi_1+\mathrm{O}(1/N^2)\,. \tag{25, 26}$$

A few trivial manipulations lead to

$$\frac{NT}{\chi_0}=m_0^2(NT,L)\,,\quad -\frac{\chi_1}{\chi_0}=\frac{\Sigma_1(0,NT,L)}{m_0^2(NT,L)}\,, \tag{27}$$

therefore $f_0^{(\chi)}(m_0^2L^2)=f_0^{(m)}(m_0^2L^2)$ and, after substitution in eq. (7b),

$$\frac{\Sigma_1(0,NT,L)}{m_L^2}-\frac{\Sigma_1(0,NT,\infty)}{m_0^2}=-\left(1+\frac{1}{z^2\partial\omega(z^2)/\partial z^2}\right)\frac{m_1^2(NT,\infty)}{m_0^2(NT,\infty)}+\frac{f_1^{(\chi)}(m_0^2L^2)}{f_0^{(\chi)}(m_0^2L^2)}\,. \tag{28}$$

Recalling that [2]

$$m_1^2=\Sigma_1(-m_0^2,NT,\infty)\,, \tag{29}$$

we therefore obtain

$$\frac{f_1^{(\chi)}}{f_0^{(\chi)}}=\frac{\Sigma_1(0,NT,L)}{m_L^2}+\frac{\Sigma_1(-m_0^2,NT,\infty)-\Sigma_1(0,NT,\infty)}{m_0^2}+\frac{\Sigma_1(-m_0^2,NT,\infty)}{m_0^2m_L^2\,\partial\omega(m_L^2L^2)/\partial m_L^2}\,. \tag{30}$$

For the purpose of analytical computation we may also notice that, in the scaling region [7],

$$\frac{\Sigma_1(-m_0^2,NT,\infty)-\Sigma_1(0,NT,\infty)}{m_0^2}=-\left.\frac{\partial\Sigma_1(p^2,NT,\infty)}{\partial p^2}\right|_{p^2=-m_0^2}+3c_1-1-\ln\frac{\pi}{2}\,, \tag{31}$$

the constant c_1 is defined in ref. [2]

$$c_1=\ln\frac{\Gamma(\frac{1}{3})\Gamma(\frac{7}{6})}{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})}=0.4861007...\,. \tag{32}$$

For the purpose of completeness we just mention that a similar analysis performed on the formal definition of the zero of the correlation function leads to

$$\frac{f_1^{(m)}}{f_0^{(m)}}=\frac{m_1^2(NT,L)}{m_0^2(NT,L)}-\frac{m_1^2(NT,\infty)}{m_0^2(NT,\infty)}\frac{1}{z^2\partial\omega(z^2)/\partial z^2}\,. \tag{33}$$

The function $f_1^{(\chi)}$ is in principle completely determined by eq. (30) once all quantities on the RHS are computed in the scaling region in the large-L limit. In practice, we may take two different approaches, and compare them in search for consistency.

The first approach is that of performing a finite lattice weak coupling expansion of the susceptibility [3] and express the results in a scaling form akin to eq. (30). In ref. [3] one can find the following result, exact to $\mathrm{O}(T^2)$:

$$\chi=L^2\{1-(N-1)TS_1+(N-1)T^2[\tfrac{1}{2}S_1-\tfrac{1}{4}S_1(1-1/L^2)]+(N-1)(N-2)S_2T^2\}\,, \tag{34}$$

$$S_n=\frac{a^2}{L^2}\sum_{p\neq 0}\frac{1}{(\hat{p}^2a^2)^n}\,. \tag{35}$$

Let us now perform a $1/N$, large-L expansion of eq. (34) to obtain

$$\frac{\chi}{NT}\simeq L^2\{1/NT-(1-1/N)\tilde{S}_1+T(\tfrac{1}{2}\tilde{S}_1^2-\tfrac{1}{4}\tilde{S}_1)+(1-3/N)d_2NT\}\,, \tag{36}$$

where $\tilde{S}_1$ is simply the large-L limit value of S_1:

$$\tilde{S}_1 = \frac{1}{4\pi} \ln \frac{32L^2}{a^2 z_c^2} . \tag{37}$$

By making use of eq. (18) we obtain

$$\frac{\chi}{NT} \simeq \frac{L^2}{z^2} + \frac{1}{N} L^2 [\tilde{S}_1 + z^2 (\tfrac{1}{2}\tilde{S}_1^2 - \tfrac{1}{4}\tilde{S}_1 - 3d_2)] , \tag{38}$$

and immediately

$$\frac{\chi_0(L)}{NT} = \frac{1}{m_L^2}, \quad \frac{\chi_1(L)}{\chi_0(L)} \cong z^2 [S_1 + z^2 (\tfrac{1}{2}\tilde{S}_1^2 - \tfrac{1}{4}\tilde{S}_1 - 3d_2)] . \tag{39,40}$$

We also need representations in terms of z^2 of the following functions (see ref. [2]):

$$\frac{\Sigma_1(-m_0^2, NT, \infty)}{m_0^2} \equiv \frac{m_1^2}{m_0^2} \cong -\frac{8\pi}{NT} + 2(\ln 16\pi NT + \gamma_E) - 2 + \pi + (-1/2\pi - \pi/8 + 4\pi G_1) NT + O(N^2 T^2) , \tag{41}$$

$$\frac{\Sigma_1(-m_0^2, NT, \infty) - \Sigma_1(0, NT, \infty)}{m_0^2} = \frac{m_1^2}{m_0^2} + \frac{\chi_1}{NT} m_0^2$$

$$\cong -\ln 4\pi NT + 3c_1 - \gamma_E + (\tfrac{1}{4} + 1/2\pi) NT - (\tfrac{1}{32} - G_1) N^2 T^2 + O(N^3 T^3) ,$$

$$G_1 = 0.04616363\ldots . \tag{42}$$

Once all substitutions are performed in eq. (30) we obtain

$$\frac{f_1^{(\chi)}}{f_0^{(\chi)}} = -\ln \frac{4\pi}{z^2} + 2 + 3c_1 - \gamma_E - \frac{z^2}{2\pi} \left(\ln \frac{16\pi}{z^2} + \gamma_E \right) + \frac{z^4}{8\pi^2} + O(z^6) . \tag{43}$$

We notice the complete cancellation of all dependence on L other than the scaling dependence included in the definition of z. This is a nontrivial check of consistency between the $1/N$ and the standard perturbative approach to finite size scaling. Eq. (43) also provides a benchmark for all numerical evaluation of finite size behaviour at least in the regime $z \ll 2\pi$.

The second approach to the determination of the function $f_1^{(\chi)}$ is the numerical evaluation of the RHS of eq. (30) for several values of L/a and $\beta \equiv 1/NT$. This procedure has the appreciable advantage of providing a direct evidence of finite size scaling and a quantitative determination of the onset of scaling, both in L and in β. Some of the quantities appearing in eq. (30) are infinite lattice values of the inverse two-point function at special values of the momentum. In order to compute them we made use of the integral representation introduced in ref. [2] and resorted to accurate algorithms of numerical integration (300 point Gauss integration).

Finite lattice summations were performed for values of L/a ranging from 3 to 160 and β from 0.7 to 0.95, corresponding via eq. (12) to values of z from 1 to 10.

In fig. 2 we present the numerical results for $f_1^{(\chi)}/f_0^{(\chi)}$ by drawing lines through the data at fixed β, together with the weak coupling result of eq. (43). the convergence to a universal finite size scaling function is rather fast in the variable L/a, as already observed in ref. [4]. More crucial is the dependence on β: for values smaller than the onset of (infinite volume) scaling at $\beta \approx 0.8$ we see a significant departure from a universal finite size scaling behaviour. This phenomenon is expected and perfectly understandable from the point of view of Wilson's renormalization group approach.

Let us now imagine to have a purely perturbative determination of f_1/f_0: this would imply the ignorance of the constant appearing in eq. (43). This constant (as well as the ratio $m/\Lambda_{\overline{\rm MS}}$) could be in principle determined by matching the perturbative series for the finite size scaling function to the expected exponential behaviour at large z. However our computation shows that these functions have non monotonic behaviour at intermediate values of z, and a very slow approach to the asymptotic regime. It seems therefore very hard to get good extrap-

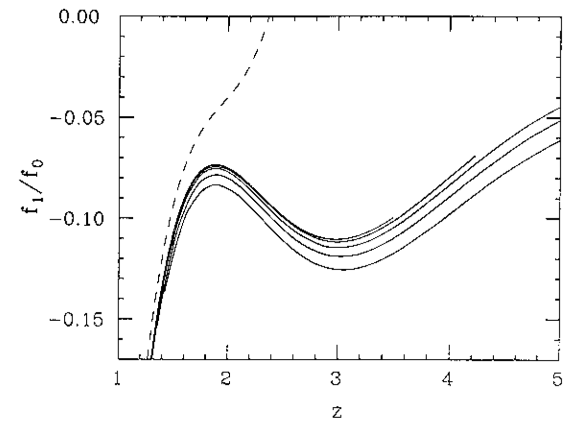

Fig. 2. $f_1^{(z)}/f_0^{(z)}$ as a function of z. The solid lines are numerical results for $\beta=0.70$, 0.75, 0.80, 0.85, 0.90, 0.95 respectively, from bottom to top. the dashed line is the result of eq. (43).

olations to large z values from perturbative results, at least to the satisfactory level one can reach in the large-N limit [5].

We thank H. Flyvbjerg for many interesting conversations on this subject and for communicating us his finite size results prior to publication.

References

[1] J.L. Cardy, ed., Finite-size scaling (North-Holland, Amsterdam, 1988).
[2] P. Biscari, M. Campostrini and P. Rossi, Phys. Lett. B 242 (1990) 225.
[3] H. Flyvbjerg, J. Phys. A 22 (1989) 3393.
[4] M. Campostrini, G. Curci and P. Rossi, Nucl. Phys. B 314 (1989) 467.
[5] M. Lüscher, Phys. Lett. B 118 (1982) 391.
[6] H. Flyvbjerg, Phys. Lett. B 245 (1990) 533.
[7] P. Rossi, unpublished.

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ABSTRACT

Two classes of renormalizable $1/N$ expandable two dimensional models are analyzed to $O(1/N)$ and the asymptotic behavior of the renormalized two-point functions is nonperturbatively evaluated.

These results are taken as a benchmark to study the applicability of dimensional regularization and perturbative minimal subtraction renormalization to the context of the $1/N$ expansion. Perturbation theory is applied to $O(1/N)$ diagrams to all orders in the weak coupling constant, and after resummation the same finite renormalization group invariant asymptotic amplitudes are obtained.

As a byproduct, the $O(1/N)$ contributions to renormalization group Z functions in the minimal subtraction scheme are extracted and the critical index η is evaluated and compared to previous nonperturbative results, finding complete agreement.

An Appendix is devoted to the extension of these results to a supersymmetric version of the models.

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1. Introduction

The importance of dimensional regularization in the context of perturbative quantum field theory cannot be overstressed. Not only it offers the possibility of defining a systematic renormalization algorithm, but also one can exploit its methods and results in the analysis of critical phenomena by the use of Wilson's ε-expansion.

There has however been till now a rather restricted use of this regularization scheme in the context of the so-called $1/N$ expansion approach. This is essentially due to the fact that in the $1/N$ formulation the propagators of the effective excitations, when computed in d dimensions and with nonvanishing mass parameters, can only be expressed in terms of hypergeometric functions, which are quite intractable when integrations or ε-expansions must be performed. One example of this intractability is the fact that a dimensionally regularized version of the $1/N$ expansion may develop $\ln|\varepsilon|$ singularities in addition to poles [1,2].

In turn the $1/N$ expansion can be certainly performed (at least in vector models) by sticking to two dimensions and choosing such regularization schemes as a sharp momentum or a lattice cutoff [3].

Moreover, in the context of the evaluation of critical indices, most difficulties can be bypassed by performing all calculations directly at the critical value of the coupling constant (seen as a function of d) and exploiting the scale properties of correlation functions at criticality [4]. In this way not only critical exponents were computed to $O(1/N^2)$ [4,5], but also one could reconstruct the renormalization group functions β and γ themselves starting from the values of the exponents [6]. In principle, by a generalization of the same algorithm, one might also reconstruct, order by order in $1/N$, the renormalization factors Z, at least in the minimal subtraction scheme.

However it would certainly be important to be able to perform, at least in principle, a direct calculation of the dimensionally regularized Green's functions within the $1/N$ expansion, in order to verify consistency with standard perturbation theory not only in the computation of divergences but also in the renormalized convergent contributions. It would also be pleasant to check the renormalization group invariant structure of the renormalized Green's functions and to study the summability properties of their perturbative series.

In the attempt to close the gap between dimensional regularization and $1/N$ expansion, we started from the observation that most interesting properties of the Green's functions are already contained in their asymptotic behavior, i.e. in the limit when all mass parameters are very small compared to external momenta. These asymptotic behaviors are well defined both in the $1/N$ expansion and in standard perturbation theory in models not plagued by infrared pathologies. Actually, in asymptotically free theories where the appearance of mass is a dynamical phenomenon, the prediction of asymptotic behaviors is all that conventional perturbation theory can reasonably provide.

In the massless limit, the propagators of the effective excitations can be expressed in terms of elementary functions: their mathematical structure is such that, with appropriate manipulations, the correspondence with perturbation theory can be made manifest. Moreover, in this limit it is possible to reduce calculations to typical dimensionally regularized integrals and to resum the results in a compact and renormalization group invariant form ready for comparison with $1/N$ calculations in different schemes.

In the present paper, we consider for definiteness two classes of renormalizable two-dimensional models, O(N) nonlinear sigma and Gross–Neveu, whose structure and $1/N$ expansion are discussed in Sect. 2.

In order to establish a benchmark, we computed the relevant $O(1/N)$ Green's functions in a sharp momentum cutoff scheme, for arbitrary nonzero values of the mass parameter m_0, and presented the results in a renormalization group invariant form in Sect. 3.

The asymptotic behavior of these Green's functions obtained in the $m_0 \to 0$ limit is presented in Sect. 4.

Finally, Sects. 5 and 6 are devoted to the analysis of O(N) and Gross–Neveu models respectively in the context of dimensional regularization. We generated the massless perturbative series, evaluated the relevant Feynman integrals and proceeded to two different resummations of our results, corresponding to an interchange in the order of some limits.

We could show that the two resummations differ only in their divergent parts, corresponding to different choices of renormalization factors Z, while the convergent contribution is universal, renormalization group invariant and corresponds to the asymptotic expansion presented in Sect. 4. The first resummation, which we denote by "nonperturbative renormalization," leads to the appearance of the $\ln|\varepsilon|$ singularities already noticed in [1,2] and is in rather direct correspondence with the sharp momentum regularization scheme. The other one ("perturbative renormalization") is just the standard minimal subtraction scheme pushed to all orders in the coupling. Its divergent parts are the $1/N$ contributions to Z factors and allow for the determination of β and γ functions, which in turn leads to a perturbative evaluation of the $O(1/N)$ contribution to the critical exponent η in full agreement with previously derived and conceptually independent calculations.

The extension of our methods and results to the supersymmetric O(N) models is briefly discussed in the Appendix.

2. The models and their $1/N$ expansion

We are explicitly interested in two classes of models: O(N) nonlinear sigma models [7,8] and U(N) Gross-Neveu models [9] in two dimensions. Beyond obvious differences, these models share a few common features that suggest a common treatment:

- asymptotic freedom;
- dynamic mass generation;
- existence of exact factorized S matrices;
- $1/N$ expandability.

The bare continuum Euclidean lagrangian of the O(N) models is

$$S = \frac{N\beta}{2}\int d^2x\, \partial_\mu \vec{S}\cdot\partial_\mu\vec{S}, \tag{2.1}$$

where $\vec{S}^2 = 1$. It can be turned into an effective action for the Lagrange multiplier field $\alpha(x)$ introduced in order to implement the constraint:

$$S_{\text{eff}}(\alpha) = \frac{N}{2}\left\{\mathrm{Tr}\ln\left[-\Box + i\alpha(x)\right] - i\beta\alpha(x)\right\}, \tag{2.2}$$

This effective action is suitable for a $1/N$ expansion around the classical translation invariant saddle point $\langle\alpha(x)\rangle_0$ satisfying the unrenormalized gap equation

$$\int\frac{d^2p}{(2\pi)^2}\,\frac{1}{p^2 + i\langle\alpha\rangle_0} = \beta \equiv \frac{1}{2\pi f}, \tag{2.3}$$

where we have introduced the (rescaled) weak coupling constant f. The propagator of the quantum fluctuations around $\langle\alpha\rangle_0 = -im_0^2$ is

$$\frac{1}{N}\Delta(k,m_0) = \frac{1}{N}\,\frac{4\pi k^2\xi}{\ln\dfrac{\xi+1}{\xi-1}}, \qquad \xi = \sqrt{1+\frac{4m_0^2}{k^2}}. \tag{2.4}$$

In a sharp momentum (SM) cutoff scheme the gap equation takes the form:

$$\frac{1}{f} = \frac{1}{2}\ln\frac{M^2}{m_0^2} \tag{2.5}$$

The mass gap m^2 is a $1/N$ expandable quantity:

$$m^2 = m_0^2 + \frac{1}{N}m_1^2 + O\left(\frac{1}{N^2}\right) \tag{2.6}$$

and m_1^2 can be analytically computed by evaluating the pole of the invariant two-point function:

$$\frac{1}{p^2 + m_0^2 + \frac{1}{N}\Sigma(p^2)} \tag{2.7}$$

In the SM scheme one obtains [3,10]:

$$m_1^2 = 2m_0^2\left[-\frac{2}{f} + \ln\frac{8}{f} + \gamma_E\right] \tag{2.8}$$

The scaling part of the free energy density, in turn, can be cast into the form:

$$\mathcal{F} = \frac{N-2}{8\pi}m^2 \tag{2.9}$$

Let's now recall some results for the renormalization group functions in the SM scheme from Ref. [3]:

$$\begin{aligned}\beta(f) &= -\frac{N-2}{N}f^2\left[1 + \frac{f}{N} + O\left(\frac{1}{N^2}\right)\right],\\ \gamma(f) &= \frac{N-1}{N}f\left[1 + \frac{f}{N} + O\left(\frac{1}{N^2}\right)\right].\end{aligned} \tag{2.10}$$

The bare continuum Euclidean lagrangian of the U(N) Gross-Neveu models is:

$$S = \int \mathrm{d}^2 x \left[\bar\psi \partial\!\!\!/ \psi - \tfrac{1}{2} g \left(\bar\psi\psi\right)^2 \right]. \tag{2.11}$$

The effective action for the Lagrange multiplier field $\sigma(x)$ is:

$$S_{\mathrm{eff}}(\sigma) = -N \operatorname{Tr} \ln \left[\partial\!\!\!/ - \sigma(x)\right] + \frac{N}{2\pi f} \sigma^2(x) \tag{2.12}$$

where $f \equiv Ng/\pi$. The bare saddle point equation is:

$$\frac{1}{2\pi f} = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{1}{p^2 + \langle\sigma\rangle_0^2}. \tag{2.13}$$

The propagator of the quantum fluctuations around $\langle\sigma\rangle_0 = m_0$ is:

$$\frac{1}{N} \Delta(k, m_0) = \frac{1}{N} \frac{2\pi}{\xi \ln \dfrac{\xi + 1}{\xi - 1}} \tag{2.14}$$

and in the SM regularization scheme:

$$\frac{1}{f} = \ln \frac{M}{m_0}. \tag{2.15}$$

The $1/N$ expansion of the mass gap is:

$$m = m_0 + \frac{1}{N} m_1 + O\left(\frac{1}{N^2}\right) \tag{2.16}$$

and the $1/N$ correction is computed from the pole of the fermion two-point function:

$$\frac{1}{-i p\!\!\!/ - m_0 + \frac{1}{N}\Sigma(p)} \tag{2.17}$$

Its value is:

$$m_1 = -\frac{1}{2} m_0 \left[\frac{2}{f} + \ln \frac{1}{2f} + \gamma_E\right] \tag{2.18}$$

The free energy density satisfies:

$$\mathcal{F} = -\frac{N-1}{4\pi} m^2. \tag{2.19}$$

Finally the $O(1/N)$ renormalization group functions in the SM scheme are:

$$\beta(f) = -\frac{N-1}{N} f^2 \left[1 - \frac{f}{2N} + O\left(\frac{1}{N^2}\right)\right]$$
$$\gamma(f) = O\left(\frac{1}{N^2}\right). \tag{2.20}$$

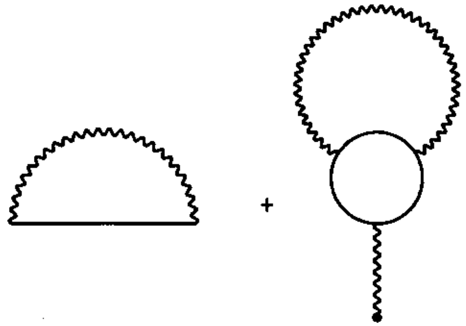

Fig. 1. Feynman graphs corresponding to the $O(1/N)$ contribution to $\Sigma(p)$.

There are deep correspondences between the fermionic and the bosonic models. However, in order to compare them properly one must keep in mind that Gross–Neveu models actually possess a O$(2N)$ symmetry. Therefore comparisons should be made only after a rescaling $N \to N/2$ in the fermionic results. In particular, in Appendix A the U$(N/2)$ Gross–Neveu model will appear as the fermionic part of a supersymmetric action whose bosonic component is the O(N) sigma model.

In these classes of models almost all relevant $O(1/N)$ results can be obtained from the study of the two point correlation function of the fundamental excitations. In turn this function can be parametrized in terms of a self-energy function $\Sigma(p)$, introduced in Eq. (2.7) and Eq. (2.17).

The $O(1/N)$ Feynman graphs contributing to $\Sigma(p)$ have in both classes the same structure, drawn in Fig. 1. The wavy line is the graphical representation of the propagator $\Delta(k, m_0)$, defined in Eq. (2.4) and Eq. (2.14), while the solid line is the bare propagator of the fundamental fields (bosons and fermions respectively).

As a consequence we obtain in the O(N) case [11]:

$$\Sigma_1(p) \equiv \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{\Delta(k)}{(p+k)^2 + m_0^2} + \frac{\Delta(0)}{2} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \Delta(k) \frac{\partial}{\partial m_0^2} \Delta^{-1}(k) \tag{2.21}$$

and in the Gross–Neveu case:

$$\Sigma_1(p) \equiv \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{\Delta(k)}{i p\!\!\!/ + i k\!\!\!/ + m_0} + \frac{\Delta(0)}{2} \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \Delta(k) \frac{\partial}{\partial m_0} \left[\Delta^{-1}(k) - \frac{1}{\pi f}\right], \tag{2.22}$$

where we made use of the gap equation and

$$-\frac{\partial}{\partial m_0} \frac{1}{f} = \frac{1}{m_0}.$$

These results are unavoidably regularization dependent. As we shall see, it is however not too difficult, starting from the first principles of renormalization, to isolate a finite, calculable, scheme independent term and separately compute all divergent regularization dependent contributions.

3. SM regularization and renormalization

The simplest way of computing $1/N$ contributions to the self-energy is the SM scheme. Since the Δ propagators are finite, the Feynman integrals appearing in $\Sigma_1(p)$ are one-loop integrals of regular functions. They can therefore be regularized by the procedure of subtracting explicitly the highest powers of the integration variable appearing in the Taylor expansion of the integrand.

This procedure obviously introduces a dependence on the cutoff value M^2 appearing as a lower limit for the integration of the subtraction terms. However, via the gap equation, this dependence can be completely eliminated in favor of an explicit dependence on the (renormalized) coupling constant, which in turn can be reabsorbed in the renormalization group invariant definition of the physical mass and in the wave function renormalization.

Let's exemplify this procedure by considering the O(N) models. In Eq. (2.21) the function $\Delta(k)$ is rotationally invariant in momentum space. We can therefore perform the angular integration in the two-dimensional momentum plane by applying:

$$\int_0^{2\pi} \frac{d\theta}{2\pi} \frac{1}{(p+k)^2 + m_0^2} = \frac{1}{\sqrt{(p^2+m_0^2+k^2)^2 - 4p^2k^2}}. \qquad (3.1)$$

By applying Eq. (2.4) and Eq. (3.1) and taking derivatives we obtain from Eq. (2.21) the following (non regularized) representation of the self-energy:

$$\Sigma_1(p) = \int_0^\infty \mathrm{d}k^2 \left[\frac{1}{\ln\dfrac{\xi+1}{\xi-1}} \left(\frac{k^2\xi}{\sqrt{(p^2+m_0^2+k^2)^2 - 4p^2k^2}} - \frac{1}{\xi} \right) - \frac{2m_0^2}{4m_0^2+k^2} \right]. \qquad (3.2)$$

A regularized version of Eq. (3.2) is obtained by applying the above described procedure:

$$\Sigma_1^{\mathrm{reg}}(p) = \Sigma_1(p) - \int_{M^2}^\infty \mathrm{d}k^2 \frac{p^2+3m_0^2}{k^2\ln(k^2/m_0^2)} + \int_{M^2}^\infty \mathrm{d}k^2 \frac{2m_0^2}{k^2}. \qquad (3.3)$$

It is convenient to introduce the following identities:

$$\int_0^\infty \mathrm{d}k^2 \frac{1}{\ln\dfrac{\xi+1}{\xi-1}} \left(1 - \frac{1}{\xi}\right) - \int_{M^2}^\infty \mathrm{d}k^2 \frac{2m_0^2}{k^2\ln(k^2/m_0^2)} = 2m_0^2\left(\ln\ln\frac{M^2}{m_0^2} + \gamma_E\right), \qquad (3.4)$$

$$\int_0^\infty \mathrm{d}k^2 \frac{m_0^2}{4m_0^2+k^2} - \int_{M^2}^\infty \mathrm{d}k^2 \frac{m_0^2}{k^2} = \ln\frac{M^2}{4m_0^2}. \qquad (3.5)$$

We can now exploit the scale properties of the dynamical variables and parametrize the self-energy by:

$$\Sigma_1^{\mathrm{reg}}(p) = m_0^2\hat\Sigma_1\left(\frac{p^2}{m_0^2}\right) + (p^2+m_0^2)\left(\ln\ln\frac{M^2}{m_0^2} + \gamma_E\right) + 2m_0^2\left(\ln\frac{4m_0^2}{M^2} + \ln\ln\frac{M^2}{m_0^2} + \gamma_E\right); \qquad (3.6)$$

where, by introducing the rescaled variables

$$x = \frac{p^2}{m_0^2}, \qquad y = \frac{k^2}{m_0^2}, \qquad \xi = \sqrt{1+\frac{4}{y}}$$

we can easily derive from Eq. (3.3) by the use of Eq. (3.4) and Eq. (3.5) the representation:

$$\hat\Sigma_1(x) = \int_0^\infty \mathrm{d}y \frac{1}{\ln\dfrac{\xi+1}{\xi-1}} \left[\frac{\xi y}{\sqrt{(1+y-x)^2+4x}} - 1 + \frac{x+1}{2}\left(\frac{1}{\xi} - 1\right) \right]. \qquad (3.7)$$

By comparing Eq. (3.6) with Eq. (2.8) it seems natural to rewrite our results in the form of a renormalized two-point function:

$$\frac{1 + \frac{1}{N}\left(\ln\frac{1}{2}f - \gamma_E\right)}{p^2 + m^2 + \frac{1}{N}m^2\hat\Sigma_1(p^2/m^2)} \qquad (3.8)$$

satisfying the pole condition $\hat\Sigma_1(-1) = 0$.

The analysis of the Gross–Neveu models goes along the same lines. Let's just quote the following representation of the non regularized self-energy function:

$$\Sigma_1(p) = \frac{-i\not p}{4p^2}\int_0^\infty \mathrm{d}k^2 \frac{1}{\xi\ln\dfrac{\xi+1}{\xi-1}} \left(1 + \frac{p^2 - m_0^2 - k^2}{\sqrt{(p^2+m_0^2+k^2)^2 - 4p^2k^2}}\right)$$
$$+ \frac{m_0}{2}\int_0^\infty \mathrm{d}k^2 \left[\frac{1}{\xi\ln\dfrac{\xi+1}{\xi-1}} \frac{1}{\sqrt{(p^2+m_0^2+k^2)^2 - 4p^2k^2}} + \frac{1}{k^2+4m_0^2} \right], \qquad (3.9)$$

and its regularized counterpart:

$$\Sigma_1^{\mathrm{reg}}(p) = \Sigma_1(p) - \int_{M^2}^\infty \mathrm{d}k^2 \frac{m_0}{2k^2\ln(k^2/m_0^2)} - \int_{M^2}^\infty \mathrm{d}k^2 \frac{m_0}{2k^2}. \qquad (3.10)$$

It is worth noticing that the term proportional to $-i\not p$ requires no regularization, i.e. no wave-function renormalization is needed in this scheme to this order of approximation, consistently with the results presented in Eq. (2.20).

By applying the abovementioned identities and Eq. (2.18) we can reformulate our results in the form of a renormalized two-point function:

$$\frac{1}{-i\not p - m + \frac{1}{N}\left[-i\not p A(p^2/m^2) + mB(p^2/m^2)\right]} \qquad (3.11)$$

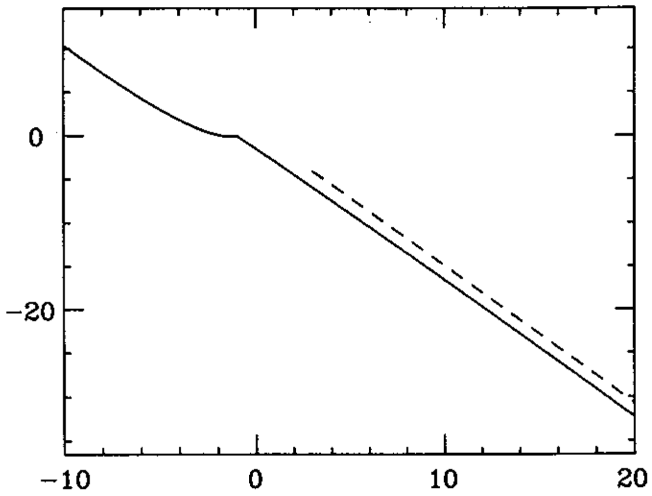

Fig. 2. The $O(1/N)$ contribution to the self-energy of the O(N) models $\hat{\Sigma}_1(x)$ (solid line); its asymptotic expansion (dashed line).

where

$$A(x) = \frac{1}{4x}\int_0^\infty \mathrm{d}y \, \frac{1}{\xi \ln \dfrac{\xi+1}{\xi-1}} \left[1 - \frac{1+y-x}{\sqrt{(1+y-x)^2+4x}}\right] \tag{3.12}$$

and

$$B(x) = \tfrac{1}{2}\int_0^\infty \mathrm{d}y \, \frac{1}{\xi \ln \dfrac{\xi+1}{\xi-1}} \left[\frac{1}{\sqrt{(1+y-x)^2+4x}} + \frac{1-\xi}{2}\right]. \tag{3.13}$$

The pole condition is now reflected in the relationship:

$$A(-1) + B(-1) = 0. \tag{3.14}$$

One may check that the following identity is satisfied:

$$\hat{\Sigma}_1(x) + 4\big(xA(x) - B(x)\big) = 2(1+x)\,B(x). \tag{3.15}$$

Eq. (3.15) finds a natural interpretation in the context of the supersymmetric O(N) models, as shown in the Appendix.

Eqs. (3.7), (3.12) and (3.13) are ready for numerical computation, as well as analytical evaluation at special values of the argument [3]. We plot the self-energy functions in Figs. 2, 3 together with their asymptotic behaviors, that we shall extract in the next section.

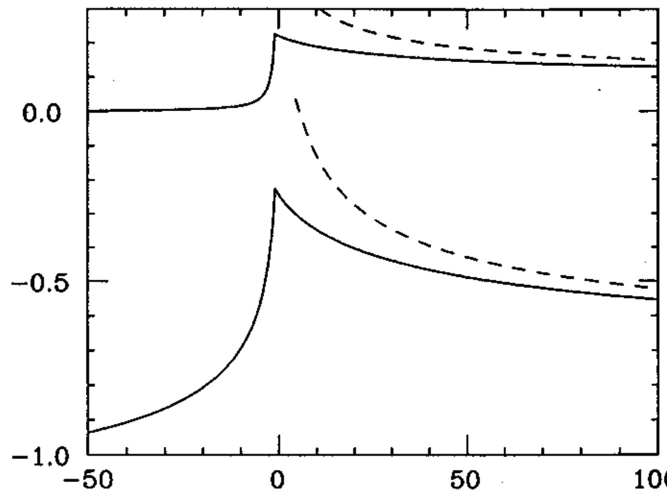

Fig. 3. The two components of the $O(1/N)$ contribution to the self-energy of the Gross–Neveu models $A(x)$ (upper solid line) and $B(x)$ (lower solid line); their asymptotic expansions (dashed lines).

4. Asymptotic Behavior

On the ground of general field-theoretical arguments we expect to be able to obtain the asymptotic behavior of any Green function from perturbative considerations only, and to express it in terms of such renormalization group invariants as the so called "running coupling constant."

One of our aims is to show that such a perturbative approach does indeed converge to the same asymptotic behavior as one may obtain from strictly nonperturbative considerations, like those based on the $1/N$ expansion. Let's therefore first extract the "true" asymptotic behavior from the results of the previous sections.

According to a reasonable definition of asymptopia, and focusing on the O(N) case, we shall define to be "asymptotic" that part of the function $\hat{\Sigma}_1(x)/x$ that is *not* depressed by powers of $1/x$ in the large x limit. We shall however allow for an x dependence that may be formally expressed as a series in the powers of $1/\ln x$.

An analysis of Eq. (3.7) shows that in the asymptotic regime $x \to \infty$ the integral is dominated by the region $x \approx y \gg 1$. We can therefore obtain the relationship

$$\hat{\Sigma}_1(x) \underset{x\to\infty}{\longrightarrow} \int_0^\infty \frac{\mathrm{d}y}{\ln y}\left[\frac{y}{\sqrt{(y-x)^2+4x}} - 1\right] - \int_x^\infty \frac{\mathrm{d}y}{\ln y}\,\frac{x}{y} - x\,(\ln\ln x + \gamma_{\mathrm{E}}), \tag{4.1}$$

where we kept the subleading term $4x$ under the square root in order to regularize the expression in the region $y \approx x$. Changing integration variables to $z = y/x$ we obtain

$$\frac{1}{x}\hat{\Sigma}_1(x) \underset{x\to\infty}{\longrightarrow} \int_0^\infty \frac{dz}{\ln z + \ln x}\left[\frac{z}{\sqrt{(z-1)^2+4/x}} - 1\right] - \int_1^\infty \frac{dz}{\ln z + \ln x}\frac{1}{z} - (\ln\ln x + \gamma_E)\,. \tag{4.2}$$

By trivial manipulations of Eq. (4.2) we get the expression

$$\begin{aligned}\frac{1}{x}\hat{\Sigma}_1(x) \underset{x\to\infty}{\longrightarrow} &\int_0^\infty \frac{dz}{\ln x}\left[\frac{z}{\sqrt{(z-1)^2+4/x}} - 1\right] - \int_1^\infty \frac{dz}{\ln x}\frac{1}{z} - (\ln\ln x + \gamma_E)\\ &- \int_0^\infty \frac{dz}{\ln z + \ln x}\frac{\ln z}{\ln x}\left[\frac{z}{\sqrt{(z-1)^2+4/x}} - 1\right] + \int_1^\infty \frac{dz}{\ln z + \ln x}\frac{\ln z}{\ln x}\frac{1}{z}\,.\end{aligned} \tag{4.3}$$

The first two integrations can be performed analytically, and the term $4/x$ can be safely neglected in the third integral. Therefore we conclude that

$$\begin{aligned}\frac{1}{x}\hat{\Sigma}_1(x) \underset{x\to\infty}{\longrightarrow} &-(\ln\ln x + \gamma_E) + 1 - \frac{2}{\ln x} - \fint_0^\infty \frac{dz}{\ln z + \ln x}\frac{\ln z}{\ln x}\left[\frac{z}{1-z} - 1\right]\\ &- \int_1^\infty \frac{dz}{\ln z + \ln x}\frac{\ln z}{\ln x}\left[\frac{z}{z-1} - 1 - \frac{1}{z}\right]\\ &= -(\ln\ln x + \gamma_E) + 1 - 2\fint_0^1 \frac{dz}{\ln z + \ln x} + 2\fint_0^1 \frac{dz}{\ln x}\frac{\ln^2 z}{\ln^2 x - \ln^2 z}\frac{1}{1-z}\,.\end{aligned} \tag{4.4}$$

This expression admits an asymptotic expansion in the form

$$-(\ln\ln x + \gamma_E) + 1 - 2\sum_{k=0}^\infty \frac{k!}{(\ln x)^{k+1}} + 2\sum_{k=1}^\infty \frac{(2k)!}{(\ln x)^{2k+1}}\zeta(2k+1)\,. \tag{4.5}$$

The first two terms of Eq. (4.5) were evaluated in [12] and a few more terms can be easily deduced from the renormalization group improved expression of the perturbative two point function.

The corresponding computation in the Gross–Neveu model is much simpler and leads to

$$\begin{aligned}A(x) \underset{x\to\infty}{\longrightarrow} &\frac{1}{4x}\int_0^\infty \frac{dy}{\ln y}\left[1 - \frac{y-x}{\sqrt{(y-x)^2+4x}}\right]\\ \longrightarrow &\frac{1}{4}\fint_0^\infty \frac{dz}{\ln z + \ln x}\left[1 - \frac{z-1}{|z-1|}\right]\\ = &\frac{1}{2}\fint_0^1 \frac{dz}{\ln z + \ln x} \equiv \frac{\mathrm{Li}(x)}{2x}\,.\end{aligned} \tag{4.6}$$

The asymptotic behavior of $B(x)$ can easily be obtained from Eq. (3.15).

In Figs. 2, 3 we have drawn the asymptotic behaviors as dashed lines; the agreement of the asymptotic and exact behaviors at rather low values of the variable is as good as one might expect.

This asymptotic analysis is certainly worth some comment. We must observe that, whilst the results are perfectly calculable functions (in the Gross–Neveu case even a known special function), they are certainly not expressible on terms of Borel-summable series in the running coupling constant $f_R \sim 1/\ln x$. This indicates that Borel summability is by no means a necessary (or even desirable) feature of the perturbative series: actually the existence of a Borel ambiguity is strictly related to the very possibility of occurrence of the phenomenon of dynamical mass generation.

5. Dimensional regularization in the $1/N$ expansion: the $O(N)$ models

In order to compare the results presented in the previous sections with standard perturbation theory, it is certainly convenient to assume the dimensional regularization scheme, which will not only allow for a number of extremely useful computation tricks, but also set the stage for an analysis of critical behaviors for real d ($\neq 2$) which we shall develop at the end of our work.

In order to generate the perturbative series keeping only $O(1/N)$ contributions in the limit $m_0 \to 0$ without any formal trouble generated by the bad infrared behavior of the individual Feynman diagrams, we found it convenient to start from our Eq. (2.21) and modify it by adding the quantity

$$\beta\Delta(p) - \int \frac{d^d k}{(2\pi)^d}\frac{\Delta(p)}{(p+k)^2 + m_0^2} = 0\,, \tag{5.1}$$

vanishing because of the gap equation. As a consequence,

$$\Sigma_1(p) = \beta\Delta(p) + \int \frac{d^d k}{(2\pi)^d}\left[\frac{\Delta(k) - \Delta(p)}{(p+k)^2 + m_0^2} + \frac{\Delta(0)}{2}\Delta(k)\frac{\partial}{\partial m_0^2}\Delta^{-1}(k)\right], \tag{5.2}$$

and in the limit $m_0 \to 0$

$$\Delta(k) \longrightarrow \Delta_0(k) = \frac{k^2}{\beta + A_0(k)}\,, \tag{5.3}$$

where

$$A_0(k) = \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\left[\frac{1}{p^2}\frac{k^2}{(p+k)^2} - \frac{1}{p^2} - \frac{1}{(p+k)^2}\right].$$

In the same limit

$$\frac{\Delta(0)}{2}\frac{\partial}{\partial m_0^2}\Delta^{-1}(k) \equiv -\frac{\partial}{\partial\beta}\Delta^{-1}(k) \longrightarrow -\frac{1}{k^2}\,. \tag{5.4}$$

Therefore

$$\Sigma_1(p) \underset{m_0\to 0}{\longrightarrow} \Sigma_{10}(p) = \int \frac{d^dk}{(2\pi)^d} \frac{\Delta_0(k+p) - \Delta_0(p) - \Delta_0(k)}{k^2} + \beta\Delta_0(p) . \tag{5.5}$$

This representation is completely general and free of infrared divergences. Even more important, one may check that its expansion in powers of $1/\beta$ reproduces the $O(1/N)$ diagrams appearing in the standard perturbative series.

Let's however consider dimensional regularization: then by definition the second and third term in the integral vanish and we obtain

$$\Sigma_{10}(p,\beta,d) = \int \frac{d^dk}{(2\pi)^d} \frac{\Delta_0(k)}{(p+k)^2} + \beta\Delta_0(p) , \tag{5.6}$$

where now

$$A_0(k,d) = \tfrac{1}{2}\int \frac{d^dp}{(2\pi)^d} \frac{k^2}{p^2(p+k)^2} = \frac{k^\varepsilon}{\varepsilon} S_d , \qquad S_d = 2\frac{\Gamma\left(2-\frac{1}{2}d\right)}{(4\pi)^{d/2}} \frac{\Gamma\left(\frac{1}{2}d\right)\Gamma\left(\frac{1}{2}d\right)}{\Gamma(d-1)} \longrightarrow \frac{1}{2\pi} , \tag{5.7}$$

and $d = 2+\varepsilon$. To simplify notation, and with no loss of generality, we have chosen the subtraction scale μ of dimensional regularization to be equal to 1.

In order to make correspondence with standard weak coupling perturbation theory [8,13] it is convenient to introduce the (bare) coupling $F \equiv NT \equiv S_d/\beta$ and represent the $O(1/N)$ two point function in the form

$$G(p,F) \cong \frac{F}{p^2}\left[1 - \frac{1}{N}\frac{\Sigma_{10}(p,F)}{p^2}\right] , \tag{5.8}$$

where

$$\frac{\Sigma_{10}(p,F)}{p^2} = \frac{1}{1+Fp^\varepsilon/\varepsilon} + \frac{1}{S_d}\int \frac{d^dk}{(2\pi)^d} \frac{k^2}{p^2(p+k)^2} \frac{F}{1+Fk^\varepsilon/\varepsilon} . \tag{5.9}$$

We can now proceed to an explicit evaluation of the integral by series expansion:

$$\begin{aligned} \frac{\Sigma_{10}(p,F)}{p^2} &= 1 + \sum_{n=1}^{\infty}(-1)^n \left(\frac{F}{\varepsilon}\right)^n p^{\varepsilon n}\left[1 - \frac{\varepsilon}{S_d}\int \frac{d^dk}{(2\pi)^d} \frac{k^2(p^2)^{-1-\frac{\varepsilon n}{2}}}{(p+k)^2}(k^2)^{\frac{\varepsilon(n-1)}{2}}\right] \\ &= 1 + \sum_{n=1}^{\infty}(-1)^n \left(\frac{F}{\varepsilon}\right)^n p^{\varepsilon n}\left[1 - \frac{n-1}{n}\frac{1+\frac{1}{2}\varepsilon(n-1)}{1+\frac{1}{2}\varepsilon(n+1)}\right. \\ &\quad \times \left.\frac{\Gamma\left(1-\frac{1}{2}\varepsilon n\right)\Gamma\left(1+\frac{1}{2}\varepsilon n\right)}{\Gamma\left(1-\frac{1}{2}\varepsilon(n-1)\right)\Gamma\left(1+\frac{1}{2}\varepsilon(n+1)\right)} \frac{\Gamma(1+\varepsilon)}{\Gamma\left(1-\frac{1}{2}\varepsilon\right)\Gamma\left(1+\frac{1}{2}\varepsilon\right)}\right] , \end{aligned} \tag{5.10}$$

where we have applied standard results of integration in d-dimensional space. Eq. (5.10) can be substituted into Eq. (5.8) to obtain

$$G(p,F) \cong \frac{N-1}{Np^2} F\left[1 - \frac{1}{N}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{F}{\varepsilon}\right)^n p^{\varepsilon n} D(\varepsilon,n)\right] , \tag{5.11}$$

where

$$\begin{aligned} D(\varepsilon,n) &= n - (n-1)\frac{1+\frac{1}{2}\varepsilon(n-1)}{1+\frac{1}{2}\varepsilon(n+1)} C(\varepsilon,n) , \\ C(\varepsilon,n) &= \frac{\Gamma\left(1-\frac{1}{2}\varepsilon n\right)\Gamma\left(1+\frac{1}{2}\varepsilon n\right)}{\Gamma\left(1-\frac{1}{2}\varepsilon(n-1)\right)\Gamma\left(1+\frac{1}{2}\varepsilon(n+1)\right)} \frac{\Gamma(1+\varepsilon)}{\Gamma\left(1-\frac{1}{2}\varepsilon\right)\Gamma\left(1+\frac{1}{2}\varepsilon\right)} \\ &\approx 1 + \varepsilon^3\frac{n^2-1}{4}\zeta(3) + O(\varepsilon^4) , \end{aligned} \tag{5.12}$$

and we can easily check that all known perturbative results [13] are correctly reproduced.

In order to renormalize this result we recall that

$$G_R(p,f) = Z^{-1}G(p, F{=}Z_1 f) , \tag{5.13}$$

where f is the renormalized coupling and

$$Z = \frac{1}{1-f/\varepsilon} + O\left(\frac{1}{N}\right), \qquad Z_1 = \frac{1}{1-f/\varepsilon} + O\left(\frac{1}{N}\right). \tag{5.14}$$

Therefore

$$\begin{aligned} G_R(p,f) &= \frac{N-1}{Np^2}\left(Z^{-1}Z_1\right) f\left[1 - \frac{1}{N}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{Z_1 f}{\varepsilon}p^\varepsilon\right)^n D(\varepsilon,n)\right] \\ &= \frac{N-1}{Np^2} f\left[1 - \frac{1}{N}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{f/\varepsilon}{1-f/\varepsilon}p^\varepsilon\right)^n D(\varepsilon,n)\right]\left[1+O\left(\frac{1}{N}\right)\right] . \end{aligned} \tag{5.15}$$

The next step requires some exchange between series summation and $\varepsilon \to 0$ limit, and we shall only consider it as heuristic; later on we shall offer a more rigorous derivation. Let's therefore consider that Eq. (5.15) is a power series in the powers of

$$\frac{-fp^\varepsilon/\varepsilon}{1-f/\varepsilon} = \frac{p^\varepsilon}{1-\varepsilon/f} \underset{\varepsilon\to 0}{\longrightarrow} 1 + \varepsilon\left[\frac{1}{f} + \ln p\right] + O(\varepsilon^2) = 1 + \varepsilon\ln\frac{p}{m_0} + O(\varepsilon^2) , \tag{5.16}$$

while the coefficients of the series are polynomials in n and ε, and the minimum degree in ε associated with degree k in n is $k+1$. However the power series enjoy the property

$$\sum_{n=0}^{\infty} n^k x^n = \frac{k!}{(1-x)^{k+1}} + O\left(\frac{1}{(1-x)^k}\right) \tag{5.17}$$

and therefore performing separate summations over the different powers of n appearing in the coefficients we obtain the minimum degree in ε associated with each summation from

$$\varepsilon^{k+1}\sum_{n=0}^{\infty} n^k \left(1+\varepsilon \ln \frac{p}{m_0}\right)^n \longrightarrow \frac{k!\,(-1)^{k+1}}{(\ln p/m_0)^{k+1}} + O(\varepsilon)\,. \tag{5.18}$$

Hence in the $\varepsilon \to 0$ limit the only surviving contributions are these generated by the leading terms in the coefficients proportional to $\varepsilon^{k+1}n^k$. Expanding the coefficients in εn and keeping only the leading terms we obtain

$$\begin{aligned} D(\varepsilon,n) &\cong n-(n-1)\left(1-\frac{\varepsilon}{1+\frac{1}{2}\varepsilon n}\right)\frac{1}{1+\frac{1}{2}\varepsilon\,\psi\!\left(1-\frac{1}{2}\varepsilon n\right)}\,\frac{1}{1+\frac{1}{2}\varepsilon\,\psi\!\left(1+\frac{1}{2}\varepsilon n\right)}\,(1-\varepsilon\gamma_{\rm E}) \\ &\approx 1+\varepsilon n\left[\frac{1}{1+\frac{1}{2}\varepsilon n}+\tfrac{1}{2}\psi\!\left(1-\tfrac{1}{2}\varepsilon n\right)+\tfrac{1}{2}\psi\!\left(1+\tfrac{1}{2}\varepsilon n\right)+\gamma_{\rm E}\right] \\ &= 1+\varepsilon n\left[\sum_{k=0}^{\infty}\left(-\tfrac{1}{2}\varepsilon n\right)^k-\sum_{k=1}^{\infty}\left(-\tfrac{1}{2}\varepsilon n\right)^{2k}\zeta(2k{+}1)\right]. \end{aligned} \tag{5.19}$$

Substituting in Eq. (5.15) we therefore find

$$\begin{aligned} G_R(p,f) = \frac{N-1}{Np^2}\,f\left\{1+\frac{1}{N}\left[\ln\ln\frac{p^2}{m_0^2}+2\sum_{k=0}^{\infty}\frac{k!}{(\ln p^2/m_0^2)^{k+1}}\right.\right. \\ \left.\left. -2\sum_{k=1}^{\infty}\frac{(2k)!\,\zeta(2k{+}1)}{(\ln p^2/m_0^2)^{2k+1}}+\ln\frac{f}{2(1-f/\varepsilon)}\right]\right\}\left[1+O\!\left(\frac{1}{N}\right)\right], \end{aligned} \tag{5.20}$$

which fixes wavefunction renormalization in this scheme to be

$$Z^{-1}Z_1 = 1+\frac{1}{N}\ln\left(1-\frac{f}{\varepsilon}\right), \tag{5.21}$$

while the asymptotic behavior found in the SM scheme is reproduced exactly.

We would like to call this procedure "nonperturbative renormalization." Due to our manipulations we do not expect it to be fully equivalent to standard MS renormalization; it must however reproduce the same renormalization group invariant asymptotic behavior, as it actually seems to do. To be fully convinced, we must however find a way of performing "perturbative renormalization." This is not actually too difficult to achieve [14].

Let's go back to Eq. (5.9) and notice that, when expressed in terms of the renormalized coupling, the self-energy takes the form

$$\frac{\Sigma_{10}(p,f)}{p^2} = \left(1-\frac{f}{\varepsilon}\right)\frac{1}{1+\frac{f}{\varepsilon}(p^\varepsilon-1)}+\frac{1}{S_d}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{k^2}{(p+k)^2\,p^2}\,\frac{f}{1+\frac{f}{\varepsilon}(k^\varepsilon-1)}\,. \tag{5.22}$$

This expression can be represented in terms of a double summation

$$\begin{aligned} \frac{\Sigma_{10}(p,f)}{p^2} &= \sum_{n=0}^{\infty}\left(1-\frac{f}{\varepsilon}\right)\left(\frac{f}{\varepsilon}\right)^n\sum_{m=0}^{\infty}(-1)^m\binom{n}{m}p^{\varepsilon m} \\ &\quad+\sum_{n=1}^{\infty}\left(\frac{f}{\varepsilon}\right)^n\sum_{m=0}^{n-1}(-1)^m\binom{n-1}{m}\frac{\varepsilon}{S_d}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{k^2k^{\varepsilon m}}{p^2\,(p+k)^2} \\ &= 1+\sum_{n=1}^{\infty}\left(\frac{f}{\varepsilon}\right)^n\sum_{m=0}^{n}\frac{m}{n}(-1)^m\binom{n}{m}p^{\varepsilon m} \\ &\quad-\sum_{n=1}^{\infty}\left(\frac{f}{\varepsilon}\right)^n\sum_{m=1}^{n}(-1)^m\binom{n-1}{m-1}p^{\varepsilon m}\,\frac{m-1}{m}\,\frac{1+\frac{1}{2}\varepsilon(m-1)}{1+\frac{1}{2}\varepsilon(m+1)}\,C(\varepsilon,m) \\ &= 1+\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{f}{\varepsilon}\right)^n\sum_{m=0}^{n}(-1)^m\binom{n}{m}p^{\varepsilon m}D(\varepsilon,m)-\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{f}{\varepsilon}\right)^n\frac{1-\frac{1}{2}\varepsilon}{1+\frac{1}{2}\varepsilon}\,C(\varepsilon,0)\,, \end{aligned} \tag{5.23}$$

where

$$C(\varepsilon,0) = \frac{\Gamma(1+\varepsilon)}{\Gamma^3\!\left(1+\frac{1}{2}\varepsilon\right)\Gamma\!\left(1-\frac{1}{2}\varepsilon\right)}\,.$$

It is important to notice that the summation on n is just a formal representation of the perturbative series in powers of f, while the summation on m is, order by order in m, a finite sum corresponding to an n-th order Feynman diagram and its perturbative counterterms.

We can therefore proceed to conventional MS renormalization by first performing the finite summations on m and disregarding all contributions vanishing in the limit $\varepsilon \to 0$. Let's keep in mind the expansion (5.19) and let's observe that, thanks to the relationships

$$\sum_{m=0}^{n}(-1)^m\binom{n}{m}m^k = \delta_{kn}(-1)^n\,n! \qquad \text{for } k \le n, \tag{5.24}$$

we can easily prove that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^n}\sum_{m=0}^{n}(-1)^m\binom{n}{m}p^{\varepsilon m}(\varepsilon m)^k = \left(\frac{\mathrm{d}}{\mathrm{d}\ln p}\right)^k(-\ln p)^n \tag{5.25}$$

and

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{f}{\varepsilon}\right)^n\sum_{m=0}^{n}(-1)^m\binom{n}{m}p^{\varepsilon m}(\varepsilon m)^k = (-1)^k\,\frac{(k-1)!}{(\ln p/m_0)^k}\,. \tag{5.26}$$

As a consequence we obtain

$$\begin{aligned} \frac{\Sigma_{10}(p,f)}{p^2} &= 1-\ln\ln\frac{p^2}{m_0^2}-2\sum_{k=0}^{\infty}\frac{k!}{(\ln p^2/m_0^2)^{k+1}}-2\sum_{k=1}^{\infty}\frac{(2k)!\,\zeta(2k{+}1)}{(\ln p^2/m_0^2)^{2k+1}} \\ &\quad-\ln\frac{f}{2}+\triangleleft\frac{1-\frac{1}{2}\varepsilon}{1+\frac{1}{2}\varepsilon}\,C(\varepsilon,0)\ln\left(1-\frac{f}{\varepsilon}\right)\triangleright, \end{aligned} \tag{5.27}$$

where we adopted the notation $\lhd\phi(\varepsilon)\rhd$ to indicate the pole part of a Laurent expandable function $\phi(\varepsilon)$. Again we get the same asymptotic behavior of the renormalized Green's functions as in the SM scheme and in our nonperturbative renormalization approach.

Wavefunction renormalization in the MS scheme is then easily found to be

$$Z^{-1}Z_1 = 1 + \frac{1}{N}\lhd\frac{1-\frac{1}{2}\varepsilon}{1+\frac{1}{2}\varepsilon}C(\varepsilon,0)\ln\left(1-\frac{f}{\varepsilon}\right)\rhd. \tag{5.28}$$

Eq. (5.28) has far-reaching consequences. It is a closed form evaluation of the first subleading (in $1/N$) contribution to the renormalization function $Z^{-1}Z_1$ in the context of standard perturbation theory and the MS subtraction scheme. It also offers the possibility of throwing some light on the relationship with nonperturbative renormalizations: indeed if we take the limit $\varepsilon \to 0$ only after the resummation we find that

$$\ln\frac{f}{2} + \frac{1-\frac{1}{2}\varepsilon}{1+\frac{1}{2}\varepsilon}C(\varepsilon,0)\ln\left(1-\frac{f}{\varepsilon}\right) \underset{\varepsilon\to 0}{\longrightarrow} \ln\left[\frac{2}{f}\left(1-\frac{f}{\varepsilon}\right)\right] \longrightarrow -\ln\left|\frac{\varepsilon}{2}\right|, \tag{5.29}$$

a result consistent with Eq. (5.21) and with [1,2].

Finally we can exploit the special properties of the MS scheme to extract some informations on the renormalization group functions of the model. Let's introduce the expansions

$$\begin{aligned}\beta(f) &= \beta_0(f) + \frac{1}{N}\beta_1(f) + O\left(\frac{1}{N^2}\right) = \frac{\varepsilon f}{1 + f\dfrac{\partial}{\partial f}\ln Z_1}, \\ \gamma(f) &= \gamma_0(f) + \frac{1}{N}\gamma_1(f) + O\left(\frac{1}{N^2}\right) = \beta(f)\frac{\partial \ln Z}{\partial f},\end{aligned} \tag{5.30}$$

where

$$\begin{aligned}\beta_0(f) &= \varepsilon f - f^2, \\ \gamma_0(f) &= f.\end{aligned}$$

For any function ϕ analytic in a neighborhood of 0 one can show that

$$(f-\varepsilon)\frac{\partial}{\partial f}\lhd\phi(\varepsilon)\ln\left(1-\frac{f}{\varepsilon}\right)\rhd = \phi(f). \tag{5.31}$$

By use of Eq. (5.31) we can easily prove that

$$\frac{1}{N}\left[\gamma_1(f) + \frac{\beta_1(f)}{f}\right] = \beta_0(f)\frac{\partial}{\partial f}\ln\left(ZZ_1^{-1}\right) = \frac{1}{N}\left[f\frac{1-\frac{1}{2}f}{1+\frac{1}{2}f}C(f,0)\right]. \tag{5.32}$$

As a consequence we can compute the $O(1/N)$ contribution to the critical index η. Indeed from the condition $\beta(f_c) = 0$ we find

$$f_c \cong \varepsilon + \frac{1}{N}\frac{\beta_1(\varepsilon)}{\varepsilon}, \tag{5.33}$$

and from

$$\eta = -\varepsilon + \gamma(f_c) \cong -\varepsilon + f_c + \frac{1}{N}\gamma_1(\varepsilon)$$

we obtain

$$\eta \cong \frac{1}{N}\left[\frac{\beta_1(\varepsilon)}{\varepsilon} + \gamma_1(\varepsilon)\right] = \frac{1}{N}\varepsilon\frac{1-\frac{1}{2}\varepsilon}{1+\frac{1}{2}\varepsilon}C(\varepsilon,0). \tag{5.34}$$

Eq. (5.34) is a known result [15]. However to the best of our knowledge this is the first derivation of the leading (in $1/N$) contribution to η obtained within standard perturbation theory, and therefore showing that weak coupling and $1/N$ expansion commute to second nontrivial order in $1/N$ and to all orders in f.

6. Dimensional regularization in the $1/N$ expansion: the Gross–Neveu models

We may repeat this discussion in the case of the Gross–Neveu models, where some computational simplification occurs. The $m_0 \to 0$ limit of the two point function is already manifestly infrared convergent

$$\Sigma_{10}(p) = \int\frac{d^dk}{(2\pi)^d}\Delta_0(k)\frac{1}{i\not p + i\not k}, \tag{6.1}$$

where

$$\Delta_0(k) = \frac{1}{\frac{1}{Ng} + A_0(k)}, \qquad A_0(k) = \mathrm{tr}\int\frac{d^dq}{(2\pi)^d}\frac{1}{i\not k + i\not q}\frac{1}{i\not q} = 2S_d\frac{k^\varepsilon}{\varepsilon},$$

in dimensional regularization. After the definition $F = 2S_dNg$ one obtains

$$\Sigma_0(p,F) = \frac{1}{2S_d}\int\frac{d^dk}{(2\pi)^d}\frac{1}{i\not p + i\not k}\frac{F}{1+Fk^\varepsilon/\varepsilon}. \tag{6.2}$$

Evaluating the integral by series expansion one obtains

$$\Sigma_{10}(p,F) = \frac{i\not p}{4}\sum_{n=0}^{\infty}\frac{(-1)^n}{n}\left(\frac{F}{\varepsilon}\right)^n p^{\varepsilon n}\varepsilon(n-1)\frac{C(\varepsilon,n)}{1+\frac{1}{2}\varepsilon(n+1)}. \tag{6.3}$$

Nonperturbative renormalization is obtained by the replacement

$$F = \frac{f}{1-f/\varepsilon}, \tag{6.4}$$

leading via the already discussed manipulations to

$$\Sigma_{10}(f,p) = -\frac{i\not p}{2}\left(\frac{m_0^2}{p^2}\right)\mathrm{Li}\left(\frac{p^2}{m_0^2}\right), \tag{6.5}$$

exactly as in the SM scheme.

Perturbative renormalization in turn is obtained starting from

$$
\begin{aligned}
\Sigma_{10}(p,f) &= \frac{1}{2S_d}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\frac{1}{i\not p + i\not k}\,\frac{f}{1+\frac{f}{\varepsilon}(k^\epsilon - 1)} \\
&= \frac{i\not p}{4}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{f}{\varepsilon}\right)^n \sum_{m=1}^{n}(-1)^m \binom{n}{m} p^{\varepsilon m}\frac{\varepsilon(m-1)}{1+\frac{1}{2}\varepsilon(m+1)}\,C(\varepsilon,m) \\
&= \frac{i\not p}{4}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{f}{\varepsilon}\right)^n \sum_{m=0}^{n}(-1)^m \binom{n}{m} p^{\varepsilon m}\frac{\varepsilon(m-1)}{1+\frac{1}{2}\varepsilon(m+1)}\,C(\varepsilon,m) \\
&\quad + \frac{i\not p}{4}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{f}{\varepsilon}\right)^n \frac{\varepsilon}{1+\frac{1}{2}\varepsilon}\,C(\varepsilon,0)\,.
\end{aligned}
\tag{6.6}
$$

Therefore in the MS scheme, removing all terms vanishing in the $\varepsilon \to 0$ limit, we are left with

$$
\begin{aligned}
\Sigma_{10}(p,f) &\longrightarrow \frac{i\not p}{4}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{f}{\varepsilon}\right)^n \sum_{m=0}^{n}(-1)^m \binom{n}{m} p^{\varepsilon m}\frac{\varepsilon m}{1+\frac{1}{2}\varepsilon m} \\
&\qquad - \frac{i\not p}{4}\triangleleft \frac{\varepsilon}{1+\frac{1}{2}\varepsilon}\,C(\varepsilon,0)\ln\left(1-\frac{f}{\varepsilon}\right)\triangleright \\
&\longrightarrow -\frac{i\not p}{2}\left(\frac{m_0^2}{p^2}\right)\mathrm{Li}\left(\frac{p^2}{m_0^2}\right) - \frac{i\not p}{4}\triangleleft \frac{\varepsilon}{1+\frac{1}{2}\varepsilon}\,C(\varepsilon,0)\ln\left(1-\frac{f}{\varepsilon}\right)\triangleright.
\end{aligned}
\tag{6.7}
$$

It is obvious that the last term would vanish if we interchanged summation on n and $\varepsilon \to 0$ limit. From Eq. (6.7) we can extract the wavefunction renormalization factor

$$
Z = 1 - \frac{1}{N}\triangleleft \frac{\varepsilon}{4}\,\frac{1}{1+\frac{1}{2}\varepsilon}\,C(\varepsilon,0)\ln\left(1-\frac{f}{\varepsilon}\right)\triangleright + O\left(\frac{1}{N^2}\right) \tag{6.8}
$$

and the anomalous dimension of the fermionic field

$$
\gamma = \frac{1}{N}\,\frac{f^2}{4}\,\frac{1}{1+\frac{1}{2}f}\,C(f,0)\,. \tag{6.9}
$$

In turn Eq. (6.9) can be used to derive the critical exponent

$$
\eta = \gamma(f_c) \cong \frac{1}{N}\,\frac{\varepsilon^2}{4}\,\frac{1}{1+\frac{1}{2}\varepsilon}\,C(\varepsilon,0) + O\left(\frac{1}{N^2}\right), \tag{6.10}
$$

confirming the results of Refs. [5] and [16].

Appendix A. The supersymmetric O(N) models

The extension of the formalism developed in the present work to the supersymmetric O(N) models requires no major effort. The action of the models [2] is

$$
S = \frac{N}{2\pi f}\int \mathrm{d}^2x\left[\tfrac{1}{2}\,\partial_\mu S_i \partial_\mu S_i + \tfrac{1}{2}\overline{\psi}_i \not\partial \psi_i - \tfrac{1}{8}\left(\overline{\psi}_i\psi_i\right)^2\right] \tag{A.1}
$$

and the fields are submitted to the constraints

$$
S_i S_i = 1, \qquad S_i\psi_i = 0, \qquad \psi_i = \psi_i^\star. \tag{A.2}
$$

The purely bosonic part of the model is just O(N) sigma model while the purely fermionic one reduces to U$(N/2)$ Gross–Neveu model.

Introducing a supermultiplet of Lagrange multipliers (α, u, σ) and integrating over the original fields we obtain the $1/N$ expandable effective action

$$
\begin{aligned}
S_{\mathrm{eff}}(\alpha) &= \frac{N}{2}\left\{\mathrm{Tr}\ln\left[-\Box + i\alpha\right] - \mathrm{Tr}\ln\left[\not\partial - \sigma\right] + \mathrm{Tr}\ln\left[1 - \frac{1}{-\Box + i\alpha}\,\bar u\,\frac{1}{\not\partial - \sigma}\,u\right]\right\} \\
&\quad + \frac{N}{4\pi f}\left[\sigma^2 - i\alpha\right].
\end{aligned}
\tag{A.3}
$$

The gap equation takes the form

$$
\frac{1}{2\pi f} = \int \frac{\mathrm{d}^2p}{(2\pi)^2}\,\frac{1}{p^2 + i\langle\alpha\rangle_0} = \int \frac{\mathrm{d}^2p}{(2\pi)^2}\,\frac{1}{p^2 + \langle\sigma\rangle_0^2}, \tag{A.4}
$$

implying $i\langle\alpha\rangle_0 = \langle\sigma\rangle_0^2 = m_0^2$, while $\langle u\rangle_0 = 0$; in the SM scheme

$$
\frac{1}{f} = \frac{1}{2}\ln\frac{M^2}{m_0^2}. \tag{A.5}
$$

The propagators of the Lagrange multiplier fields can be easily computed; they are related to the corresponding quantities in the bosonic and fermionic models and, with obvious notation, they can be shown to satisfy the following (d-independent) identities:

$$
\Delta_\alpha(k) = (i\not k - 2m_0)\,\Delta_u(k) = \left(k^2 + 4m_0^2\right)\Delta_\sigma(k), \tag{A.6a}
$$

$$
\Delta_\sigma^{-1} = \left(k^2 + 4m_0^2\right)\tfrac{1}{2}\int \frac{\mathrm{d}^dq}{(2\pi)^d}\,\frac{1}{q^2 + m_0^2}\,\frac{1}{(q+k)^2 + m_0^2} \underset{d\to 2}{\longrightarrow} \frac{1}{4\pi}\,\xi\,\ln\frac{\xi+1}{\xi-1}. \tag{A.6b}
$$

It is certainly worth observing that Eq. (A.6a) not depending on d means that supersymmetry is not broken by dimensional regularization, at least to $O(1/N)$. One may immediately apply Eq. (A.6a) to the calculation of the free energy $\mathcal{F}$ in the $1/N$ expansion. It is very easy to show that, consistently with supersymmetry, $\mathcal{F} = 0$, due to the cancellation between bosonic and fermionic one-loop contributions.

The $1/N$ corrections to the propagators of the fundamental fields can be computed along the lines adopted in the previous sections. The diagrammatic representation of the contributions is the same as in Fig. 1, but we must remember that the wavy line may now represent also the exchange of the fermionic Lagrange multiplier. Without belaboring on the evaluation of the individual diagrams, we can collect all results and write down the scalar and fermion self-energy contributions:

$$\begin{aligned}\overset{\text{scalar}}{\Sigma_1(p)} &= \left(p^2+m_0^2\right)\int\frac{d^dk}{(2\pi)^d}\frac{\Delta_\sigma(k)}{(p+k)^2+m_0^2}-4m_0^2\Delta_\sigma(0)\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+4m_0^2},\\ \overset{\text{fermion}}{\Sigma_1(p)} &= -\left(i\not p+m_0\right)\int\frac{d^dk}{(2\pi)^d}\frac{\Delta_\sigma(k)}{(p+k)^2+m_0^2}+2m_0^2\Delta_\sigma(0)\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+4m_0^2}.\end{aligned} \tag{A.7}$$

Therefore the scalar and fermion propagators are renormalized by the same mass and wave-function renormalization; in particular in the MS scheme one obtains the following renormalized Green's functions:

$$\begin{aligned}\overset{\text{scalar}}{G_R(p,f)} &= \frac{1+\frac{1}{N}\left(\ln\frac{1}{2}f-\gamma_E\right)}{1+\frac{1}{N}2B\left(p^2/m^2\right)}\frac{1}{p^2+m^2},\\ \overset{\text{fermion}}{G_R(p,f)} &= \frac{1+\frac{1}{N}\left(\ln\frac{1}{2}f-\gamma_E\right)}{1+\frac{1}{N}2B\left(p^2/m^2\right)}\frac{1}{-i\not p-m},\end{aligned} \tag{A.8}$$

where

$$m=m_0\left[1-\frac{1}{N}\left(\frac{2}{f}-\ln 4\right)\right]. \tag{A.9}$$

Again one immediately verifies from Eq. (A.7) that supersymmetry is unbroken to this order by dimensional regularization. In such schemes as SM and MS one obtains

$$\beta(f)=-\frac{N-2}{N}f^2\left[1+O\left(\frac{1}{N^2}\right)\right]. \tag{A.10a}$$

Moreover in the SM scheme one gets

$$\gamma(f)=\frac{N-1}{N}f\left[1+O\left(\frac{1}{N^2}\right)\right]. \tag{A.10b}$$

It is interesting to notice that in the light of these results, after proper rescalings, the identity (3.15) reflects in a very pictorial way the amalgam nature of the supersymmetric O(N) models.

Finally let's comment about asymptotic behavior and MS renormalization. From Eq. (3.15) one gets

$$\lim_{x\to\infty}2B(x)=\lim_{x\to\infty}\frac{\hat\Sigma_1(x)}{x}+4A(x)=-(\ln\ln x+\gamma_E)+1+2\sum_{k=1}^{\infty}\frac{(2k)!}{(\ln x)^{2k+1}}\zeta(2k+1). \tag{A.11}$$

In dimensional regularization one may show that

$$\begin{aligned}\lim_{m_0\to 0}\int\frac{d^dk}{(2\pi)^d}\frac{\Delta_\sigma(k)}{(p+k)^2+m_0^2} &= \frac{S_d}{F}\Delta_{\sigma 0}(k)+\int\frac{d^dk}{(2\pi)^d}\frac{\Delta_{\sigma 0}(k)}{(p+k)^2},\\ S_d\,\Delta_{\sigma 0}(k) &= \frac{F}{1+Fk^\varepsilon/\varepsilon},\end{aligned} \tag{A.12}$$

and, after by now standard manipulations, one may get the following representation of the above expression:

$$1+\sum_{n=1}^{\infty}(-1)^n\left(\frac{F}{\varepsilon}\right)^n p^{\varepsilon n}\left[1-\frac{n-1}{n}C(\varepsilon,n)\right], \tag{A.13}$$

leading to the asymptotic behavior (A.11). By applying perturbative renormalization we may also extract the relationship

$$Z^{-1}Z_1=1+\frac{1}{N}\triangleleft C(\varepsilon,0)\ln\left(1-\frac{f}{\varepsilon}\right)\triangleright+O\left(\frac{1}{N^2}\right), \tag{A.14}$$

implying

$$\eta\cong\frac{1}{N}\varepsilon C(\varepsilon,0), \tag{A.15}$$

in agreement with [17].

References

1. C. Rim and W.I. Weisberger, Phys. Rev. Lett. 53 (1984) 965;
   Phys. Rev. D30 (1984) 1763.
2. A.C. Davis, J.A. Gracey, A.J. Macfarlane and M.G. Mitchard,
   Nucl. Phys. B314 (1989) 439.
3. P. Biscari, M. Campostrini and P. Rossi, Phys. Lett. 242B (1990) 225.
4. A.N. Vasil'ev, Yu.M. Pis'mak and Yu.R. Honkonen,
   Theor. Math. Phys. 46 (1981) (157); 47 (1981) 291; 50 (1982) 195.
5. J.A. Gracey, Int. Journ. Mod. Phys. A6 (1991) 395.
6. S. Hikami and E. Brezin, J. Phys. A: Math. Gen. 11 (1978) 1141.
7. A. M. Polyakov, Phys. Lett. 59B (1975) 79;.
8. E. Brezin and J. Zinn-Justin, Phys. Rev. Lett. 36 (1975) 691;
   E. Brezin, J. Zinn-Justin and J.C. Le Guillou, Phys. Rev. D14 (1976) 2615.
9. D. J. Gross and A. Neveu, Phys. Rev. D10 (1974) 3235.
10. H. Flyvbjerg, Phys. Lett. 245B (1990) 533.
11. V.F. Müller, T. Raddatz and W. Rühl, Nucl. Phys. B251 [FS 13] (1985) 212;
    Nucl. Phys. B259 (1985) 745 (E);
    G. Cristofano, R. Musto, F. Nicodemi, R. Pettorino and F. Pezzella,
    Nucl. Phys. B257 [FS 14] (1985) 505.
12. J. Orloff and R. Brout, Nucl. Phys. B270 [FS16] (1986) 273.
13. D.J. Amit and G.B. Kotliar, Nucl. Phys. B170 [FS 1](1980)187.
14. G. Paffuti and P. Rossi, Phys. Lett. 234B (1990) 85.
15. R. Abe and S. Hikami, Prog. Theor. Phys. 49 (1973) 442;
    E. Brezin and O.J. Wallace, Phys. Rev. B7 (1973) 1967;
    S.K. Ma, Phys. Rev. A7 (1973) 2172.
16. S. Hikami and T. Muta, Prog. Theor. Phys. 57 (1977) 785.
17. J.A. Gracey, J. Phys. A: Math. Gen. 23 (1990) 2183.