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$1/N$ Expansion of the topological susceptibility in the CP^{N-1} models

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ABSTRACT

We compute the $O(1/N^2)$ correction to the topological susceptibility χ_t of the two-dimensional continuum CP^{N-1} models. We define a measurable, $1/N$ expandable correlation length ξ and evaluate the dimensionless ratio $\chi_t\xi^2 \cong 1/(2\pi N)(1-0.38/N)$. We also compute the second moment of the topological susceptibility.

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1. Introduction

Two dimensional CP^{N-1} models [1,2] can play an important rôle as a theoretical laboratory for the test of analytical and numerical methods in confining, asymptotically free quantum field theories. A pleasant feature of these models is the possibility of performing a systematic $1/N$ expansion around the large N saddle point solution.

A systematic presentation of the results obtained by computing the first nonleading order of the $1/N$ expansion was given in Ref. [3]. In this letter we add an important piece of calculation by evaluating the $O(1/N^2)$ correction to the topological susceptibility of the CP^{N-1} models. At the same time we briefly review some of our previous results explicitly extracting those predictions that are independent of the renormalization and regularization scheme and can therefore be directly tested by numerical simulations on the lattice.

2. The $1/N$ expansion of the CP^{N-1} models

The bare continuum lagrangian of the two-dimensional CP^{N-1} models is [1,2]

$$S = \frac{N}{2f}\int d^2x\, \overline{D_\mu z}\, D_\mu z\,, \tag{2.1}$$

where z is an N-component complex vector field subject to the constraint $\bar{z}z = 1$ and a covariant derivative $D_\mu \equiv \partial_\mu + iA_\mu$ has been defined in terms of the composite gauge fields

$$A_\mu = \tfrac{1}{2}i\{\bar{z}\,\partial_\mu z - \partial_\mu\bar{z}\,z\}\,. \tag{2.2}$$

The $1/N$ expansion of the generating functional is obtained by introducing Lagrange multiplier fields and integrating over the vector fields z. The resulting effective action is

$$S_{\rm eff}(\alpha,\lambda_\mu) = N\,\mathrm{Tr}\ln\{-\partial_\mu\partial_\mu - i\{\partial_\mu,\lambda_\mu\} + m_0^2 + i\alpha_q\} + \frac{N}{2f}\int d^2x\,\{-i\alpha_q + \lambda_\mu\lambda_\mu\}\,. \tag{2.3}$$

$-im_0^2$ is the large N vacuum expectation value of the α field; it is determined as a function of f from the saddle point condition

$$\left.\frac{\delta S_{\rm eff}}{\delta\alpha}\right|_{\alpha=0} \equiv N\int\frac{d^2p}{(2\pi)^2}\,\frac{1}{p^2+m_0^2} - \frac{N}{2f} = 0, \tag{2.4}$$

leading in the sharp-momentum (SM) cutoff regularization scheme [4] to the relationship

$$\frac{\pi}{f} = \tfrac{1}{2}\ln\frac{M^2}{m_0^2}\,. \tag{2.5}$$

The propagators of the Lagrange multiplier fields are obtained by taking the second derivatives of the effective action at the saddle point:

$$\Delta^{-1}_{(\alpha)} = \int\frac{d^2q}{(2\pi)^2}\,\frac{1}{q^2+m_0^2}\,\frac{1}{(p+q)^2+m_0^2} = \frac{1}{2\pi p^2\xi}\ln\frac{\xi+1}{\xi-1}\,, \tag{2.6}$$

$$\Delta^{(\lambda)-1}_{\mu\nu} = 2\delta_{\mu\nu}\int \frac{d^2q}{(2\pi)^2}\,\frac{1}{q^2+m_0^2} - \int \frac{d^2q}{(2\pi)^2}\,\frac{(p_\mu+2q_\mu)(p_\nu+2q_\nu)}{[q^2+m_0^2]\left[(p+q)^2+m_0^2\right]}$$
$$= \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{1}{2\pi}\left(\xi\ln\frac{\xi+1}{\xi-1} - 2\right) \equiv \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\Delta^{-1}_{(\lambda)}, \tag{2.7}$$

where $\xi = \sqrt{1+4m_0^2/p^2}$. The appearance of a massless pole in the vector propagator (2.7) leads to the systematic appearance of infrared divergencies in all expectation values of operators that are not gauge-invariant. However these divergencies cancel when one computes gauge-invariant expectation values. The Feynman rules of the $1/N$ expansion are drawn in Fig. 1.

In order to compute higher order corrections to large N expectation values one also needs to evaluate the effective vertices (which are nothing but one-loop integrals over the z field propagators). For our purposes we shall introduce the vertices shown in Fig. 2 and corresponding to the following expressions:

$$V_3(p,k) = \int \frac{d^2q}{(2\pi)^2}\,\frac{1}{q^2+m_0^2}\,\frac{1}{(p+q)^2+m_0^2}\,\frac{1}{(k+q)^2+m_0^2}; \tag{2.8}$$

$$V_{3\,\mu\nu}(p,k) = \int \frac{d^2q}{(2\pi)^2}\,\frac{(2q_\mu+k_\mu)(2q_\nu+p_\nu)}{[q^2+m_0^2]\left[(p+q)^2+m_0^2\right]\left[(k+q)^2+m_0^2\right]}; \tag{2.9}$$

$$V_4^{(a)}(p,k) = \int \frac{d^2q}{(2\pi)^2}\,\frac{1}{[q^2+m_0^2]^2}\,\frac{1}{(p+q)^2+m_0^2}\,\frac{1}{(k+q)^2+m_0^2}; \tag{2.10a}$$

$$V_4^{(b)}(p,k) = \int \frac{d^2q}{(2\pi)^2}\,\frac{1}{q^2+m_0^2}\,\frac{1}{(p+q)^2+m_0^2}\,\frac{1}{(k+q)^2+m_0^2}\,\frac{1}{(p+k+q)^2+m_0^2}; \tag{2.10b}$$

$$V_{\mu\nu}^{(a)}(p,k) = \int \frac{d^2q}{(2\pi)^2}\,\frac{1}{[q^2+m_0^2]^2}\,\frac{(2q_\mu+k_\mu)(2q_\nu+k_\nu)}{\left[(p+q)^2+m_0^2\right]\left[(k+q)^2+m_0^2\right]}; \tag{2.11a}$$

$$V_{\mu\nu}^{(b)}(p,k) = \int \frac{d^2q}{(2\pi)^2}\,\frac{1}{q^2+m_0^2}\,\frac{2q_\mu+k_\mu}{(k+q)^2+m_0^2}\,\frac{1}{(p+q)^2+m_0^2}\,\frac{2q_\nu+2p_\nu+k_\nu}{(p+k+q)^2+m_0^2}; \tag{2.11b}$$

$$V_{\mu\nu,\rho\sigma}^{(a)}(p,k) = \int \frac{d^2q}{(2\pi)^2}\,\frac{(2q_\mu+k_\mu)(2q_\nu+k_\nu)(2q_\rho+p_\rho)(2q_\sigma+p_\sigma)}{[q^2+m_0^2]^2\left[(p+q)^2+m_0^2\right]\left[(k+q)^2+m_0^2\right]}; \tag{2.12a}$$

$$V_{\mu\nu,\rho\sigma}^{(b)}(p,k) = \int \frac{d^2q}{(2\pi)^2}\,\frac{(2q_\mu+2p_\mu+k_\mu)(2q_\nu+k_\nu)(2q_\rho+p_\rho)(2q_\sigma+2k_\sigma+p_\sigma)}{[q^2+m_0^2]\left[(k+q)^2+m_0^2\right]\left[(p+q)^2+m_0^2\right]\left[(p+k+q)^2+m_0^2\right]}. \tag{2.12b}$$

Some of these quantities have been explicitly computed in Ref. [3], and we refer the interested reader to it for the resulting (rather cumbersome) expressions. Suffice to say that all vertices can be analytically computed in terms of elementary functions and that all physically relevant (gauge-invariant) combinations can be expressed in terms of the scalar vertices (2.8), (2.10). We shall later come back to this point.

3. Renormalization group invariant quantities

From a conceptual and computational point of view, most interesting are those quantities Q_i that, besides being gauge-invariant, are also solutions of the homogeneous renormalization group equation

$$\left[M\frac{\partial}{\partial M} + \beta(f)\frac{\partial}{\partial f}\right] Q_i = 0, \tag{3.1}$$

where the renormalization group β-function has been computed to $O(1/N)$ within the SM renormalization scheme in Ref. [3] and it turns out to be

$$\beta(f) = -\frac{f^2}{\pi}\left\{1 + \frac{1}{N}\frac{f}{2\pi}\left(1+\frac{3}{1-f/\pi}\right)\right\} + O\left(\frac{f^4}{N^2}\right). \tag{3.2}$$

All solutions of Eq. (3.1) can be expressed in the general form

$$Q_i = C_i(N)\left[m^2(M^2,f)\right]^{\Delta_i}, \tag{3.3}$$

where Δ_i is the dimension of Q_i in units of square mass, C_i is a purely numerical coefficient and

$$m^2 = M^2 \exp\left[-2\int^f \frac{df'}{\beta(f')}\right]. \tag{3.4}$$

Eq. (3.2) immediately implies that

$$m^2 = M^2 \exp\left(-\frac{2\pi}{f}\right)\left\{1 + \frac{1}{N}\left[\ln\frac{\pi}{f} + 3\ln\left(\frac{\pi}{f}-1\right)\right] + O\left(\frac{f}{N^2}\right)\right\}$$
$$= m_0^2\left\{1 + \frac{1}{N}\left[\ln\ln\frac{M}{m_0} + 3\ln\left(\ln\frac{M}{m_0}-1\right)\right] + O\left(\frac{1}{N^2}\right)\right\}, \tag{3.5}$$

where we made use of the mass-gap equation (2.5).

We now want to define a physical correlation length ξ. However it has been shown in Ref. [3] that the natural definition $\overline{\xi}$, related to the large distance exponential decay of correlation functions, leads to a coefficient $C_{\overline{\xi}}(N)$ which is a nonanalytic function of $1/N$ around $N=\infty$, a quite unpleasant behavior from a computational point of view.

We therefore take the following alternative definition:

$$\xi^2 = \frac{\int d^2x\,\frac{1}{4}x^2\,\langle \mathrm{Tr}P(x)\,P(0)\rangle}{\int d^2x\,\langle \mathrm{Tr}P(x)\,P(0)\rangle}, \tag{3.6}$$

where $P_{ij}(x) = \bar{z}_i(x)\,z_j(x)$ is a local gauge-invariant composite operator. To justify our choice, let's notice that for large N Eq. (3.6) implies $\xi = \sqrt{2/3}\,\overline{\xi}$, while for $N=2$ (O(3) nonlinear σ model) the agreement between the two definitions is within 1%. ξ and $\overline{\xi}$ never disagree more than 20%, and a study of the nonanalytic behavior of the true mass gap shows that the agreement improves fast with decreasing N.

In order to compute ξ^2 in the $1/N$ expansion, we only remind from Ref. [3] that the following relationship holds:

$$\langle \mathrm{Tr} P(x) P(0) \rangle = \frac{4f^2}{N^3}\left(N^2-1\right)\left(N\Delta^{-1}_{(\alpha)} + \Delta^{-1}_{\mathrm{T}(\alpha)}\right), \tag{3.7}$$

where $\Delta^{-1}_{\mathrm{T}(\alpha)}$ is the sum of the "tadpole" contributions to the full inverse propagator of the quantum field α.

Finally we want to define the renormalization group invariant topological susceptibility

$$\chi_t = \int \mathrm{d}^2 x \,\langle q(x)\, q(0)\rangle\,, \tag{3.8}$$

where

$$q(x) = \frac{\mathrm{i}}{2\pi}\,\varepsilon_{\mu\nu}\,\overline{D_\mu z}\,D_\nu z = \frac{1}{2\pi}\,\varepsilon_{\mu\nu}\partial_\mu A_\nu\,. \tag{3.9}$$

From Eqs. (3.8) and (3.9) one can easily show that

$$\chi_t = \lim_{p^2\to 0}\frac{1}{(2\pi)^2}\,p^2\,\tilde{\Delta}_{(\lambda)}(p)\,, \tag{3.10}$$

where $\tilde{\Delta}^{-1}_{(\lambda)}(p)\left(\delta_{\mu\nu} - p_\mu p_\nu/p^2\right)$ is the full inverse propagator of the quantum field λ_μ.

In the large N limit, Eq. (2.7) trivially implies [5]

$$\chi_t \cong \frac{3}{\pi N}\,m_0^2 + O\!\left(\frac{1}{N^2}\right). \tag{3.11}$$

4. $1/N$ computations of ξ and χ_t

The $O(1/N)$ contributions to the inverse propagators of the α and λ_μ fields are drawn in Figs. 3 and 4 respectively; sets of diagrams whose contributions sum up trivially to zero by some generalization of Furry's theorem have been removed.

Let's now write down the $1/N$ contributions to the Lagrange multiplier field propagators. After a few simple manipulations, and keeping in mind that we are only interested in the "tadpole" contributions to $\tilde{\Delta}_{(\alpha)}$, we obtain

$$\begin{aligned}\Delta^{-1}_{1(\alpha)}(p) = &-\frac{1}{N}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\,\Delta_{(\alpha)}(k)\,W(p,k)\\ &+\frac{1}{N}\Delta_{(\alpha)}(0)\,\frac{\partial}{\partial m_0^2}\Delta^{-1}_{(\alpha)}(p)\int\frac{\mathrm{d}^2k}{(2\pi)^2}\,\Delta_{(\alpha)}(k)\,\tfrac{1}{2}\,\frac{\partial}{\partial m_0^2}\Delta^{-1}_{(\alpha)}(k)\\ &+\frac{1}{N}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\,\Delta^{(\lambda)}_{\mu\nu}(k)\,W_{\mu\nu}(p,k)\\ &+\frac{1}{N}\Delta_{(\alpha)}(0)\,\frac{\partial}{\partial m_0^2}\Delta^{-1}_{(\alpha)}(p)\int\frac{\mathrm{d}^2k}{(2\pi)^2}\,\Delta_{(\lambda)}(k)\,\tfrac{1}{2}\,\frac{\partial}{\partial m_0^2}\Delta^{-1}_{(\lambda)}(k)\,,\end{aligned} \tag{4.1}$$

$$\begin{aligned}\Delta^{(\lambda)-1}_{1\,\mu\nu}(k) = &\frac{1}{N}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\,\Delta_{(\alpha)}(p)\,W_{\mu\nu}(p,k)\\ &-\frac{1}{N}\Delta_{(\alpha)}(0)\,\frac{\partial}{\partial m_0^2}\Delta^{(\lambda)-1}_{\mu\nu}(k)\int\frac{\mathrm{d}^2p}{(2\pi)^2}\,\Delta_{(\alpha)}(p)\,\tfrac{1}{2}\,\frac{\partial}{\partial m_0^2}\Delta^{-1}_{(\alpha)}(p)\\ &-\frac{1}{N}\int\frac{\mathrm{d}^2p}{(2\pi)^2}\,\Delta^{(\lambda)}_{\rho\sigma}(p)\,W_{\mu\nu,\rho\sigma}(p,k)\\ &+\frac{1}{N}\Delta_{(\alpha)}(0)\,\frac{\partial}{\partial m_0^2}\Delta^{(\lambda)-1}_{\mu\nu}(k)\int\frac{\mathrm{d}^2p}{(2\pi)^2}\,\Delta_{(\lambda)}(p)\,\tfrac{1}{2}\,\frac{\partial}{\partial m_0^2}\Delta^{-1}_{(\lambda)}(p)\,,\end{aligned} \tag{4.2}$$

where we have introduced the following combinations of vertices:

$$W(p,k) = V_4^{(a)}(p,k) + V_4^{(a)}(p,-k) + V_4^{(b)}(p,k)\,, \tag{4.3}$$

$$W_{\mu\nu}(p,k) = V_{\mu\nu}^{(a)}(p,k) + V_{\mu\nu}^{(a)}(p,-k) + V_{\mu\nu}^{(b)}(p,k) + \delta_{\mu\nu}\,\frac{\partial}{\partial m_0^2}\Delta^{-1}_{(\alpha)}(p)\,, \tag{4.4}$$

$$\begin{aligned}W_{\mu\nu,\rho\sigma}(p,k) = &V^{(a)}_{\mu\nu,\rho\sigma}(p,k) + V^{(a)}_{\mu\nu,\rho\sigma}(p,-k) + V^{(b)}_{\mu\nu,\rho\sigma}(p,k) - 2\delta_{\mu\nu}\delta_{\rho\sigma}\,\frac{\partial}{\partial m_0^2}\Delta^{-1}_{(\alpha)}(0)\\ &-\delta_{\mu\nu}\,\frac{\partial}{\partial m_0^2}\Delta^{(\lambda)-1}_{\rho\sigma}(p) - \delta_{\rho\sigma}\,\frac{\partial}{\partial m_0^2}\Delta^{(\lambda)-1}_{\mu\nu}(k)\\ &-2V_{3\,\mu\rho}(k,p)\,V_{3\,\nu\sigma}(k,p)\,\Delta_{(\alpha)}(p-k)\\ &-2V_{3\,\mu\rho}(k,-p)\,V_{3\,\nu\sigma}(k,-p)\,\Delta_{(\alpha)}(p+k)\,.\end{aligned} \tag{4.5}$$

These combinations are the same as those appearing in the corresponding four-point elastic scattering amplitudes, and possess the crucial property of gauge invariance. One can indeed verify that

$$k_\mu W_{\mu\nu}(p,k) = k_\nu W_{\mu\nu}(p,k) = 0, \tag{4.6}$$

$$k_\mu W_{\mu\nu,\rho\sigma}(p,k) = k_\nu W_{\mu\nu,\rho\sigma}(p,k) = W_{\mu\nu,\rho\sigma}(p,k)\,p_\rho = W_{\mu\nu,\rho\sigma}(p,k)\,p_\sigma = 0. \tag{4.7}$$

In two dimensions, Eqs. (4.6) and (4.7) lead to a unique parametrization of the vertices (4.3) and (4.4):

$$W_{\mu\nu}(p,k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) W_1(p,k)\,, \tag{4.8}$$

$$W_{\mu\nu,\rho\sigma}(p,k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\left(\delta_{\rho\sigma} - \frac{p_\rho p_\sigma}{p^2}\right) W_2(p,k)\,. \tag{4.9}$$

In turn, using the explicit form of the scalar vertices V_3, $V_4^{(a)}$ and $V_4^{(b)}$, one can show that

$$\begin{aligned}W_1(p,k) = &-\left(k^2+4m_0^2\right)\left[V_4^{(a)}(p,k) + V_4^{(a)}(p,-k)\right]\\ &-\left(k^2+2p^2+4m_0^2\right)V_4^{(b)}(p,k) + 4\left[V_3(p,k) + V_3(p,-k)\right]\end{aligned} \tag{4.10}$$

and

$$W_2(p,k) = \left(k^2+4m_0^2\right)\left(p^2+4m_0^2\right)\left[V_4^{(a)}(p,k)+V_4^{(a)}(p,-k)\right] + \left(k^2+2p^2+4m_0^2\right)\left(p^2+2k^2+4m_0^2\right)V_4^{(b)}(p,k) - 4\left(k^2+p^2+4m_0^2\right)\left[V_3(p,k)+V_3(p,-k)\right] - 2\,\frac{p^2k^2}{(p\cdot k)^2}\left[\Delta_{(\alpha)}(p+k)\,Z_+^2+\Delta_{(\alpha)}(p-k)\,Z_-^2\right], \tag{4.11}$$

where

$$Z_\pm = \left(k^2+p^2\pm 4m_0^2\right)V_3(\mp p,k) - \Delta^{-1}_{(\alpha)}(p) - \Delta^{-1}_{(\alpha)}(k). \tag{4.12}$$

Analyticity leads to the following properties, that we explicitly verified:

$$\lim_{k^2\to 0} W_1(p,k) = 0, \tag{4.13}$$

$$\lim_{k^2\to 0} W_2(p,k) = \lim_{p^2\to 0} W_2(p,k) = 0. \tag{4.14}$$

Eqs. (4.13) and (4.14), together with gauge invariance, are crucial in making Eqs. (4.1) and (4.2) gauge-independent and free of infrared divergences. Moreover they imply the transversality of the vector field propagator and the survival of the massless pole.

In conclusion, we have the following representation of our results:

$$\Delta^{-1}_{1(\alpha)}(p) = -\frac{1}{N}\int\frac{d^2k}{(2\pi)^2}\,\Delta_{(\alpha)}(k)\,W(p,k) + \frac{1}{N}\int\frac{d^2k}{(2\pi)^2}\,\Delta_{(\lambda)}(k)\,W_1(p,k) - \frac{1}{N}\int\frac{d^2k}{(2\pi)^2}\,\frac{\Delta_{(\alpha)}(k)+k^2\Delta_{(\lambda)}(k)}{k^2+4m_0^2}\,\frac{\partial}{\partial m_0^2}\Delta^{-1}_{(\alpha)}(p), \tag{4.15}$$

$$\Delta^{(\lambda)\,-1}_{1\,\mu\nu}(k) = \left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\Delta^{-1}_{1(\lambda)}(k), \tag{4.16}$$

$$\Delta^{-1}_{1(\lambda)}(k) = \frac{1}{N}\int\frac{d^2p}{(2\pi)^2}\,\Delta_{(\alpha)}(p)\,W_1(p,k) - \frac{1}{N}\int\frac{d^2p}{(2\pi)^2}\,\Delta_{(\lambda)}(k)\,W_2(p,k) - \frac{1}{N}\int\frac{d^2p}{(2\pi)^2}\,\frac{\Delta_{(\alpha)}(p)+p^2\Delta_{(\lambda)}(p)}{p^2+4m_0^2}\,\frac{\partial}{\partial m_0^2}\Delta^{-1}_{(\lambda)}(k). \tag{4.17}$$

Eqs. (4.15) and (4.17) still require an ultraviolet regularization. Let's first define the following (asymptotic) functions:

$$\Delta^{(0)}_{(\alpha)}(k) = \frac{2\pi k^2}{\ln\left(k^2/m_0^2\right)}, \qquad \Delta^{(0)}_{(\lambda)}(k) = \frac{2\pi}{\ln\left(k^2/m_0^2\right)-2}. \tag{4.18}$$

Now, according to the rules of the SM scheme, we obtain:

$$\Delta^{-1\,\mathrm{ren}}_{1(\alpha)}(p) = \Delta^{-1}_{1(\alpha)}(p) + \frac{4}{N}\int_{M^2}\frac{d^2k}{(2\pi)^2}\left(\frac{\Delta^{(0)}_{(\alpha)}(k)}{k^4}-\frac{\Delta^{(0)}_{(\lambda)}(k)}{k^2}\right)\Delta^{-1}_{(\alpha)}(p) - \frac{2}{N}\int_{M^2}\frac{d^2k}{(2\pi)^2}\left(\frac{\Delta^{(0)}_{(\alpha)}(k)}{k^4}+3\frac{\Delta^{(0)}_{(\lambda)}(k)}{k^2}\right)m_0^2\,\frac{\partial}{\partial m_0^2}\Delta^{-1}_{(\alpha)}(p), \tag{4.19}$$

$$\Delta^{-1\,\mathrm{ren}}_{1(\lambda)}(k) = \Delta^{-1}_{1(\lambda)}(k) + \frac{1}{N}\int_{M^2}\frac{d^2p}{(2\pi)^2}\left(\frac{\Delta^{(0)}_{(\alpha)}(p)}{p^4}-\frac{\Delta^{(0)}_{(\lambda)}(p)}{p^2}\right)(2\pi)\left[\Delta^{-1}_{(\lambda)}(k)\right]^2 - \frac{2}{N}\int_{M^2}\frac{d^2p}{(2\pi)^2}\left(\frac{\Delta^{(0)}_{(\alpha)}(p)}{p^4}+3\frac{\Delta^{(0)}_{(\lambda)}(p)}{p^2}\right)m_0^2\,\frac{\partial}{\partial m_0^2}\Delta^{-1}_{(\lambda)}(k). \tag{4.20}$$

The direct evaluation of the counterterms appearing in Eqs. (4.19) and (4.20) is somewhat cumbersome and required an explicit knowledge of the asymptotic behavior of the scalar vertices. However the structure of the counterterms is dictated by the renormalizability of the model and might have been predicted on purely theoretical grounds. In particular one recognizes the appearance of wavefunction and mass renormalization effects, and may trivially check that mass renormalization is consistent with Eq. (3.5).

The regulated expressions (4.19) and (4.20) are ready for numerical evaluation. Moreover Eq. (4.19) can be expanded in powers of p^2 and Eq. (4.20) in powers of k^2. The necessary ingredients are the expansions of $V_3(p,k)$, $V_4^{(a)}(p,k)$, $V_4^{(b)}(p,k)$ and $\Delta_{(\alpha)}(p+k)$ in powers of p or k; these expansion can easily be obtained by expanding the integrands in Eqs. (2.6), (2.8) and (2.10) before performing the q integration. Applying the definitions (3.7) and (3.10) we obtain the following results:

$$\xi^2 = \frac{1}{6m_0^2}\left\{1-\frac{1}{N}\left[\ln\ln\frac{M}{m_0}+3\ln\left(\ln\frac{M}{m_0}-1\right)+c_\xi\right]+O\!\left(\frac{1}{N^2}\right)\right\} = \frac{1}{6m^2(M^2,f)}\left[1-\frac{c_\xi}{N}+O\!\left(\frac{1}{N^2}\right)\right], \tag{4.21}$$

$$c_\xi = 12.265001\ldots \tag{4.22}$$

$$\chi_t = \frac{3}{\pi N}m_0^2\left\{1+\frac{1}{N}\left[\ln\ln\frac{M}{m_0}+3\ln\left(\ln\frac{M}{m_0}-1\right)+c_\chi\right]+O\!\left(\frac{1}{N^2}\right)\right\} = \frac{3}{\pi N}m^2(M^2,f)\left[1+\frac{c_\chi}{N}+O\!\left(\frac{1}{N^2}\right)\right]. \tag{4.23}$$

$$c_\chi = 11.884913\ldots \tag{4.24}$$

Out of the two dimensionful physical quantities ξ^2 and χ_t it is possible to construct a dimensionless ratio

$$R \equiv \chi_t\xi^2 \cong \frac{1}{2\pi N}\left[1+\frac{c_R}{N}\right], \tag{4.25}$$

$$c_R = -0.380088\ldots \tag{4.26}$$

c_R can be evaluated without making reference to any specific regularization scheme and it is therefore the most directly testable prediction resulting from our analysis.

In passing we note that it is possible to define another dimensionless physical quantity by considering the second moment of the topological susceptibility:

$$\chi_t' = \int d^2x\,\tfrac{1}{4}x^2\,\langle q(x)\,q(0)\rangle = -\frac{1}{(2\pi)^2}\lim_{k^2\to 0}\frac{\partial}{\partial k^2}\left[k^2\bar{\Delta}_{(\lambda)}(k)\right]. \tag{4.27}$$

Expanding Eq. (4.27) in powers of $1/N$ one obtains

$$\chi_t' = -\frac{3}{10\pi N} + \frac{1}{N^2}\frac{1}{(2\pi)^2}\lim_{k^2\to 0}\frac{\partial}{\partial k^2}\left[k^2 \Delta^2_{(\lambda)}(k)\,\Delta^{-1}_{1(\lambda)}(k)\right] + O\left(\frac{1}{N^3}\right). \tag{4.28}$$

Comparing Eq. (4.28) with Eq. (4.20) one easily recognizes that no regularization is needed, since the mass counterterm does not contribute to Eq. (4.28) while the other counterterm vanishes in the limit $M^2 \to \infty$ and therefore, if present, it can be thought as the signal of a removable perturbative tail. Numerical evaluation along the lines of Eq. (4.20) leads to

$$\chi_t' \cong -\frac{3}{10\pi N} + \frac{c_{\chi'}}{N^2} + O\left(\frac{1}{N^3}\right), \tag{4.29}$$

$$c_{\chi'} = 1.53671... \tag{4.30}$$

5. Conclusions

The main result of our analysis is Eqs. (4.25) and (4.26). Since the coefficient c_R is reasonably small, we may expect our predictions to hold up to very small values of N. On the other side, we believe the two definitions of the correlation length to agree well for all not too large values of N. Therefore Eq. (4.25) is also in some sense an interpolating expression for the ratio between the topological susceptibility and the mass gap, which we expect to hold within a few percent in a wide range of values of N.

It will be interesting to compare our results to those of accurate numerical simulations in the scaling region.

References

1. A. D'Adda, P. Di Vecchia and M. Lüscher, Nucl. Phys. B146 (1978) 63; Nucl. Phys. B152 (1979) 125.
2. E. Witten, Nucl. Phys. B149 (1979) 285.
3. M. Campostrini and P. Rossi, CP^{N-1} *models in the* $1/N$ *Expansion*, Pisa preprint IFUP-TH 25/91.
4. M. Campostrini and P. Rossi, *Dimensional regularization in the* $1/N$ *expansion*, Pisa preprint IFUP-TH 9/91, to appear in Int. J. Mod. Phys. A.
5. M. Lüscher, Phys. Lett. 78B (1978) 465.

Figure Captions

Fig. 1. The Feynman rules of the models.

Fig. 2. Three and four-point effective vertices of the models. All momenta are entering in the diagrams.

Fig. 3. Diagrams contributing to $\Delta^{-1}_{1(\alpha)}$ ($O(1/N)$ corrections to the scalar propagator).

Fig. 4. Diagrams contributing to $\Delta^{(\lambda)-1}_{1\,\mu\nu}$ ($O(1/N)$ corrections to the vector propagator).

Fig. 1

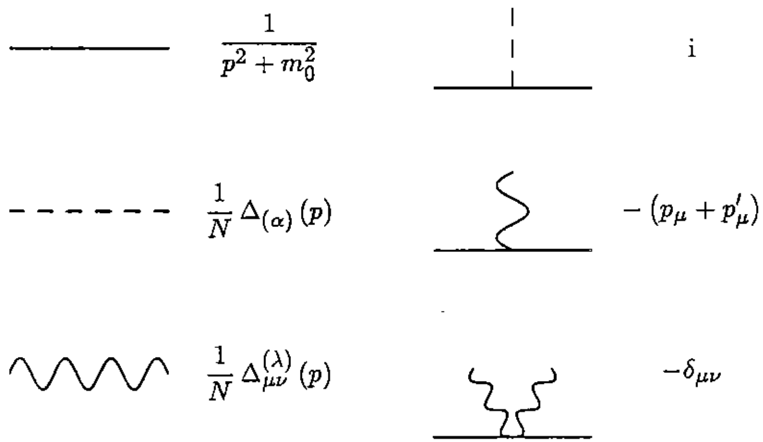

Fig. 2

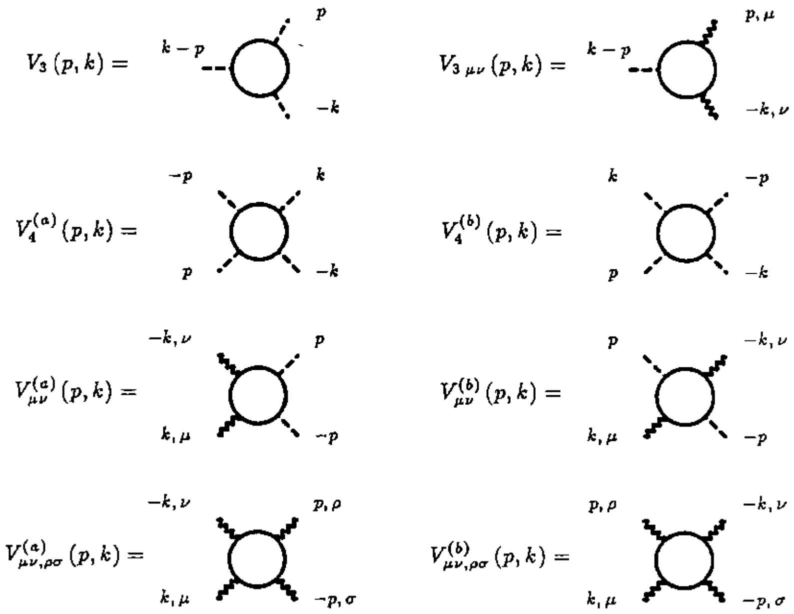

Fig. 3

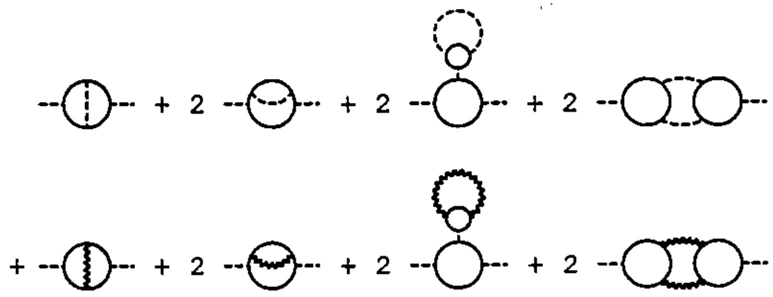

Fig. 4