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Four-point renormalized coupling constant in O(N) models

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Abstract

The renormalized zero-momentum four-point coupling g_r of O(N)-invariant scalar field theories in d dimensions is studied by applying the $1/N$ expansion and strong-coupling analysis. The $O(1/N)$ correction to the β-function and to the fixed point value g_r^* are explicitly computed. Strong-coupling series for lattice non-linear σ models are analyzed near criticality in $d = 2$ and $d = 3$ for several values of N and the corresponding values of g_r^* are extracted. Large-N and strong-coupling results are compared with each other, finding a good general agreement. For small N the strong-coupling analysis in 2d gives the best determination of g_r^* to date (for $N = 2, 3$ it is comparable with the best Monte Carlo estimates); in 3d it is consistent with available ϕ^4 field theory results.

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1. Introduction

The study of the fixed-point behavior is a crucial problem of quantum and statistical field theories, not only from a purely theoretical point of view, but also in order to clarify such phenomenologically relevant issues as the existence (and quantitative estimates) of triviality bounds.

For understandable reasons most theoretical effort has up to now been directed towards the analysis of a few selected models, including O(0), O(1), O(2) and O(3) in three dimensions and O(4) ("Higgs") models in four dimensions. In our view, it is very useful to extend the analysis to the case of a generic symmetry group O(N) and to models living in an arbitrary number of space dimensions d. Apart from opening the way to new possible physical applications, such a generalization may offer the possibility of testing and cross-checking the several different methods that have been applied to the

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problem at hand, thus putting on firmer grounds results that often rely only on a single approach and whose generality cannot therefore be fully understood.

In making these statements, we have especially in mind the possibility of systematically extending the application of two very well known techniques: strong-coupling and $1/N$ expansion. These two techniques enjoy relevant advantages: finite convergence radius for the strong-coupling expansion and non-perturbative interpretation of results for the $1/N$ expansion. Nonetheless they suffer from some drawbacks: lack of control on the accuracy of the resummation techniques in the strong-coupling case, poor information on the convergence properties and technical difficulty in the extending the series for the $1/N$ expansion; these problems have often discouraged people from pursuing these approaches.

Nevertheless we think, and we hope to show convincingly in the present paper, that a renewed effort in these directions may prove very fruitful. We emphasize that the systematic comparison of the two classes of results can be crucial in establishing their reliability: agreement of the $1/N$ expansion with strong-coupling results for $N \geqslant \bar{N}$ can be taken as evidence of its summability down to $\bar{N}$.

The convergence properties of the $1/N$ expansion in O(N)-symmetric models have been analyzed in the past. Avan and de Vega [1] showed the Borel summability of the $1/N$ expansion in less than four dimensions, and argued in favor of convergence (for $N > 2$) in two-dimensional non-linear σ models, where the known exact S-matrix is analytic in $1/N$, even in the absence of a convergent weak-coupling expansion. Moreover, Kupiainen [2] has proven rigorously that the $1/N$ expansion of O(N)-symmetric models in the lattice strong-coupling phase is an asymptotic series.

The present paper is organized as follows.

In Section 2 we introduce our notation for the general O(N)-invariant scalar field theory with quartic interaction in the continuum formulation and define the quantity that we are going to study: the renormalized zero-momentum four-point coupling g_r, whose behavior in the scaling region and fixed-point value for arbitrary N and d is the object of our investigations. This quantity is related to the so-called Binder cumulant. We compute the next-to-leading $1/N$ correction to the renormalized coupling as a function of the bare coupling and the renormalized mass by reducing it to a set of Feynman integrals that can be evaluated in the continuum $1/N$ expanded model without any regularization for all $d < 4$. We show how to compute the $1/N$ correction to the β-function $\beta(g_r) = m_r\, dg_r/dm_r$ and the fixed point value g_r^* such that $\beta(g_r^*) = 0$.

In Section 3 we give an exact evaluation of g_r in one dimension by solving exactly the one-dimensional non-linear σ models for arbitrary N, and draw from this example some general indication about the possible dependence of g_r^* on the parameter N.

In Section 4 we review the available results on non-linear σ models in two and three dimensions on the lattice and present our explicit computations of $1/N$ effects for the above models at criticality. We briefly comment on the case $d = 4$ and $O(1/N)$ logarithmic deviations from scaling.

In Section 5 we analyze and discuss the strong-coupling series for the renormalized coupling of the non-linear σ models in two and three dimensions on the lattice and for arbitrary N, which we extracted from the results of Lüscher and Weisz [3] as elaborated on by Butera et al. [4]. Strong-coupling results are compared to all available calculations presented in the literature (ϕ^4 field theory at fixed dimensions and Monte Carlo simulations) and to our $1/N$ results, finding a good general agreement.

Finally in Section 6 we draw some conclusions.

An explicit representation of the $1/N$ correction to the β-function is exhibited and discussed in Appendix A.

2. The renormalized coupling and its $1/N$ expansion

According to the previous discussion it is interesting to form a renormalization-group invariant dimensionless combination of vacuum expectation values playing the rôle of a renormalized four-point coupling and to study its behavior in the proximity of a critical point. In particular we are interested in $O(N)$-invariant scalar field theories in arbitrary dimensions $d \leqslant 4$ and we wish to apply $1/N$ expansion techniques to the above-mentioned problem.

From the point of view of the $1/N$ expansion the standard notation is somewhat inconvenient: we shall therefore define our own conventions, trying to establish correspondence with the literature as far as possible, and especially trying to make all relationships with Refs. [3,5–8] as transparent as we can.

The usual $O(N)$-invariant Euclidean continuum Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu \boldsymbol{\phi}\, \partial_\mu \boldsymbol{\phi} + \frac{1}{2}\mu_0^2 \boldsymbol{\phi}^2 + \frac{g_0}{4!}(\boldsymbol{\phi}^2)^2. \tag{1}$$

It is however convenient to redefine the quartic coupling (both bare and renormalized) according to the definition

$$\widehat{g} \equiv \frac{Ng}{3}. \tag{2}$$

We shall also define

$$\beta \equiv \frac{-2\mu_0^2}{\widehat{g}_0}, \qquad \gamma \equiv \frac{1}{\widehat{g}_0}, \qquad s \equiv \frac{\boldsymbol{\phi}}{\sqrt{N\beta}}, \tag{3}$$

and introduce an auxiliary field α.

The resulting effective Lagrangian is

$$\mathcal{L} = \frac{N}{2}\left[\beta \partial_\mu s\, \partial_\mu s + i\beta\alpha(s^2 - 1) + \gamma\alpha^2\right], \tag{4}$$

and after performing a Gaussian integration over the field s we obtain

$$\mathcal{L} = \frac{N}{2}\left[\operatorname{Tr}\ln\beta\left(-\partial_\mu\partial_\mu + i\alpha\right) - i\beta\alpha + \gamma\alpha^2\right], \tag{5}$$

which reduces to the usual effective large-N action for the non-linear σ model in the limit $\gamma \to 0$.

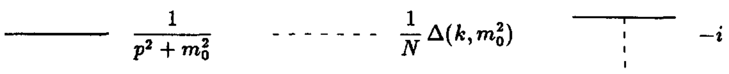

Fig. 1. Feynman rules for the $1/N$ expansion.

Correspondence with Refs. [3,5-8] is established by the relationships

$$\beta = \frac{2\kappa}{N}, \qquad \gamma = \frac{1}{N}\frac{\kappa^2}{2\lambda}. \tag{6}$$

Renormalization is performed according to the following prescriptions for the two- and four-point correlation functions of the field ϕ:

$$\begin{aligned} &\Gamma^{(2)}(p,-p)_{\alpha\beta} = \Gamma^{(2)}(p^2)\delta_{\alpha\beta}, \\ &\qquad \Gamma^{(2)}(p^2) = Z_r^{-1}\left[m_r^2 + p^2 + O(p^4)\right], \end{aligned} \tag{7}$$

and

$$\begin{aligned} &\Gamma^{(4)}(0,0,0,0)_{\alpha\beta\gamma\delta} = \Gamma^{(4)}(0)\left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right), \\ &\qquad \Gamma^{(4)}(0) = -Z_r^{-2}\frac{\widehat{g}_r}{N}(m_r^2)^{2-d/2}. \end{aligned} \tag{8}$$

Let us now notice that

$$-N\frac{\Gamma^{(4)}(0,0,0,0)_{\alpha\alpha\gamma\gamma}}{\left[\Gamma^{(2)}(0,0)_{\alpha\alpha}\right]^2}(m_r^2)^{d/2} = \left(1 + \frac{2}{N}\right)\widehat{g}_r. \tag{9}$$

Eq. (9) will be our working definition of the renormalized four-point coupling.

In order to compute the leading and next-to-leading contributions to $\widehat{g}_r$ in the continuum $1/N$ expansion, we shall need an evaluation of the corresponding contribution to the two-point function and to the zero-momentum four-point function.

The evaluation of the Feynman rules shown in Fig. 1 is essentially straightforward. We only mention that the bare propagator of the ϕ field is expressed in terms of a "bare" large-N mass parameter m_0^2 introduced by the gap equation

$$\int \frac{d^d p}{(2\pi)^d}\frac{1}{p^2 + m_0^2} = \beta + 2\gamma m_0^2, \tag{10}$$

while the propagator of the Lagrange multiplier field α is defined to be $(1/N)\Delta(k, m_0^2)$, where in turn

$$\begin{aligned} \Delta^{-1}(k, m_0^2) &= \frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\frac{1}{p^2 + m_0^2}\frac{1}{(p+k)^2 + m_0^2} + \gamma \\ &= \frac{1}{2}\frac{\Gamma(2 - d/2)}{(4\pi)^{d/2}}\left(\frac{k^2}{4} + m_0^2\right)^{d/2-2} F\left[2 - \frac{d}{2}, \frac{1}{2}, \frac{3}{2}, \left(1 + \frac{4m_0^2}{k^2}\right)^{-1}\right] \\ &\quad + \gamma. \end{aligned} \tag{11}$$

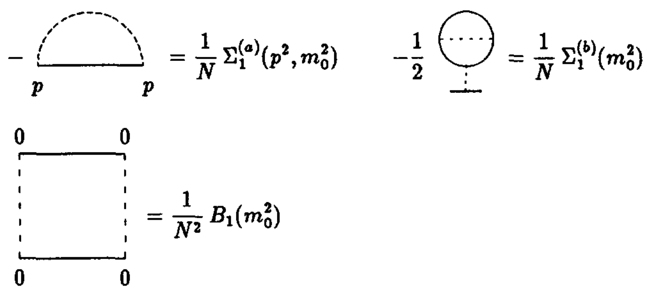

Fig. 2. Graphical definition of fundamental integrals.

The relevant higher-order Lagrangian effective vertices are obtained by taking derivatives of $\Delta^{-1}(k)$ with respect to m_0^2, according to the correspondence table

$$V^{(3)}(0,k,k) = -\frac{\partial}{\partial m_0^2}\Delta^{-1}(k),$$

$$2V^{(4)}(0,0,k,k) + V^{(4)}(0,k,0,k) = -\frac{\partial}{\partial m_0^2}V^{(3)}(0,k,k) = \frac{\partial^2}{(\partial m_0^2)^2}\Delta^{-1}(k), \quad (12)$$

where the mass dependence is suppressed in the arguments. The derivatives appearing in Eq. (12) may be evaluated by a generalization of the so-called "cutting rule" of Ref. [9], whose d-dimensional form is

$$\frac{\partial}{\partial m_0^2}\Delta^{-1}(k) = -\frac{2}{k^2+4m_0^2}\left[(3-d)\Delta^{-1}(k) + \Delta^{-1}(0) + \gamma(d-4)\right]. \quad (13)$$

In writing Eqs. (10) and (11) some ultraviolet regularization, when needed, is assumed. Actually our final results will turn out to be independent of the regularization as expected on physical grounds.

Eq. (9) shows that in order to compute $\widehat{g}_r$ to any definite order in the $1/N$ expansion we must be able to compute the quantities $\Gamma^{(2)}(p)$, $\Gamma^{(4)}(0)$ and m_r^2 with the same precision. Leading order calculations are straightforward. Next-to-leading contributions may be formally represented in terms of a few fundamental integrals, which are graphically represented in Fig. 2 and listed below:

$$\Sigma_1^{(a)}(p^2, m_0^2) = \int \frac{d^dk}{(2\pi)^d}\frac{\Delta(k,m_0^2)}{(p+k)^2+m_0^2}, \quad (14)$$

$$\Sigma_1^{(b)}(m_0^2) = -\frac{1}{2}\Delta(0,m_0^2)\int \frac{d^dk}{(2\pi)^d}V^{(3)}(0,k,k)\Delta(k,m_0^2), \quad (15)$$

$$B_1(m_0^2) = \int \frac{d^dk}{(2\pi)^d}\frac{\Delta(k,m_0^2)^2}{(k^2+m_0^2)^2}. \quad (16)$$

It is very easy to show that the two-point function and the renormalized mass are respectively

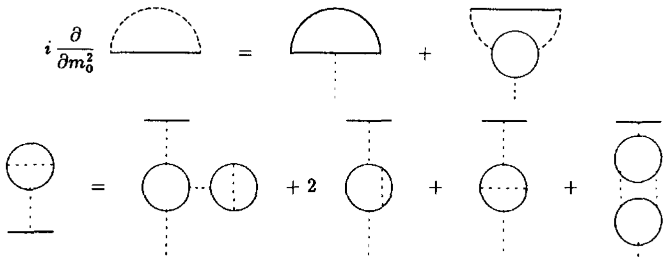

Fig. 3. Identities among Feynman graphs.

$$\Gamma^{(2)}(p^2) = p^2 + m_0^2 + \frac{1}{N}\left[\Sigma_1^{(a)}(p^2, m_0^2) + \Sigma_1^{(b)}(m_0^2)\right] + O\left(\frac{1}{N^2}\right) \tag{17}$$

and

$$m_r^2 = m_0^2 + \frac{1}{N}\left[\Sigma_1^{(a)}(0, m_0^2) + \Sigma_1^{(b)}(m_0^2) - m_0^2 \left.\frac{\partial \Sigma_1^{(a)}(p^2, m_0^2)}{\partial p^2}\right|_{p^2=0}\right] + O\left(\frac{1}{N^2}\right). \tag{18}$$

Explicit use of Eqs. (12), leading to the graphical identities drawn in Fig. 3, allows us to obtain the representation

$$-N\Gamma^{(4)}(0) = \Delta(0, m_0^2)\left[1 + \frac{2}{N}\frac{\partial}{\partial m_0^2}\Sigma_1^{(a)}(0, m_0^2) + \frac{1}{N}\frac{\partial}{\partial m_0^2}\Sigma_1^{(b)}(m_0^2)\right] - \frac{2}{N}B_1(m_0^2) + O\left(\frac{1}{N^2}\right). \tag{19}$$

It is now important to notice that, in order to obtain a finite result, $\hat{g}_r$ must be expressed in terms of the renormalized mass m_r^2. This is achieved by inverting Eq. (18), which leads to

$$m_0^2 = m_r^2 - \frac{1}{N}\left[\Sigma_1^{(a)}(0, m_r^2) + \Sigma_1^{(b)}(m_r^2) - m_r^2 \left.\frac{\partial \Sigma_1^{(a)}(p^2, m_r^2)}{\partial p^2}\right|_{p^2=0}\right] + O\left(\frac{1}{N^2}\right). \tag{20}$$

and, as a consequence,

$$\Gamma^{(2)}(p^2) = p^2 + m_r^2 + \frac{1}{N}\left[\Sigma_1^{(a)}(p^2, m_r^2) - \Sigma_1^{(a)}(0, m_r^2) + m_r^2 \left.\frac{\partial \Sigma_1^{(a)}(p^2, m_r^2)}{\partial p^2}\right|_{p^2=0}\right] + O\left(\frac{1}{N^2}\right). \tag{21}$$

$$\Delta(0, m_0^2) = \Delta(0, m_r^2) + \frac{1}{N}\Delta(0, m_r^2)\left[1 - \gamma\Delta(0, m_r^2)\right]\left(\frac{d}{2} - 2\right)$$

$$\times\left[\frac{\Sigma_1^{(a)}(0,m_r^2)}{m_r^2}+\frac{\Sigma_1^{(b)}(m_r^2)}{m_r^2}-\left.\frac{\partial\Sigma_1^{(a)}(p^2,m_r^2)}{\partial p^2}\right|_{p^2=0}\right]. \tag{22}$$

Collecting all the above results and substituting into Eq. (9) we obtain the following representation of the renormalized coupling:

$$\begin{aligned}\widehat{g}_r=(m_r^2)^{d/2-2}\Delta(0,m_r^2)\Bigg\{1&+\frac{1}{N}\left[1-\gamma\Delta(0,m_r^2)\right]\left(\frac{d}{2}-2\right)\\&\times\left[\frac{\Sigma_1^{(a)}(0,m_r^2)}{m_r^2}+\frac{\Sigma_1^{(b)}(m_r^2)}{m_r^2}-\left.\frac{\partial\Sigma_1^{(a)}(p^2,m_r^2)}{\partial p^2}\right|_{p^2=0}\right]\\&+\frac{1}{N}\left[2\frac{\partial\Sigma_1^{(a)}(0,m_r^2)}{\partial m_r^2}+\frac{\partial\Sigma_1^{(b)}(m_r^2)}{\partial m_r^2}-2\left.\frac{\partial\Sigma_1^{(a)}(p^2,m_r^2)}{\partial p^2}\right|_{p^2=0}\right.\\&\left.-2\Delta^{-1}(0,m_r^2)B_1(m_r^2)\right]\Bigg\}.\end{aligned} \tag{23}$$

From now on, we suppress the m_r^2 dependence in Δ. Substituting Eqs. (14), (15) and (16) and making explicit use of Eq. (13), one obtains the following representation:

$$\begin{aligned}\widehat{g}_r=(m_r^2)^{d/2-2}\Delta(0)\Bigg\{1&+\frac{1}{N}\left[1-\gamma\Delta(0)\right](3-d)2^{d-1}\\&+\frac{1}{N}\int\frac{d^dk}{(2\pi)^d}\Delta(k)\left[\frac{2m_r^2}{(k^2+m_r^2)^3}+\frac{3m_r^2}{(k^2+m_r^2)(k^2+4m_r^2)}\left(\frac{d/2-4}{k^2+m_r^2}-\frac{2(d-1)}{k^2+4m_r^2}\right)\right]\\&-\frac{1}{N}(d-4)\gamma\Delta(0)\int\frac{d^dk}{(2\pi)^d}\Delta(k)\frac{1}{d}\left[\frac{2m_r^2}{(k^2+m_r^2)^3}\right.\\&\left.+\frac{3m_r^2}{(k^2+m_r^2)(k^2+4m_r^2)}\left(\frac{d/2-2}{k^2+m_r^2}+\frac{2(d-1)}{k^2+4m_r^2}\right)\right]\\&+\frac{1}{N}(d-4)\gamma\Delta(0)\int\frac{d^dk}{(2\pi)^d}\Delta(k)\frac{2(d-1)^2}{d(k^2+4m_r^2)^2}\\&-\frac{2}{N}\Delta^{-1}(0)\int\frac{d^dk}{(2\pi)^d}\frac{\Delta(k)^2}{(k^2+4m_r^2)^2}\left[\frac{3m_r^2}{(k^2+m_r^2)}-(d-4)\gamma\Delta(0)\right]^2\Bigg\}+O\left(\frac{1}{N^2}\right).\end{aligned} \tag{24}$$

Further computational simplification is achieved by making use of the following straightforward consequence of Eq. (13):

$$\begin{aligned}\Delta^{-1}(0)\Delta(k)^2=&\frac{4m_r^2+(4-d)k^2}{4m_r^2+(4-d)\gamma\Delta(0)k^2}\Delta(k)\\&-\frac{2k^2(k^2+4m_r^2)}{4m_r^2+(4-d)\gamma\Delta(0)k^2}\frac{\partial\Delta(k)}{\partial k^2},\end{aligned} \tag{25}$$

which may be applied to Eq. (24) in order to get rid of the $\Delta(k)^2$ dependence in the integrand, while a partial integration may eliminate the dependence on $\partial\Delta(k)/\partial k^2$. It is easy to recognize that whenever $d < 4$ and $\gamma \geqslant 0$ all integrations are finite. The final result can be formally expressed by the relationship

$$\widehat{g}_r = \widehat{g}_r^{(0)}(x) + \frac{1}{N}\widehat{g}_r^{(1)}(x) + O\left(\frac{1}{N^2}\right), \tag{26}$$

where all dependence on the renormalized mass and the bare coupling can only come through the dimensionless combination $x \equiv m_r^{4-d}/\widehat{g}_0$. Specifically one obtains

$$\widehat{g}_r^{(0)} = \left(\frac{1}{g_*} + x\right)^{-1}, \tag{27}$$

where

$$\frac{1}{g_*} = \frac{m_r^{4-d}}{2}\int \frac{d^d p}{(2\pi)^d}\frac{1}{\left(p^2+m_r^2\right)^2} = \frac{1}{2}\frac{\Gamma\left(2-d/2\right)}{(4\pi)^{d/2}} \tag{28}$$

is the inverse of the large-N fixed point value of the renormalized coupling.

Eq. (26) is the obvious starting point for the construction of the $1/N$ expanded β-function of the model, via the relationship

$$\beta(\widehat{g}_r) = m_r\frac{d\widehat{g}_r}{dm_r} = (4-d)\,x(\widehat{g}_r)\left.\frac{d\widehat{g}_r}{dx}\right|_{x=x(\widehat{g}_r)}$$
$$= \beta^{(0)}(\widehat{g}_r) + \frac{1}{N}\beta^{(1)}(\widehat{g}_r) + O\left(\frac{1}{N^2}\right), \tag{29}$$

where $x(\widehat{g}_r)$ is obtained by inverting the equation $\widehat{g}_r = \widehat{g}_r(x)$, and it admits in turn a $1/N$ expansion in the form

$$x(\widehat{g}_r) = x^{(0)}(\widehat{g}_r) + \frac{1}{N}x^{(1)}(\widehat{g}_r) + O\left(\frac{1}{N^2}\right). \tag{30}$$

It is easy to recognize that Eq. (27) implies

$$x^{(0)}(\widehat{g}_r) = \frac{1}{\widehat{g}_r} - \frac{1}{g_*}, \tag{31}$$

and therefore the large-N limit of the β-function reduces to

$$\beta^{(0)}(\widehat{g}_r) = (d-4)\widehat{g}_r\left(1 - \frac{\widehat{g}_r}{g_*}\right). \tag{32}$$

This is the standard (large-N) one-loop result provided that we identify

$$\frac{4-d}{g_*} = \frac{\Gamma\left(3-d/2\right)}{(4\pi)^{d/2}} = \beta_0. \tag{33}$$

A simple consequence of Eqs. (26), (27), (30) and (31) is the relationship

$$x^{(1)}(\widehat{g}_r) = \frac{1}{\widehat{g}_r^2}\widehat{g}_r^{(1)}(x^{(0)}(\widehat{g}_r)). \tag{34}$$

We may now consider the expansion of Eq. (29) in powers of $1/N$ and notice that the derivative of $\widehat{g}_r^{(1)}(x)$ with respect to $x^{(0)}$ may be exchanged with a derivative with respect to $\widehat{g}_r$. As a consequence, after some manipulations we can prove the relationship

$$\beta^{(1)}(\widehat{g}_r) = \left[\beta^{(0)}(\widehat{g}_r)\right]^2 \frac{\partial}{\partial \widehat{g}_r} \left[\frac{\widehat{g}_r^{(1)}\left(x^{(0)}(\widehat{g}_r)\right)}{\beta^{(0)}(\widehat{g}_r)} \right], \tag{35}$$

where notable simplifications occur when evaluating $\widehat{g}_r^{(1)}$ directly as a function of $\widehat{g}_r^{-1} - g_*^{-1}$. In particular

$$m_r^{d-4}\Delta(k) \longrightarrow \frac{\widehat{g}_r}{1+\widehat{g}_r\Pi(k/m_r)}, \tag{36}$$

where

$$\Pi(k/m_r) = \frac{1}{2}\int \frac{d^dp}{(2\pi)^d}\frac{m_r^{4-d}}{p^2+m_r^2}\left[\frac{1}{(p+k)^2+m_r^2} - \frac{1}{p^2+m_r^2}\right] \tag{37}$$

is a regular dimensionless function with the property $\Pi(0) = 0$ and a finite $d \longrightarrow 4$ limit. As long as $d < 4$ one may show that $\beta^{(1)}(\widehat{g}_r)$ is well defined and finite for all $0 \leqslant \widehat{g}_r \leqslant g_*$. We obtained an explicit integral representation of $\beta^{(1)}(\widehat{g}_r)$ for arbitrary d, and showed that the series expansion of such a representation in the powers of $\widehat{g}_r$ may be obtained also in the $d \longrightarrow 4$ limit and reproduces all known results as long as comparison is allowed. The representation of $\beta^{(1)}(\widehat{g}_r)$ and a short discussion of its features are presented in Appendix A. The non-perturbative properties of $\beta^{(1)}(\widehat{g}_r)$ when $d \longrightarrow 4$ will be analyzed and discussed in a separate publication.

For what concerns the very important issue of the fixed point of $\widehat{g}_r$, we must notice that the β-function vanishes when $x \longrightarrow 0$, i.e. when

$$\widehat{g}_r^* = g_* + \frac{1}{N}\widehat{g}_r^{(1)}(0) + O\left(\frac{1}{N^2}\right). \tag{38}$$

Eq. (24), supplemented with Eq. (25), lends itself to an easy evaluation in the limit $x \longrightarrow 0$, corresponding to the limit $\gamma \longrightarrow 0$. The final result is

$$\widehat{g}_r^{(1)}(0) = g_*\Bigg[(3-d)2^{d-1} - \int \frac{d^dk}{(2\pi)^d}\frac{\Delta_0(k)}{(k^2+m_r^2)^2}\left(\frac{4m_r^2}{k^2+m_r^2} + 9\left(\frac{d}{2}-1\right)\frac{m_r^2}{k^2+4m_r^2}\right)\Bigg], \tag{39}$$

where

$$\Delta_0(k) = \lim_{x\longrightarrow 0}\Delta(k). \tag{40}$$

Notice that the fixed-point value of the renormalized coupling may be obtained directly by computing the $\gamma \longrightarrow 0$ limit of the coupling $\widehat{g}_r$ in the scaling region. However, this is nothing but the value taken by $\widehat{g}_r$ in the corresponding continuum limit field theory, which is the usual non-linear σ model in d dimensions. In turn this is the limit of the lattice non-linear σ model when $\beta \longrightarrow \beta_c$, the value of the coupling such that the renormalized mass (i.e. inverse correlation length) is equal to zero in the lattice $\gamma \longrightarrow 0$ limit.

3. Non-linear σ models in one dimension

Before discussing the general d-dimensional case, let us illustrate some features of the problem by solving the simple but not trivial one-dimensional case. One-dimensional non-linear σ models are a completely integrable system, both on the lattice and in the continuum [10,11,14]. Indeed in any lattice formulation with nearest-neighbor interactions the two- and four-point correlation functions are easily expressed in terms of two quantities that in turn are related to vacuum expectation values of the model defined on a single link.

Without belaboring on the rather trivial manipulations needed to derive these results [10,11], we simply quote that in any O(N)-invariant σ model theory satisfying the constraint $s \cdot s = 1$, one may write

$$\langle s_m^a s_n^b \rangle = B_{11}^{n-m} \frac{1}{N} \delta^{ab},$$
$$\left\langle \left(s_m^a s_m^b - \frac{1}{N}\delta^{ab} \right) \left(s_n^c s_n^d - \frac{1}{N}\delta^{cd} \right) \right\rangle = B_{22}^{n-m} \left[\frac{1}{N(N+2)} \delta^{abcd} - \frac{1}{N^2} \delta^{ab}\delta^{cd} \right] \quad (41)$$

for $n \geqslant m$, and

$$\langle s_m^a s_n^b s_p^c s_q^d \rangle_c = B_{11}^{n-m+q-p} \left[\delta^{abcd} \left(\frac{B_{22}^{p-n}}{N(N+2)} - \frac{B_{11}^{2(p-n)}}{N^2} \right) + \delta^{ab}\delta^{cd} \frac{1}{N^2} \left(B_{11}^{2(p-n)} - B_{22}^{p-n} \right) \right] \quad (42)$$

for $q \geqslant p \geqslant n \geqslant m$, where m, n, p, q are integer numbers labeling lattice sites,

$$\delta^{abcd} \equiv \delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}, \quad (43)$$

and

$$B_{11} = \langle s_1 \cdot s_0 \rangle, \qquad B_{22} = \frac{N\langle (s_1 \cdot s_0)^2 \rangle - 1}{N-1}, \quad (44)$$

where expectation values are taken in the single-link model: B_{11} and B_{22} are the character coefficients in the (pseudo) character expansion of the model [10,11], or, equivalently, the coefficient of the expansion of the theory in hyperspherical harmonics. The results corresponding to different orderings of the lattice points are obtained by trivial permutations.

Zero-momentum lattice Fourier transforms can be computed as functions of B_{11} and B_{22} by performing trivial summations of geometric series. One may then easily recognize that

$$\Gamma^{(2)}(0)_{aa} = \frac{1+B_{11}}{1-B_{11}}, \tag{45}$$

$$m_r^2 = \frac{(1-B_{11})^2}{B_{11}}, \tag{46}$$

while after some purely algebraic effort one obtains

$$N\Gamma^{(4)}(0)_{aabb} = 4(N-1)\frac{B_{22}}{1-B_{22}}\left(\frac{1+B_{11}}{1-B_{11}}\right)^2 \\ -4NB_{11}^2\frac{1+B_{11}}{(1-B_{11})^3} - \frac{16B_{11}}{(1-B_{11})^3} - 2. \tag{47}$$

As a consequence, by applying Eq. (9) one obtains

$$\left(1+\frac{2}{N}\right)\hat{g}_r = \frac{1}{\sqrt{B_{11}}}\left[\frac{2(1-B_{11})^3}{(1+B_{11})^2} + \frac{4NB_{11}^2}{1+B_{11}} \right. \\ \left. + \frac{16B_{11}}{(1+B_{11})^2} - 4(N-1)\frac{1-B_{11}}{1-B_{22}}B_{22}\right]. \tag{48}$$

We now want to take the critical limit, which, as shown in Ref. [14], depends in a complicated way on the specific Hamiltonian, as in one dimension there are infinitely many universality classes. We will restrict ourselves to those theories for which $m_r \to 0$. Thus Eq. (46) implies $B_{11} \to 1$. When $N \geqslant 1$ we have also [14] $B_{22} \to 1$, so that

$$\left(1+\frac{2}{N}\right)\hat{g}_r^* = 2(N+2) - 4(N-1)\lim_{B_{11}\to 1}\frac{1-B_{11}}{1-B_{22}}. \tag{49}$$

The quantity $(1-B_{11})/(1-B_{22})$ characterizes the universality class as being simply the ratio of the mass gap in the spin-one and spin-two channels. Within the universality class corresponding to the standard continuum limit, we have

$$\lim_{B_{11}\to 1}\frac{1-B_{11}}{1-B_{22}} = \frac{N-1}{2N} \tag{50}$$

and, as a consequence,

$$\left(1+\frac{2}{N}\right)\hat{g}_r^* = 8\left(1-\frac{1}{4N}\right) \tag{51}$$

for $N \geqslant 1$. This solution agrees perfectly with the prediction resulting from Eq. (39). When $N < 1$ we have no general argument for the behavior of B_{22} in the massless limit. We may however restrict our attention to the universality class corresponding to the standard continuum limit, and within this class we may consider the specific lattice example of the minimal nearest-neighbor coupling. For this action one may show that for arbitrary N

$$B_{11} = \frac{I_{N/2}(N\beta)}{I_{N/2-1}(N\beta)}, \qquad B_{22} = \frac{I_{N/2+1}(N\beta)}{I_{N/2-1}(N\beta)} = 1 - \frac{B_{11}}{\beta}. \tag{52}$$

When $N \geqslant 1$ B_{11} is strictly smaller than one for all finite values of β and only in the limit $\beta \to \infty$ the massless regime is attained, in which case $B_{22} \to 1$ as well, as expected

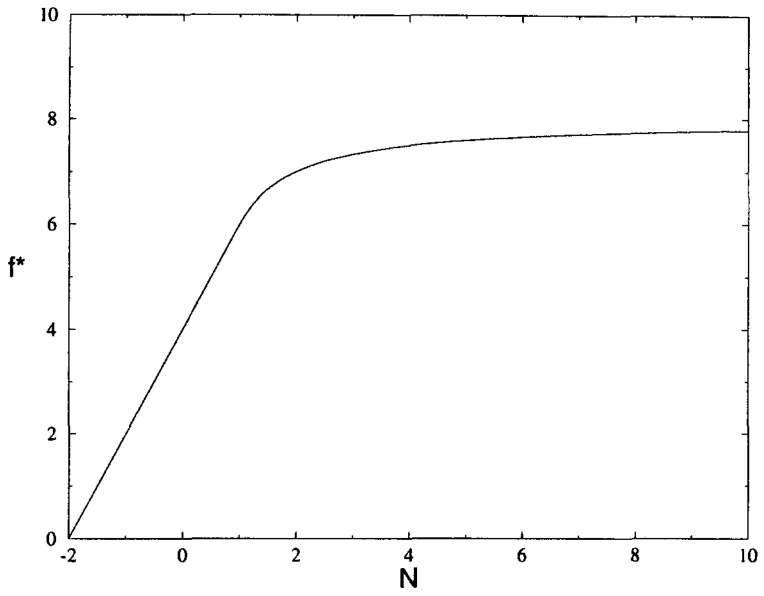

Fig. 4. f^* vs. N for 1-d O(N) models.

from the general argument. However, when $N < 1$ one may numerically check that a finite value β_c exists such that $B_{11}(\beta_c) = 1$: as a consequence, $B_{22}(\beta_c) = 1 - 1/\beta_c \neq 1$. Since B_{22} is strictly different from one in the massless limit, we get from Eq. (49) that within this universality class

$$\left(1 + \frac{2}{N}\right) \widehat{g}_r^* = 2(N+2) \tag{53}$$

for $N < 1$. Let us notice that the two solutions connect very smoothly to each other (the function and its first derivative have the same left and right limit) because of the double zero at $N = 1$ in the contribution that does not vanish when $N > 1$. Fig. 4 shows $\left(1 + 2/N\right) \widehat{g}_r^*$ vs. N.

This is very reminiscent of what is going to happen when $d = 2$: for small N β_c is finite and there is a domain of analyticity in N for $\widehat{g}_r$ around $N = -2$, while for large N β_c is infinite (asymptotic freedom) and analyticity in $1/N$ is present. The two regimes seem to meet smoothly at $N = 2$.

As a further check of our results we may consider the $N = 0$ case. This is a very simple model of self-avoiding walks in one dimension. All computations are straightforward and one obtains, with the notations adopted here,

$$\lim_{N \to 0} \left(1 + \frac{2}{N}\right) \widehat{g}_r^* = 4, \tag{54}$$

in full agreement with our general formula. Also intermediate steps are reproduced, with the identification $B_{11} = \beta$, $B_{22} = 0$.

4. Non-linear σ models in higher dimensions

Lattice non-linear σ models, which we may choose to describe in terms of the standard $O(N)$-invariant nearest-neighbor action

$$S_L = -N\beta \sum_{x,\mu} s_x \cdot s_{x+\mu} \tag{55}$$

subject to the constraint $s_x^2 = 1$, when considered on a d-dimensional lattice with $d < 4$ have a non-trivial critical point $\beta_c \leqslant \infty$ whose neighborhood (scaling region) is properly described by a renormalized continuum field theory. This theory is in turn nothing but the $\gamma \to 0$ limit of the standard $O(N)$-invariant scalar field theory (linear σ model). We may therefore study the critical properties (and in particular the fixed-point value of the renormalized coupling) of the symmetric phase of the $O(N)$ model by exploring the region $\beta \to \beta_c$ of the lattice model.

The left-hand side of Eq. (9) has a simple reinterpretation in terms of quantities defined within the associated lattice spin model. Setting

$$\chi = \sum_x \langle s_0 \cdot s_x \rangle, \qquad m_2 = \sum_x x^2 \langle s_0 \cdot s_x \rangle,$$
$$\xi^2 = \frac{m_2}{2d\chi} = \frac{1}{m_r^2}, \qquad \chi_4 = \sum_{x,y,z} \langle s_0 \cdot s_x s_y \cdot s_z \rangle_c, \tag{56}$$

one can argue that the combination

$$\frac{\chi_4}{\chi^2 \xi^d} \tag{57}$$

should either admit a non-trivial limiting value or vanish with logarithmic deviations from scaling when the critical line is approached. This is essentially a consequence of the existence of an unique diverging relevant scale in the scaling region. It is furthermore trivial to show that in the scaling region, $m_r \to 0$,

$$f \equiv -N \frac{\chi_4}{\chi^2 \xi^d} = \left(1 + \frac{2}{N}\right) \widehat{g}_r, \tag{58}$$

and in particular

$$f^* \equiv f(\beta_c) = \left(1 + \frac{2}{N}\right) \widehat{g}_r^* . \tag{59}$$

We also mention that f can be written in terms of the Binder cumulant defined on a L^d lattice

$$U_L = 1 + \frac{2}{N} - \frac{\langle (S \cdot S)^2 \rangle}{\left(\langle S \cdot S \rangle\right)^2}, \tag{60}$$

where $S = \sum_x s_x$. Indeed,

$$f = N \lim_{L \to \infty} U_L \left(\frac{L}{\xi_L}\right)^d . \tag{61}$$

As already mentioned in Section 3, there is a crucial dependence on the space dimensionality as well as on N. In two dimensions it is well known that models with $-2 \leqslant N \leqslant 2$ are well described at criticality by conformal field theories with $c \leqslant 1$. In particular at $N = -2$ the fixed point is Gaussian, $N = 0$ corresponds to a model of self-avoiding random walks, $N = 1$ is the solvable Ising model, and $N = 2$ is the XY model showing the Kosterlitz–Thouless critical phenomenon, characterized by an exponential singularity at a finite β_c.

When $-2 \leqslant N \leqslant 2$ the critical point occurs at a finite value of β_c. When $N \geqslant 3$ there is apparently no criticality for any finite value of β. This is consistent with the Mermin–Wagner theorem on the absence of spontaneous symmetry breakdown for two-dimensional continuous symmetry and with the weak-coupling (large-β) prediction of asymptotic freedom and dynamical mass generation for this class of models. Large-N results and the $1/N$ expansion are completely consistent with the above picture [12,13]. From the point of view of the renormalized coupling analysis it is however impossible to distinguish between the two behaviors, since they are both compatible with a non-zero value of f^*.

We now briefly present some large-N results regarding f and its limit at β_c. On the lattice, using the action (55), the large-N limit of $f(\beta)$ can be easily obtained from the saddle-point equation

$$\beta = \int \frac{d^d q}{(2\pi)^d} \frac{1}{\widehat{q}^2 + m_0^2}, \tag{62}$$

where $\widehat{q}^2 \equiv 4\sum_\mu \sin^2(q_\mu/2)$, and the relation

$$f(\beta) = -\frac{2}{m_0^{4-d}} \left(\frac{\partial \beta}{\partial m_0^2}\right)^{-1}. \tag{63}$$

In 2d the above equations are made more explicit by writing

$$\beta = \frac{1}{2\pi} k K(k), \qquad f = 4\pi \frac{1+k}{kE(k)}, \tag{64}$$

where $k = \left(1 + m_0^2/4\right)^{-1}$, K and E are elliptic functions. Fig. 5 shows $f(\beta)$ vs. β. In the large-β limit the continuum result (28), i.e. $f^* = 8\pi$, is recovered.

Our $1/N$ expansion analysis of Section 2 leads to the evaluation of the $O(1/N)$ correction to f^*. Indeed, in two dimensions,

$$f^* = 8\pi \left[1 + \frac{f_1}{N} + O\left(\frac{1}{N^2}\right)\right] \tag{65}$$

with $f_1 = -0.602033\ldots$. In three dimensions we face a quite different situation. The critical point occurs at a finite value β_c for all values of N. At $N = \infty$ [15]

$$\beta_c(N = \infty) = \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\widehat{q}^2} = 0.252731\ldots \tag{66}$$

The four-point renormalized coupling $f(\beta)$ at $N = \infty$ is shown in Fig. 6. At the critical

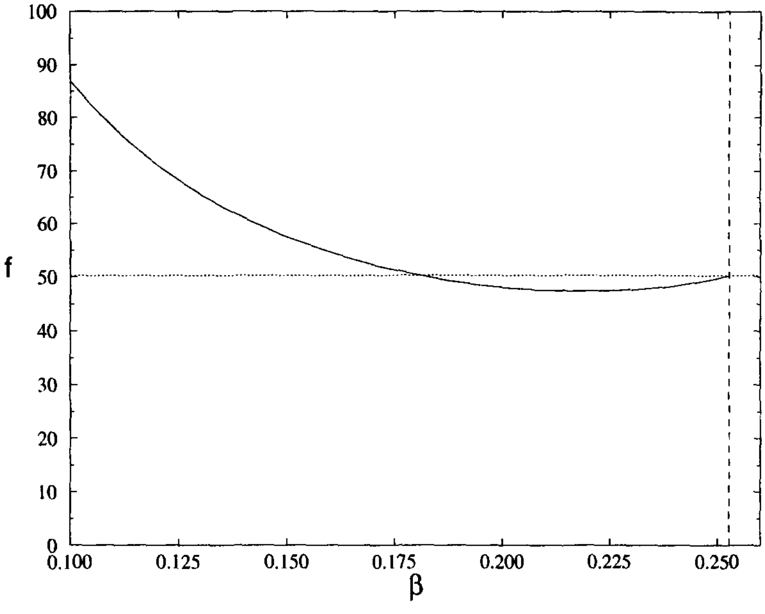

Fig. 5. $f(\beta)$ vs. β for 2d O(∞) model. The dotted horizontal line represents the continuum value $f^* = 8\pi$.

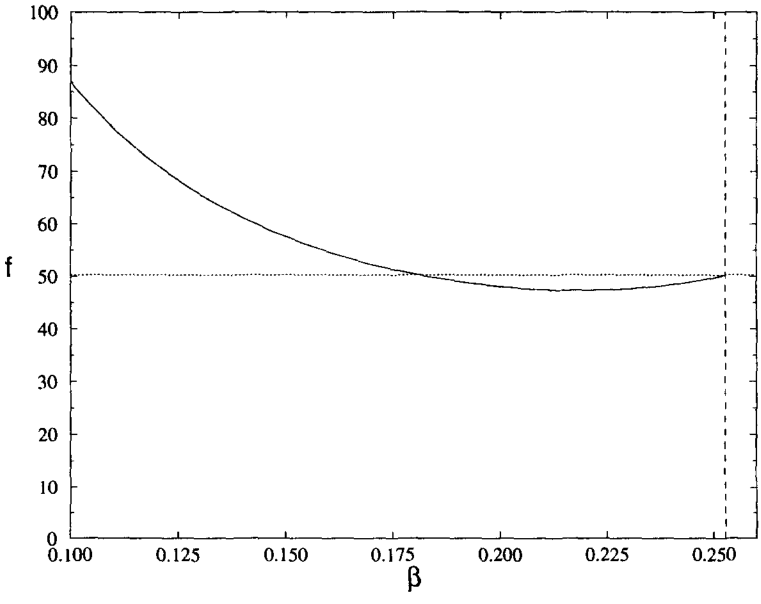

Fig. 6. $f(\beta)$ vs. β for 3d O(∞) model. The dotted horizontal line represents the continuum value $f^* = 16\pi$. The dashed vertical line indicates the critical point $\beta_c = 0.252631\ldots$

point $f^* = 16\pi$. It is also possible to estimate the deviation of $\beta_c(N)$ from Eq. (66) by the $1/N$ expansion technique presented first in Ref. [15], leading to the relationship

$$\beta_c(N) = \beta_c(\infty) + \frac{b_1}{N} + O\left(\frac{1}{N^2}\right), \tag{67}$$

where

$$b_1 = -\int \frac{d^3q}{(2\pi)^3} \Delta^{(0)}(q) \int \frac{d^3p}{(2\pi)^3} \frac{1}{(\widehat{p}^2)^2} \left[\frac{1}{2\widehat{(p+q)}^2} + \frac{1}{2\widehat{(p-q)}^2} - \frac{1}{\widehat{q}^2} \right],$$
$$\Delta^{(0)}(q)^{-1} = \frac{1}{2} \int \frac{d^3r}{(2\pi)^3} \frac{1}{\widehat{r}^2 \widehat{(r+q)}^2}. \tag{68}$$

Numerically $b_1 \simeq -0.117$. It is also important to have an estimate of the value of the internal energy E at the criticality

$$E_c(N) = E_c(\infty) + \frac{e_1}{N} + O\left(\frac{1}{N^2}\right). \tag{69}$$

We notice that

$$E = \langle s_x \cdot s_{x+\mu} \rangle = \frac{1}{Nd} \frac{\partial}{\partial \beta} \ln Z(\beta, N)$$
$$= 1 - \frac{1}{2\beta d} + \frac{m_0^2}{2d} + \frac{1}{2Nd} \left[\frac{2}{\beta} + \bar{\Sigma}_1^{(b)}(m_0^2) \right] + O\left(\frac{1}{N^2}\right), \tag{70}$$

where $\bar{\Sigma}_1^{(b)}(m_0^2)$ is the lattice counterpart of $\Sigma_1^{(b)}(m_0^2)$, defined in Eq. (15). By setting $d = 3$ and by considering the $m_0^2 \to 0$ limit we then obtain

$$E_c(N) = 1 - \frac{1}{6\beta_c(\infty)}$$
$$+ \frac{1}{6N} \left[\frac{b_1}{\beta_c(\infty)^2} + \frac{2}{\beta_c(\infty)} - \int \frac{d^3q}{(2\pi)^3} \frac{\Delta^{(0)}(q)}{\widehat{q}^2} \right] + O\left(\frac{1}{N^2}\right). \tag{71}$$

Numerically we obtained $E_c(\infty) = 0.340537\ldots$ and $e_1 \simeq -0.07$. For the three-dimensional case our $O(1/N)$ calculation of f^* gives

$$f^* = 16\pi \left[1 + \frac{f_1}{N} + O\left(\frac{1}{N^2}\right) \right] \tag{72}$$

with $f_1 = -1.54601\ldots$.

In the two- and three-dimensional O(N) models a number of techniques have been applied to the determination of f^*. In particular we mention the ϕ^4 field theoretical approach at fixed dimensions proposed by Parisi [16] and developed in Refs. [17,18], also making use of Borel resummation techniques (see for example Refs. [19,20] for a review on this approach). This method has been applied to $N = 1$ (Ising models) in 2d [18], $N = 0, 1, 2, 3$ [17,18,21] and many larger values of N [21] in 3d, leading to a rather precise estimate of f^*, especially in 3d. In order to compare our results with the field theoretical calculations we must keep in mind that it is customary to rescale the coupling in such a way that for all values of N the one-loop fixed point value of the new coupling $\bar{g}$ be exactly one [17,18]. By comparing the one-loop expression of the β-function, which in our notation would be

$$\beta(\widehat{g}_r) = (d-4)\widehat{g}_r + \frac{N+8}{N}\frac{\Gamma\left(3-d/2\right)}{(4\pi)^{d/2}}\widehat{g}_r^2 + O(\widehat{g}_r^3), \tag{73}$$

we find

$$\bar{g}^* = \frac{N+8}{N+2}\frac{\Gamma(2-d/2)}{2(4\pi)^{d/2}}f^*. \tag{74}$$

We also mention a determination of f^* for $N=0$ in 3d by working directly with the self-avoiding random walk model [22], which turns out to be in full agreement with the corresponding ϕ^4 field theoretical calculation [17,18].

Estimates of f^* can also be obtained by Monte Carlo simulations using the lattice formulation of the theory, by directly measuring $f(\beta)$. Numerical studies concerning the four-point coupling have been presented in the literature for some two-dimensional models: for $N = 1$ [23], and $N = 2,3$ [24]. The comparison with these works must take into account the extra factor N in our definition (61) of the four-point coupling f.

Finally let us briefly comment on the $d = 4$ case. In this case it is not possible to define a non-trivial limit for the non-linear σ model in the strict $\gamma = 0$ regime, at least within the $1/N$ expansion, since we obtain the naive result $\Delta(k) = 0$ implying $\widehat{g}_r^* = 0$. This is however consistent with the common expectation that O(N)-invariant models in four dimensions may only have a trivial fixed point, in which case the critical region should be characterized by logarithmic deviations from scaling. That this is the case has been shown by Lüscher and Weisz by making use of the strong-coupling expansion in a beautiful series of papers [3,5-8]. Kristjansen and Flyvbjerg [25] in turn have developed the lattice $1/N$ expansion of the O(N)-invariant models in four dimensions both in the symmetric and in the broken phase, finding substantial agreement with Refs. [3,5-8] at $N = 4$ in the region around criticality.

We may add that, by properly manipulating the expression presented in Appendix A in the limit $d \to 4$, it is possible to compute exactly (albeit only numerically) the $1/N$ correction to the β-function of the $\left(\phi^2\right)^2$ model for all values of the (running) renormalized coupling. In turn these results might be used to improve our understanding of the non-perturbative limit of a strongly interacting Higgs sector on the line traced by Refs. [26,27], where the leading order result was analyzed.

5. Strong-coupling analysis

The non-triviality issue can be also investigated by high-temperature series methods formulating the theory on the lattice. We consider the nearest-neighbor formulation (55) of O(N) vector models. Notice that in (55) we have introduced a rescaled inverse temperature β.

The strong-coupling expansion of $f(\beta)$ has the following form:

$$A_d(\beta) \equiv \beta^{d/2} f(\beta) = 2 + \sum_{i=1}^{\infty} a_i \beta^i. \tag{75}$$

Series up to 14th order of the quantities involved in the definition of $f(\beta)$, i.e. χ, m_2 and χ_4, have been calculated by Lüscher and Weisz [3], and re-elaborated on by Butera et al. [4]. From such series one can obtain $A_d(\beta)$ up to 13th order. We mention that, for other purposes, we have calculated the strong-coupling series of the Green function $G(x) \equiv \langle s_0 \cdot s_x \rangle$ up to 21st order in 2d, and up to 15th in 3d, obtaining the corresponding series of the energy, the magnetic susceptibility and the second moment correlation length to the same order [28].

We also considered strong-coupling series in the energy $f(E)$, which can be obtained by inverting the strong-coupling series of the energy $E = \beta + O(\beta^3)$ and substituting in Eq. (75)

$$f(E) = \frac{1}{E^{d/2}} \left[2 + \sum_{i=1}^{\infty} e_i E^i \right]. \tag{76}$$

From our strong-coupling series of the energy [28], we could calculate $B_d(E) \equiv E^{d/2} f(E)$ up to 13th order.

Before describing our analysis of the above series based on the Padé approximants (PA's) technique (see Ref. [29] for a review on the analysis of strong-coupling series), we recall that PA's are expected to converge well to meromorphic analytic functions. More flexibility is achieved by applying the PA analysis to the logarithmic derivative of the strong-coupling series considered (Dlog-PA analysis), and therefore enlarging the class of functions that can be reproduced to those having branch-point singularities. In general more complicated structures may arise, such as confluent singularities, which are sources of systematic errors for a PA (or Dlog-PA) analysis. In particular confluent singularities at β_c, i.e. confluent corrections to scaling arising from irrelevant operators [18,30], lead in general to a non-diverging singularity of $f(\beta)$ at β_c. Indeed in the presence of confluent singularities we would expect $f(\beta)$ to behave as

$$f(\beta) \simeq f^* + c\,(\beta_c - \beta)^{\Delta} \tag{77}$$

close to β_c with $\Delta > 0$. Such a behavior close to β_c cannot be reproduced by PA's or Dlog-PA's, while it could be detected by a first or higher order differential approximant analysis [31]. Therefore, in order to reduce systematic errors, one should turn to a more general and flexible analysis, such as differential approximants, which, on the other hand, require many terms of the series to give stable results. We tried also this type of analysis without getting stable and therefore acceptable results, very likely due to the relative shortness of the available series. We then expect PA and Dlog-PA analysis to be subject to larger systematic errors when confluent singularities are more relevant, as in 3d models at small N, where they represent a serious problem also in the determination of the critical exponents from the available strong-coupling series.

It is important to notice that the accuracy and the convergence of the PA estimates may change when considering different representations of the same quantity, according to how well the function at hand can be reproduced by a meromorphic analytic function. By comparing the results from different series representations of the same quantity one may check for possible systematic errors in the resummation procedure employed. To this purpose, in our study we will compare estimates of f^* coming from the strong-coupling series of both $f(\beta)$ and $f(E)$. Exact results at $N = \infty$ presented in the previous section, beside giving an idea of the behavior of $f(\beta)$ at finite N, represent useful benchmarks for strong-coupling methods.

The most direct way to evaluate $f^* \equiv f(\beta_c)$ would consist in computing $[l/m]$ PA's $A_{l/m}(\beta)$ from the available series of $A_d(\beta)$, and evaluating $\beta^{-d/2}A_{l/m}(\beta)$ at the critical point β_c (at least if $\beta_c < \infty$; if $\beta_c = \infty$ things are trickier as we will discuss below). This simple procedure works already reasonably well, but we found more effective a Dlog-PA analysis, which showed a greater stability and whose results will be presented in the following.

Our Dlog-PA analysis consisted in computing $[l/m]$ PA to the strong-coupling series of the logarithmic derivative of $A_d(\beta)$, indicated by $\mathrm{Dlog}_{l/m}A_d(\beta)$, and subsequently a set of corresponding approximants $f_{l/m}(\beta)$ to $f(\beta)$, which are obtained by reconstructing $f(\beta)$ from the logarithmic derivative of $A_d(\beta)$:

$$f_{l/m}(\beta) = \frac{2}{\beta^{d/2}} \exp \int_0^\beta d\beta' \, \mathrm{Dlog}_{l/m} A_d(\beta'). \tag{78}$$

All the approximants with

$$l + m \geqslant 10, \qquad m \geqslant l \geqslant 4 \tag{79}$$

were considered in order to check the stability of the procedure. Notice that, for given l, m, the number of terms of the series of $A_d(\beta)$ used by the corresponding Dlog-PA is $n = l + m + 1$. Once the approximant $f_{l/m}(\beta)$ is computed, if β_c is finite, its value at β_c provides an estimate of f^*.

This requires a rather precise determination of β_c, which is in some cases available in the literature from strong-coupling and numerical Monte Carlo studies. When β_c was not known, we estimated it from a Dlog-PA analysis of the strong-coupling series of the magnetic susceptibility (up to 21st order in 2d and 15th in 3d). Our strong-coupling determinations of β_c in 3d models at large N compare very well with the $O(1/N)$ calculation (67), as shown in Fig. 7. Let us notice that the error on the value of β is small enough not to be relevant for the estimate of f^*.

In order to better understand the analytic structure of $f(\beta)$ we have done a detailed study of the complex-plane singularities of the Dlog-PA's of $A_d(\beta)$. We have first checked hyperscaling. A violation of hyperscaling would lead to a behavior $f(\beta) \sim A_d(\beta) \sim (\beta_c - \beta)^\epsilon$ for $\beta \to \beta_c$, and thus the Dlog-PA's would show a simple pole at $\beta = \beta_c$. We recall that a Dlog-PA analysis is in general very efficient in detecting power-

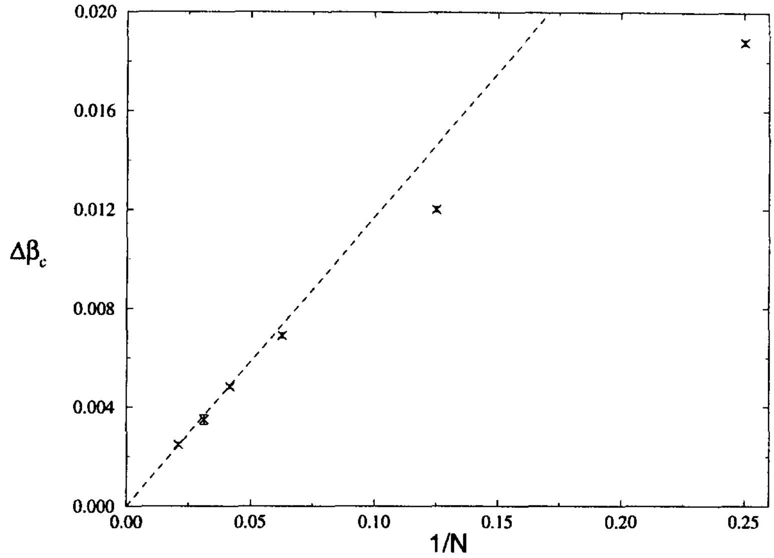

Fig. 7. $\Delta\beta_c \equiv \beta_c(\infty) - \beta_c(N)$ vs. $1/N$ in 3d models, as estimated by a strong-coupling analysis. The dashed line represents the $O(1/N)$ calculation (cf. Eq. (67)).

law singularities. We have found no evidence of such a pole, confirming hyperscaling arguments. However, notice that when $1 > \Delta > 0$ Eq. (77) implies a behavior

$$\mathrm{Dlog} A_d(\beta) \sim (\beta_c - \beta)^{\Delta - 1} \tag{80}$$

close to β_c. In two dimensions $\Delta \simeq 1$ and therefore we do not expect to find singularities around β_c. This is confirmed by the analysis of $A_2(\beta)$. In three dimensions, while at large N we have $\Delta = 1 + O(1/N)$ [32], at small N we expect $\Delta \simeq 0.5$; thus $A_3(\beta)$ should behave as in Eq. (80), and in the Dlog-PA's the singularity should be mimicked by a shifted pole at a β larger than β_c. Indeed in the analysis of $A_3(\beta)$ we have found a singularity typically at $\beta \simeq 1.1$–$1.2\,\beta_c$. This fact will eventually affect the determination of $f(\beta)$ close to β_c by a systematic error. However, since the singularity is integrable, the error must be finite and the analysis shows that such errors are actually reasonably small.

Sometime the PA's showed spurious singularities on the positive real axis (or very close to it) for $\beta \lesssim \beta_c$. We considered these approximants defective, and discarded them from the analysis. Such defective PA's were a minority, as the tables show. The only stable singularity detected by the Dlog-PA's of $A_d(\beta)$ lies in the negative β axis and closer to the origin than β_c: it turns out to be nothing but a regular zero of $A_d(\beta)$. The position of this negative zero is reported in the Tables 1, 2, 3 and 4 for several values of N.

As final estimates of f^*, reported in Tables 5 and 6, we take the average of the values $f_{l/m}(\beta_c)$ from the non-defective PA's using all available terms of the series, i.e. those with $n = l + m + 1 = 13$. The errors displayed in Tables 5 and 6 are just

Table 1
For some 2d O(N) σ models with $N < 2$ we report the critical point, β_c; the estimate of the singularity of the PA's closest to the origin, β_0, which corresponds to a regular zero of $A_2(\beta)$; the values of the approximants $f_{l/m}(\beta)$ at the critical point. Asterisks mark defective PA's

N	β_c	β_0	5/5	4/6	5/6	4/7	6/6	5/7	4/8
−1	0.3145(1)	−0.1315	5.27	5.32	5.27	5.28	5.27	5.31	*
−1/2	0.3492(1)	−0.1506	7.87	*	*	8.04	7.85	8.10	8.00
0	0.379052(1) [38]	−0.1653	10.51	*	*	10.43	10.54	10.48	*
1/2	0.408545(8)	−0.1774	12.60	12.72	12.68	*	12.66	12.61	12.63
1	0.4406867...	−0.1878	14.65	14.72	14.70	14.67	14.69	*	14.57
3/2	0.4804(1)	−0.1969	16.76	16.76	16.76	16.60	16.83	16.83	16.47

Table 2
For the 2d XY model ($N = 2$) we give some details on the analysis of the series of $f(\beta)$ (first line) and $f(E)$ (second line). We report the critical point, β_c (E_c); the estimate of the regular zero of $A_2(\beta)$ ($B_2(E)$) closest to the origin, β_0 (E_0); the values of the approximants $f_{l/m}(\beta)$ ($f_{l/m}(E)$) at β_c (E_c). Asterisks mark defective PA's

N			5/5	4/6	5/6	4/7	6/6	5/7	4/8
2	$\beta_c = 0.559(3)$ [34,35]	$\beta_0 = -0.2049$	19.27	19.44	*	18.71	*	*	18.24
	$E_c = 0.722(3)$ [34]	$E_0 = -0.2179$	18.28	18.46	18.30	18.35	18.29	*	18.17

Table 3
We give some details of the strong-coupling analysis of the series $f(\beta)$ (first line) and $f(E)$ (second line) for two asymptotically free models: $N = 3, 4$. We report the estimate of the regular zero of $A_2(\beta)$ ($B_2(E)$) closest to the origin, β_0 (E_0); the values of the approximants $f_{l/m}(\beta)$ ($f_{l/m}(E)$) at a value $\bar{\beta}$ ($\bar{E}$) corresponding to a correlation length $\xi \simeq 10$. The values of E and ξ are taken from Ref. [37] for $N = 3$, and Ref. [36] for $N = 4$. Asterisks mark defective PA's, i.e. PA's having singularities for $\beta \lesssim \bar{\beta}$ ($E \lesssim \bar{E}$)

N			ξ	5/5	4/6	5/6	4/7	6/6	5/7	4/8
3	$\beta_0 = -0.2188$	$\beta = 0.5$	11.05(1)	20.3	20.6	*	20.0	*	19.5	19.8
	$E_0 = -0.2330$	$E = 0.60157$	11.05(1)	19.9	20.0	19.9	19.9	19.9	19.9	19.8
4	$\beta_0 = -0.2305$	$\beta = 0.525$	10.32(3)	21.8	22.4	*	21.3	*	20.6	21.0
	$E_0 = -0.2456$	$E = 0.60089$	10.32(3)	21.2	21.3	21.3	21.3	21.2	21.4	21.1

indicative; they are the variance around the estimate of f^* of the results coming from all PA's considered (cf. (79)), which should give an idea of the spread of the results coming from different PA's. Such errors do not always provide a reliable estimate of the systematic errors, which may be underestimated especially when the structure of the function (or of its logarithmic derivative) is not well approximated by a meromorphic analytic function. In such cases a more reliable estimate of the systematic error would come from the comparison of results from the analysis of different series representing the same quantity, which in general are not expected to have the same structure.

For this reason we have considered the series in the energy variable, which we have analyzed exactly as $f(\beta)$. In this case instead of β_c we needed E_c, the energy at the critical point. When the value of E_c was not available in the literature, we estimated it

Table 4
For 3d $O(N)$ σ models we present a summary of the analysis of the strong-coupling series of $f(\beta)$ (first line) and $f(E)$ (second line) at some values of N. We report the critical point, β_c (E_c); the estimate of the regular zero of $A_3(\beta)$ ($B_3(E)$) closest to the origin, β_0 (E_0); the values of the approximants $f_{l/m}(\beta)$ ($f_{l/m}(E)$) at β_c (E_c). Asterisks mark defective PA's. Errors due to the uncertainty of β_c and E_c are at most of the order of one in the last digit of the numbers reported (except for some cases where they are given explicitly). We mention that at $N = 1$ and $N = 2$ our strong-coupling analysis led to $E_c = 0.332(3)$ for both

N	β_c, E_c	β_0, E_0	5/5	4/6	5/6	4/7	6/6	5/7	4/8
−1	0.19840(3)	−0.109	*	11.2	11.3	10.9	10.6	10.7	*
	0.350(5)	−0.117	9.3	9.4	9.7	*	10.1	10.5	10.3
0	0.21350(1) [38]	−0.134	19.7	19.8	19.6	19.8	*	19.4	*
	0.333(5)	−0.146	*	18.1(3)	*	18.4(2)	18.3(2)	*	*
1	0.221652(4) [39]	−0.149	25.4	26.0	25.3	25.8	25.4	24.8	*
	0.3301(1) [41]	−0.166	24.3	24.3	24.3	24.4	24.4	24.4	24.4
2	0.22710(1) [40]	−0.160	29.4	29.6	29.4	29.6	29.4	*	29.6
	0.3297(2) [42]	−0.180	28.9	28.9	28.9	29.0	28.9	28.9	28.9
3	0.231012(12) [43,44]	−0.168	32.5	32.3	*	32.4	32.5	32.4	32.4
	0.331(3)	−0.191	32.2	32.2	32.2	32.3	32.3	32.3	32.3
4	0.2339(1)	−0.175	34.9	34.5	35.3	34.6	34.9	34.7	34.7
	0.333(2)	−0.200	34.8	34.8	34.7	34.9	34.8	34.9	34.8
8	0.2407(1)	−0.19	40.5	39.9	41.1	40.2	40.6	40.4	40.3
	0.334(1)	−0.224	40.5	40.5	*	40.8	40.6	40.7	40.7
16	0.2458(1)	−0.20	44.8	44.3	45.2	44.6	44.9	44.7	44.7
	0.3370(5)	−0.246	44.8	44.8	*	45.1	44.9	45.0	45.0
24	0.2479(1)	−0.21	46.5	46.1	46.8	46.4	46.6	46.4	46.4
	0.3379(3)	−0.260	46.4	46.5	47.0	46.7	46.6	46.7	46.6
32	0.2492(2)	−0.22	47.5	47.1	47.6	47.3	47.5	47.4	47.4
	0.3384(3)	−0.268	47.3	47.4	47.7	47.5	47.5	47.5	47.5
48	0.2502(1)	−0.22	48.4	48.1	48.5	48.3	48.4	48.3	48.3
	0.3390(3)	−0.280	48.2	48.3	48.5	48.5	48.4	48.4	48.4
∞	0.252731...	−0.25	50.24	50.12	50.31	50.21	50.27	50.25	50.24
	0.340537...	−0.34	50.15	50.23	50.36	50.32	50.26	50.29	50.26

Table 5
For 2d $O(N)$ σ models we report the critical point β_c; f^* from our strong-coupling analysis, f^*_{sc}; the $O(1/N)$ calculation of f^*, $f^*_{1/N}$; f^* from ϕ^4 field theory at fixed dimensions, f^*_{ft}; f^* from Monte Carlo simulations, f^*_{mc}

N	β_c	f^*_{sc}	$f^*_{1/N}$	f^*_{ft}	f^*_{mc}
−1	0.3145(1)	5.29(3)			
−1/2	0.3492(1)	8.0(1)			
0	0.379052(1) [38]	10.51(5)			
1/2	0.408545(8)	12.63(5)			
1	0.4406867...	14.63(7)		15.5(8) [18]	14.3(1.0) [23]
3/2	0.4804(1)	16.7(2)			
2	0.559(3) [34,35]	18.2(2)			17.7(2) [24]
3	∞	19.8(4)	20.09		19.6(2) [24]
4	∞	21.2(5)	21.35		
∞	∞		25.1327...		

Table 6
For 3d O(N) σ models we report f^* as estimated by the analysis of the strong-coupling expansion of $f(\beta)$, $f^*_{sc,\beta}$; f^* from the analysis of the series of $f(E)$, $f^*_{sc,E}$; the $O(1/N)$ calculation of f^*, $f^*_{1/N}$; f^* from ϕ^4 field theory at fixed dimensions, f^*_{ft}. In Ref. [21] data of f^* were reported without errors, and differences with Refs. [17,18] should be due to a different resummation procedure

N	$f^*_{sc,\beta}$	$f^*_{sc,E}$	$f^*_{1/N}$	f^*_{ft}
−1	10.7(4)	10.3(6)		
0	19.4(3)	18.3(3)		17.86(5) [18,17] 17.62 [21]
1	25.1(5)	24.4(1)		23.72(8) [18,17] 23.47 [21]
2	29.5(1)	28.9(1)		28.27(8) [18,17] 28.03 [21]
3	32.4(1)	32.3(1)		31.78(9) [18,17] 31.60 [21]
4	34.8(3)	34.8(1)	30.84	34.41 [21]
8	40.4(4)	40.7(1)	40.55	40.93 [21]
16	44.8(3)	45.0(1)	45.41	45.50 [21]
24	46.5(2)	46.6(2)	47.03	47.13 [21]
32	47.4(2)	47.5(1)	47.84	47.94 [21]
48	48.3(1)	48.4(1)	48.65	
∞	50.25(6)	50.27(6)	50.2654...	

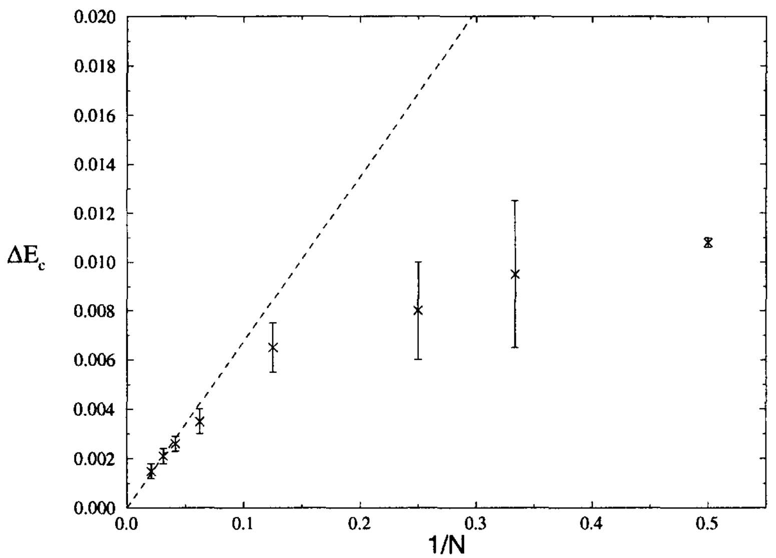

Fig. 8. $\Delta E_c \equiv E_c(\infty) - E_c(N)$ vs. $1/N$ in 3d models, as estimated by a strong-coupling analysis. The dashed line represents the $O(1/N)$ calculation (cf. Eq. (69)).

by the first real positive singularity found in the analysis of the available strong-coupling series of the magnetic susceptibility expressed in powers of E. This procedure provides an estimate of E_c much less precise than β_c (see Table 4 for the values obtained in 3d), but sufficiently good to our purposes, given that $f(E)$ is smooth around E_c. In Fig. 8 we compare our determinations of E_c in 3d models with the large-N result (71), showing agreement within the uncertainty of the strong-coupling results.

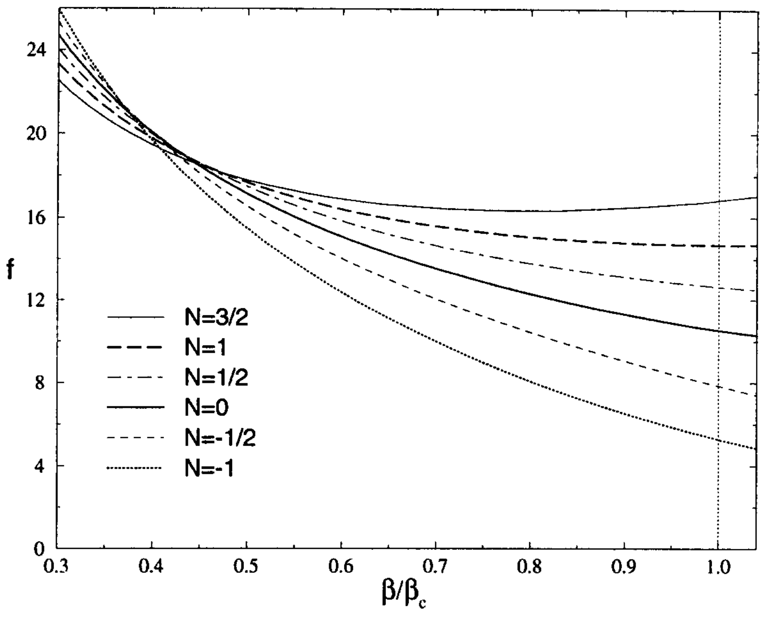

Fig. 9. $f_{6/6}(\beta)$ vs. β/β_c for various values of $N < 2$ in two-dimensional models.

In asymptotically free models where $\beta_c = \infty$, the task of determining f^* from a strong-coupling approach appears much harder. On the other hand, since at sufficiently large β we expect that

$$|f(\beta) - f^*| \sim \xi^{-2}, \tag{81}$$

a reasonable estimate of f^* could be obtained at β-values corresponding to large but finite correlation lengths, e.g. $\xi \gtrsim 10$, where the curve $f(\beta)$ should already be stable (scaling region). Notice that this is the same idea underlying numerical Monte Carlo studies. Another interesting possibility is to change variable from β to the energy E, and analyze the series in powers of E. In the energy variable the continuum limit is reached for $E \rightarrow 1$, and therefore the strong-coupling approach to the continuum limit appears more feasible. In order to reach the continuum limit from strong coupling, we believe this change of variable to be effective especially for the analysis of dimensionless ratios of physical quantities.

To begin with we present the results obtained for the 2d models. PA's for $-1 \leqslant N < 2$ are quite stable, giving estimates of f^* very close to each other as shown in Table 1, where the values of the approximants $f_{l/m}(\beta)$ at β_c are reported. For $N < -1$, due to the instability of the corresponding PA's, we could not get reliable estimates of f^*. Fig. 9 shows $f_{6/6}(\beta)$ vs. β/β_c for various values of N. Differences in the other PA's were of the order of the width of the lines drawn in the figure. Final estimates of f^* are reported in Table 5. In order to check possible systematic errors in our analysis, we applied the above procedure to the 13th order series of $B_2(E)$ of the Ising model

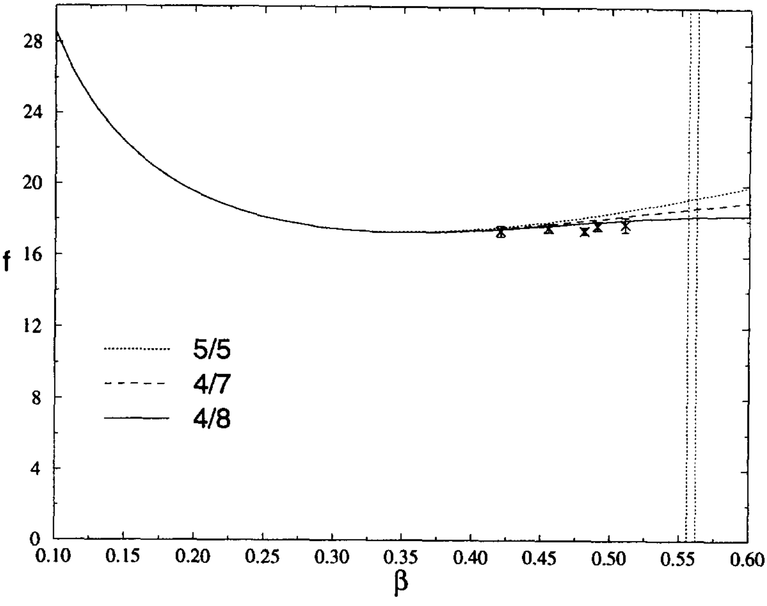

Fig. 10. Some $f_{l/m}(\beta)$ are plotted vs. β for the 2d XY model ($N = 2$). The vertical dotted lines indicate the critical point: $\beta_c = 0.559(3)$ [34,35]. Monte Carlo data from Ref. [24] are also shown.

($N = 1$). The value of $f(E)$ at $E_c = \sqrt{2}/2$, the energy value at β_c, must give again f^*. As in the analysis of $A_2(\beta)$, approximants $f_{l/m}(E)$ turn out to be rather stable, leading to the result $f^* = 14.64(10)$, which is perfectly consistent with the estimate $f^* = 14.63(7)$ coming from the analysis of $A_2(\beta)$.

For the Ising model ($N = 1$) our estimate of f^* is in agreement with the result of Ref. [33], obtained by a slightly different strong-coupling analysis (the value reported there is $f^* = 14.67(5)$), and with the estimates by ϕ^4 field theory calculations at fixed dimensions [18] and numerical Monte Carlo simulations [23] (see Table 5).

The 2d XY model ($N = 2$) is expected to follow the pattern of a Kosterlitz-Thouless critical phenomenon, whose critical region is characterized by a correlation length diverging exponentially with respect to $\tau \equiv \beta_c - \beta$: $\xi \sim \exp(b/\tau^\sigma)$ with $\sigma = 1/2$. For this model the values of $f_{l/m}(\beta)$ and $f_{l/m}(E)$ respectively at $\beta_c = 0.559(3)$ [34,35] and $E_c = 0.722(3)$ [34] are reported in Table 2. Fig. 10 shows various non-defective $f_{l/m}(\beta)$ at $N = 2$, comparing them with the Monte Carlo results obtained recently by Kim [24] for correlation lengths: $5 \lesssim \xi \lesssim 70$. The agreement is very good especially for PA's obtained using all available 13 terms of the strong-coupling series. PA's of the series in E turn out to be more stable, as shown in Table 2 and in Fig. 11, giving a perfectly consistent result for f^*. Our final estimate is $f^* = 18.2(2)$ which is slightly larger than the Monte Carlo result $f^* = 17.7(2)$ [24] (this number has been obtained by taking only data for $\xi \gtrsim 25$ of Ref. [24] and taking into account the extra factor N in our definition (58)), but definitely consistent.

When $N \geqslant 3$ the critical point moves to infinity making the determination of f^*

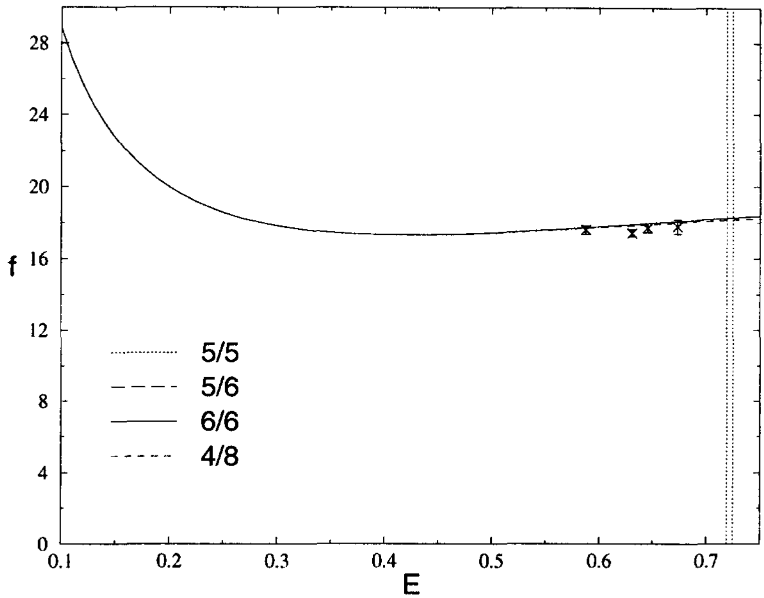

Fig. 11. Some $f_{l/m}(E)$ are plotted vs. E for the 2d XY model ($N = 2$). The vertical dotted lines indicate the value of the energy at the critical point: $E_c = 0.722(3)$, estimated from Monte Carlo data [34]. Monte Carlo data from Ref. [24] are also shown.

from strong coupling harder. For such models our analysis should be considered just exploratory due to the shortness of the available series, but as we will see the results look promising. In order to give an idea of the stability of our resummation procedure in this case, in Table 3 we report the values taken by $f_{l/m}(\beta)$ and $f_{l/m}(E)$ for pairs of β and E corresponding to a correlation length $\xi \simeq 10$. In Fig. 12 various $f_{l/m}(\beta)$ at $N = 3$ are drawn and compared with the Monte Carlo results of Ref. [24], obtained for correlation lengths $10 \lesssim \xi \lesssim 120$. The curves corresponding to different $f_{l/m}(\beta)$ are very close up to $\beta \simeq 0.5$ (see also Table 3). At $\beta \gtrsim 0.5$ we observe that curves from different PA's become more and more stable with increasing $n = l + m + 1$, improving the agreement with the Monte Carlo data. Anyway, the agreement is quite good, even for $\beta \simeq 0.6$, corresponding to $\xi \simeq 65$. Fig. 13 shows some approximants $f_{l/m}(E)$ computed from the series in the energy. At $E = 1$ they give consistent results within 5–10%. Similar results are observed for $N = 4$, as shown in Fig. 14, where some $f_{l/m}(\beta)$ are plotted. Notice that at $\beta = 0.6$ $\xi \simeq 25$ [36].

In order to get an estimate of f^* for both $N = 3$ and $N = 4$ we considered the values of $f_{l/m}(\beta)$ and $f_{l/m}(E)$ at the largest values of β and E where they are still stable, i.e. $\beta \simeq 0.5$ and $E \simeq 0.6$, which correspond to an acceptably large correlation length $\xi \simeq 10$. So, from data in Table 3, our final estimate at $N = 3$ is $f^* \simeq 19.8$, with an uncertainty of few per cent, which compares very well with the Monte Carlo result $f^* = 19.6(2)$ (obtained by fitting all data reported in Ref. [24] to a constant) and $O(1/N)$ calculation $f^* \simeq 20.09$. At $N = 4$ our estimate is $f^* \simeq 21.2$ against

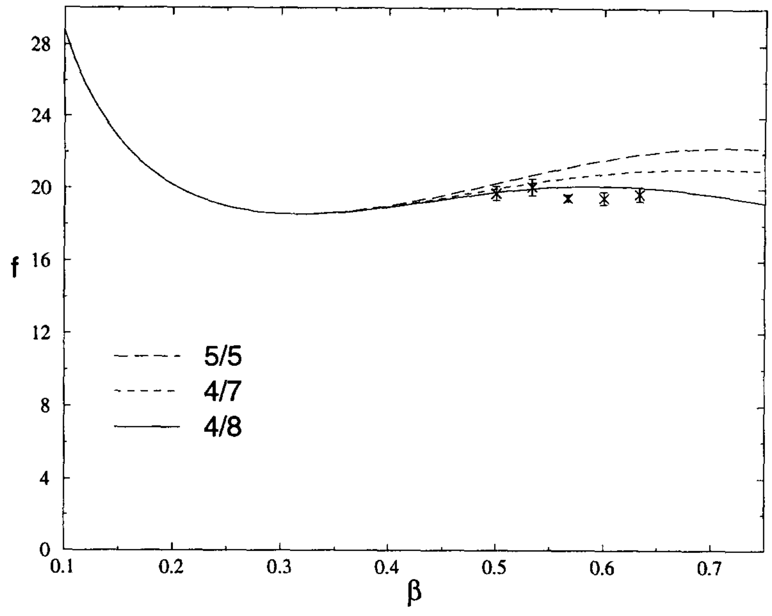

Fig. 12. Some $f_{l/m}(\beta)$ are plotted vs. β for the 2d O(3) model. Monte Carlo data from Ref. [24] are also shown.

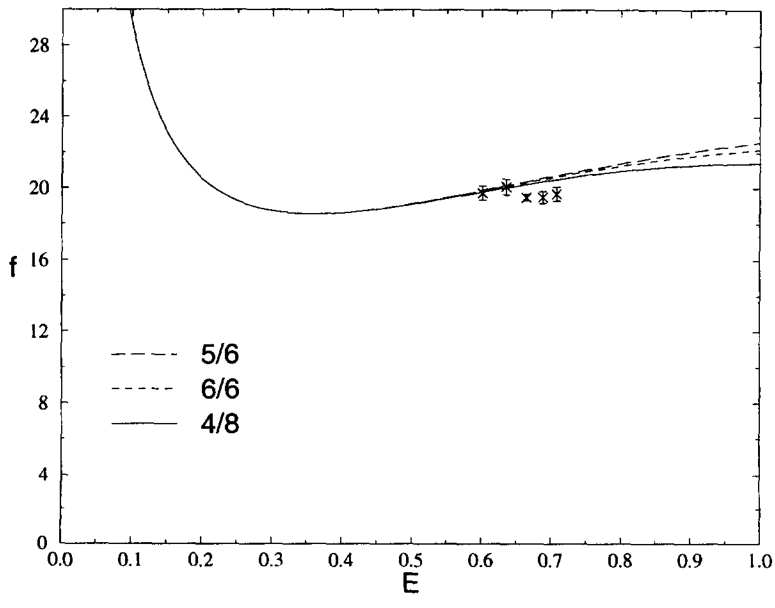

Fig. 13. Some $f_{l/m}(E)$ are plotted vs. E for the 2d O(3) model. Monte Carlo data from Ref. [24] are also shown.

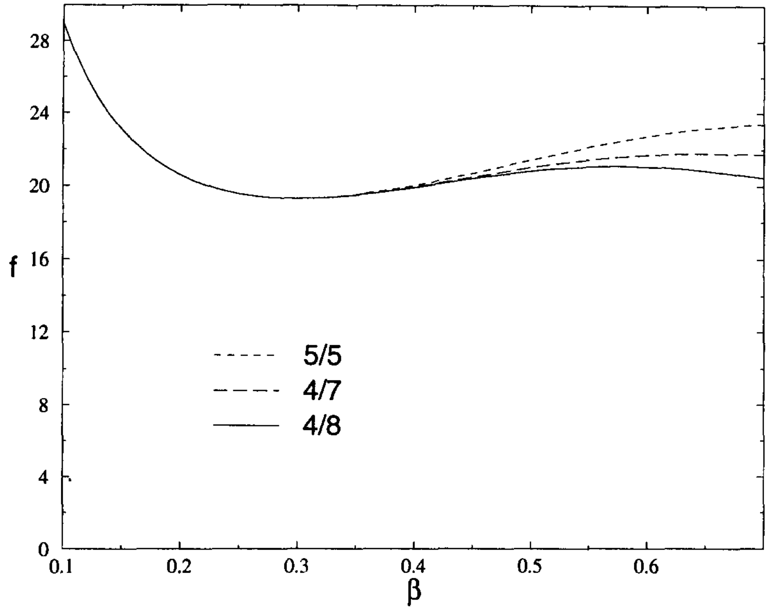

Fig. 14. Some $f_{l/m}(\beta)$ are plotted vs. β for the 2d O(4) model.

$f^* \simeq 21.35$ coming from the $O(1/N)$ calculation.

Fig. 15 summarizes our 2d results: it shows our strong-coupling estimates of f^* vs. N, comparing them with the available estimates of f^* from alternative approaches: ϕ^4 field theory, Monte Carlo and $1/N$ expansion techniques. There is a general agreement, in particular the $O(1/N)$ calculation $f^* \simeq 8\pi\left(1 - 0.602033/N\right)$ fits very well the data down to $N = 3$. Furthermore we observe the linear approach of f^* toward zero for $N \to -2$, similar to the $d = 1$ case.

Let us now consider 3d O(N) σ models, which show a critical behavior at a finite β for all values of N. In order to check possible systematic errors we analyzed both the strong-coupling series of $f(\beta)$ and $f(E)$. Table 4 shows a summary of the estimates of f^* from the values of the approximants $f_{l/m}(\beta)$ at β_c and $f_{l/m}(E)$ at E_c. Final estimates of f^* from the analysis of the strong-coupling series in β and in E are reported in Table 6. We recall that the errors displayed in Table 6 are related to the spread of the PA results, while an estimate of the true systematic errors could only come from the comparison of results from different series associated to the same quantity.

Fig. 16 shows typical curves of $f(\beta)$ obtained by [6/6] PA's (sometimes we used [7/5] PA's when the [6/6] ones were defective). The error bars displayed at $\beta/\beta_c = 1$ show the spread of the estimates of f^* from different non-defective $[l/m]$ PA's.

At large N, typically $N \geqslant 3$, both series of $f(\beta)$ and $f(E)$ give consistent results, which should be an indication of small systematic errors. As further check of our resummation procedure in the large-N region, we repeated our analysis at $N = \infty$. We found that most of the approximants $f_{l/m}(\beta)$ constructed from the 13th order series

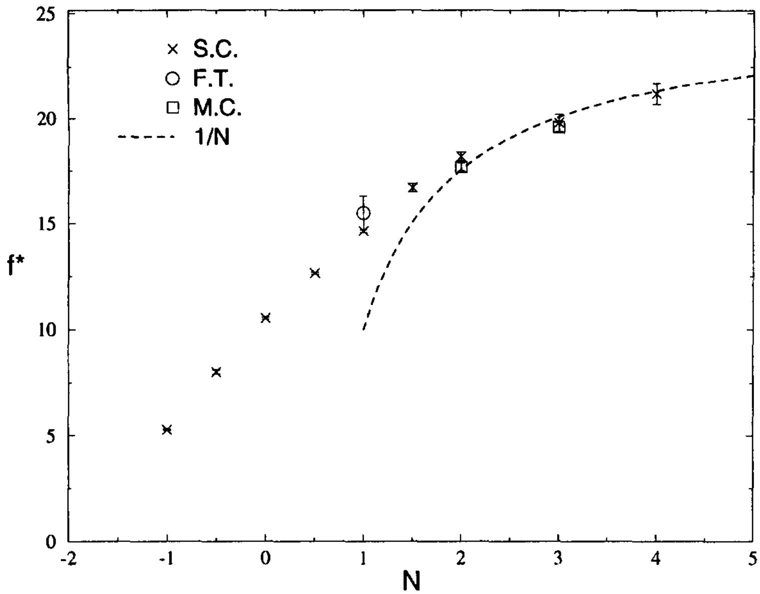

Fig. 15. For 2d models we plot f^* vs. N obtained from our strong-coupling analysis. For comparison field theoretical and Monte Carlo estimates are also shown. The dashed line represents the $O(1/N)$ calculation of f.

of $A_3(\beta)$, if plotted in Fig. 6, would not be distinguishable from the exact curve. The analysis of the $N = \infty$ 13th order series of $A_3(\beta)$ and $B_3(E)$ would have given respectively $f^* = 50.25(6)$ and $f^* = 50.27(6)$ against the exact value $f^* = 16\pi = 50.2654\ldots$. Therefore everything seems to work fine at large N. On the contrary, at small N there are discrepancies between the analysis in β and in E, which are definitely larger than the typical spread of the PA estimates of f^* from each series. Such differences give somehow an idea of the size of the systematic errors of our analysis when applied to these values of N.

In Table 6 for comparison we also give the results from ϕ^4 field theory and $1/N$ expansion. Fig. 17 summarizes all available results for f^*. There, as a strong-coupling estimate of f^*, we show the average of the results from the series $f(\beta)$ and $f(E)$, while their difference is used as an estimate of the systematic error.

At large N, $N \geqslant 8$, there is substantial general agreement: estimates from the strong-coupling approach, $O(1/N)$ calculation $f^* \simeq 16\pi\left(1 - 1.54601/N\right)$, and ϕ^4 field theory differ at most by 1% to each other. At small N, $N = 0, 1, 2$, our strong-coupling estimates show relevant discrepancies with the field theoretical calculations, which are of the size of the differences between the results coming from the analysis of $f(\beta)$ and $f(E)$, and therefore they should be caused by systematic errors in the strong-coupling analysis employed. Anyway, such discrepancies are not dramatic; indeed they are at most 5% and decrease with increasing N.

In conclusion we have seen that in two and three dimensions 13 terms of the strong-

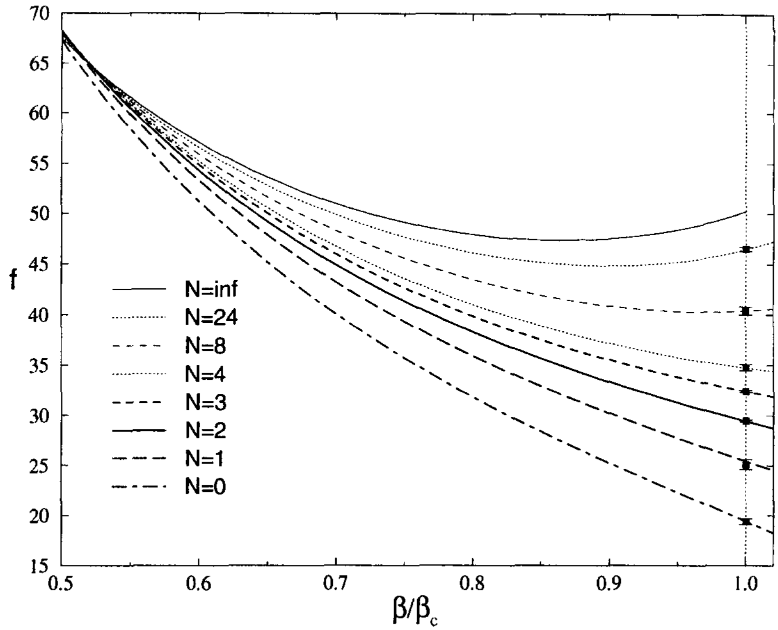

Fig. 16. $f(\beta)$ vs. β/β_c for various values of N in three-dimensional models as obtained by a [6/6] PA (or [7/5] when the [6/6] one was defective). Error bars at $\beta/\beta_c = 1$ show the spread in the determination of f^* from all PA's considered.

coupling series of $A_d(\beta)$ and $B_d(E)$ are already sufficient to give quite stable results, which compare very well with calculations from other techniques, such as ϕ^4 field theory at fixed dimensions, Monte Carlo simulations and the $1/N$ expansion. Of course an extension of the series of f would be welcome, especially for two reasons:

(i) to further stabilize the PA's in the asymptotically free models and obtain reliable estimates at values of β corresponding to large correlation lengths $\xi \gtrsim 100$, and moreover check whether the change of variable $\beta \longrightarrow E$ allows one to get a reliable strong-coupling estimate of f^* in the continuum limit $E \longrightarrow 1$;

(ii) to see whether the apparent discrepancies at small N in 3d with the more precise ϕ^4 field theory calculations get reduced.

An extension of the series of $f(\beta)$ may also allow a more accurate and flexible analysis, like differential approximants, which in general require many terms of the series in order to give stable results, and which could provide a better reconstruction of $f(\beta)$ from its strong-coupling series, taking properly into account the confluent singularities, which should be the major source of systematic error in 3d models at small N.

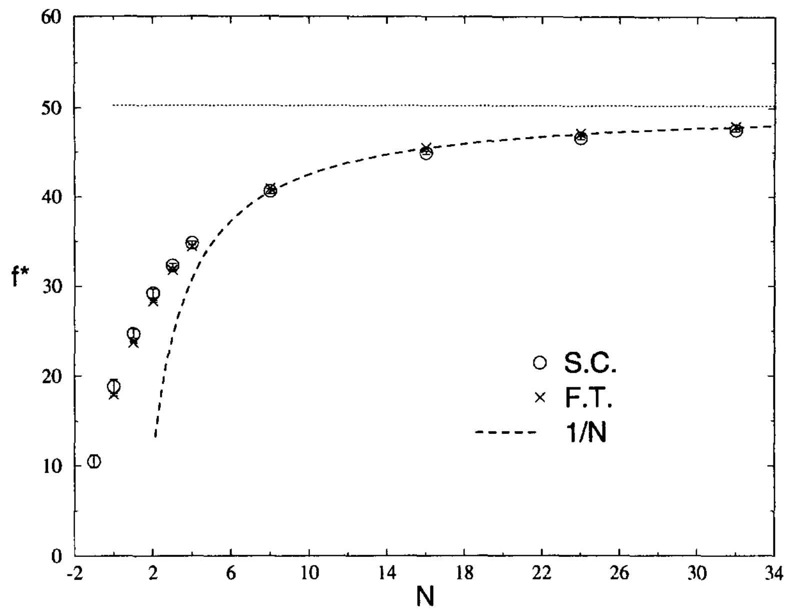

Fig. 17. f^* vs. N from our strong-coupling analysis in 3d. For comparison field theoretical estimates are also shown. The dashed line represents the $O(1/N)$ calculation of f^*. The dotted line indicates the value of f^* at $N = \infty$.

6. Conclusions

We computed the dependence of the renormalized four-point coupling g_r from the renormalized mass m_r and the bare coupling to $O(1/N)$ for O(N)-invariant $\left(\phi^2\right)^2_d$ theories ($d \leqslant 4$) in the symmetric phase. As a consequence, we obtained expressions for the β-function and its fixed point g_r^* within the same approximation.

We extracted an independent determination of g_r^* from the strong-coupling analysis of the O(N) non-linear σ models, which we performed for $d = 2, 3$ and selected values of N in the whole range $N > -2$, applying resummation techniques both in the inverse temperature variable β and in the energy variable E. In two dimensions and for N sufficiently large ($N \geqslant 3$) in three dimensions we found a good agreement with the ϕ^4 fixed-dimension field theory estimates, and we could also check consistency with the $1/N$ prediction, thus seemingly indicating good convergence properties of the $1/N$ expansion at least when applied to the above quantities. In three dimensions and for small N, however, some discrepancy between resummations of the series in β and in E occurred, which we interpreted as an indication of systematic errors, and which was also reflected into a small disagreement with results presented in the literature and obtained with other techniques, like ϕ^4 field theory at fixed dimensions. Such discrepancy might be significantly reduced by knowing a few more terms in the strong-coupling series, whose feasibility seems to be well within the range of present day strong-coupling techniques. In our opinion improving the strong-coupling analysis might lead to a determination of the fixed point value of the renormalized four-point coupling with a precision comparable to, or even better than, the best available results. We stress the crucial role played by the comparison of series in the variables β and E in order to estimate the relevance of systematic errors.

Acknowledgements

It is a pleasure to thank A.J. Guttmann and A.D. Sokal for useful and stimulating discussions.

Appendix A

By applying Eq. (35) to Eq. (24) and making explicit use of Eq. (36), we may obtain the following explicit representation of the $O(1/N)$ contribution to the β-function of O(N) models in d dimensions:

$$\begin{aligned}
\frac{\beta^{(1)}(\widehat{g}_r)}{\widehat{g}_r^2} &= (d-3)2^{d-1}\beta_0 \\
&+\frac{2}{d}(d-1)^2\left(d-4+\beta_0\widehat{g}_r\right)^2\int\frac{d^du}{(2\pi)^d}\frac{1}{\left[1+\widehat{g}_r\Pi(u)\right]^2}\frac{1}{\left(4+u^2\right)^2} \\
&+2\int\frac{d^du}{(2\pi)^d}\left[\frac{\beta_0\widehat{g}_r+d-4}{\left(1+\widehat{g}_r\Pi(u)\right)^2}-\frac{\beta_0\widehat{g}_r}{1+\widehat{g}_r\Pi(u)}\right] \\
&\quad\times\left[\frac{1}{(1+u^2)^3}+\frac{3}{(1+u^2)(4+u^2)}\left(\frac{d/4-2}{1+u^2}-\frac{d-1}{4+u^2}\right)\right] \\
&-\frac{2}{d}\int\frac{d^du}{(2\pi)^d}\frac{(\beta_0\widehat{g}_r+d-4)^2}{\left(1+\widehat{g}_r\Pi(u)\right)^2} \\
&\quad\times\left[\frac{1}{(1+u^2)^3}+\frac{3}{(1+u^2)(4+u^2)}\left(\frac{d/4-1}{1+u^2}+\frac{d-1}{4+u^2}\right)\right] \\
&-4\int\frac{d^du}{(2\pi)^d}\left[\frac{\beta_0\widehat{g}_r+d-4}{\left(1+\widehat{g}_r\Pi(u)\right)^3}-\frac{\beta_0\widehat{g}_r+d/2-2}{\left(1+\widehat{g}_r\Pi(u)\right)^2}\right] \\
&\quad\times\frac{1}{(4+u^2)^2}\left(\beta_0\widehat{g}_r+d-4-\frac{3}{1+u^2}\right)^2 \\
&-4\int\frac{d^du}{(2\pi)^d}\frac{\beta_0\widehat{g}_r(\beta_0\widehat{g}_r+d-4)}{\left(1+\widehat{g}_r\Pi(u)\right)^2}\frac{1}{(4+u^2)^2}\left(\beta_0\widehat{g}_r+d-4-\frac{3}{1+u^2}\right),
\end{aligned} \tag{A.1}$$

where we have introduced the rescaled integration variable $u \equiv k/m_r$. By noticing that, according to its definition (37),

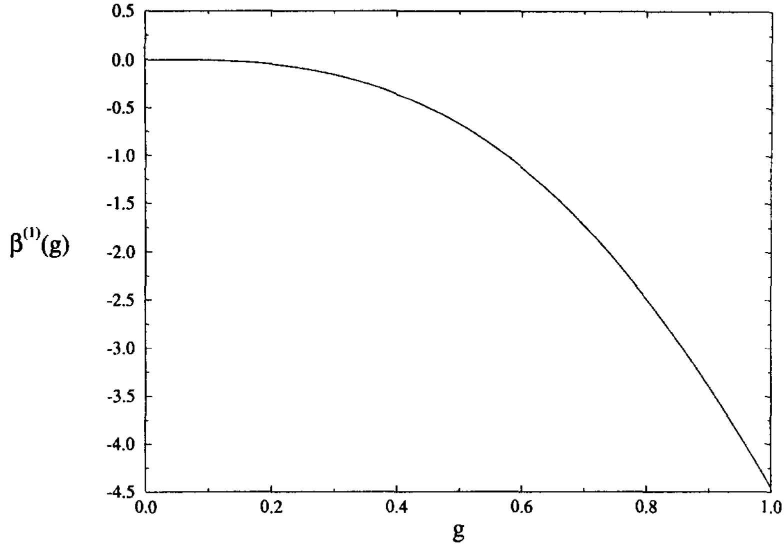

Fig. A.1. $\beta^{(1)}(\bar{g})$ vs. $\bar{g}$ for the three-dimensional case.

$$0 \geqslant \Pi(u) \geqslant -\frac{1}{g_*}, \tag{A.2}$$

it is easy to get convinced that all integrals appearing in Eq. (A.1) are well defined and finite as long as $d \leqslant 4$ and $\widehat{g}_r < g_*$. Moreover it is possible to perform a series expansion of Eq. (A.1) in the powers of $\widehat{g}_r$, reproducing order by order standard perturbation theory results [21], and in particular to leading order

$$\frac{\beta^{(1)}(\widehat{g}_r)}{\widehat{g}_r^2} \underset{\widehat{g}_r \to 0}{\longrightarrow} 8\beta_0 \tag{A.3}$$

for all values of d.

For the sake of comparison in Figs. A.1 and A.2 we plot the function $\beta^{(1)}(\bar{g})$, where $\bar{g}$ has been defined as in Refs. [17,18,21] such that $\bar{g}^* = 1$ at $N = \infty$ (see also Section 4 and Eq. (74)), respectively for $d = 3$ and $d = 2$.

We recall once more that

$$\bar{g} = \frac{N+8}{N} \frac{\beta_0}{4-d} \widehat{g}, \tag{A.4}$$

and by definition we set

$$\beta^{(1)}(\bar{g}) = (4-d) \sum_{n=1}^{\infty} \beta_n \bar{g}^{n+2}. \tag{A.5}$$

In Table A.1 we report all values in $d = 1, 2, 3$ such that $\beta_n \gtrsim 10^{-3}$. As a check of accuracy of the perturbative expansion we may employ the identity

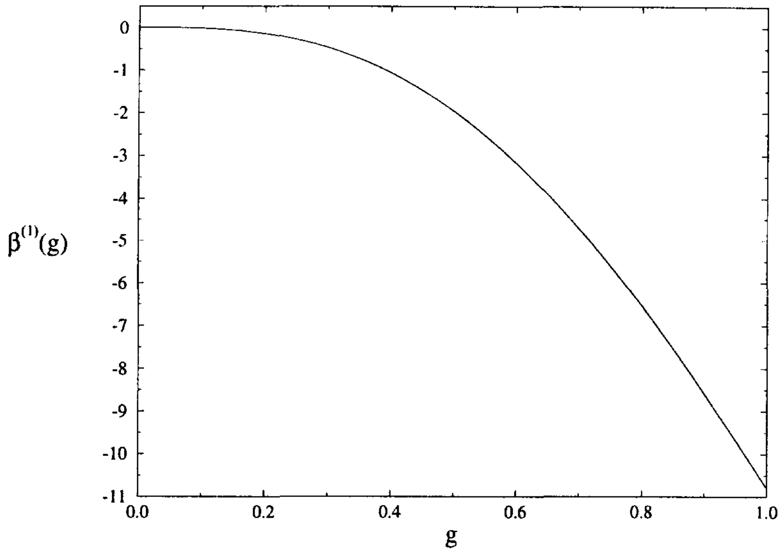

Fig. A.2. $\beta^{(1)}(\bar{g})$ vs. $\bar{g}$ for the two-dimensional case.

Table A.1
We report the values of β_n, defined in Eq. (A.5), in $d = 1, 2, 3$ such that $\beta_n \gtrsim 10^{-3}$. Notice that in two dimensions $\beta_1 = \frac{44}{3} + \frac{128}{27}\pi^2 - \frac{64}{9}\psi'(1/3) = -10.33501055$

	$d = 1$	$d = 2$	$d = 3$
β_1	$-\frac{388}{27}$	-10.33501055	$-\frac{164}{27}$
β_2	$\frac{1187}{108}$	5.00027593	1.34894276
β_3	$-\frac{335}{162}$	-0.08884297	0.15564589
β_4	$-\frac{10001}{46656}$	-0.00407962	0.05123618
β_5	$-\frac{605}{11664}$	0.00506747	0.02342417
β_6	$-\frac{20045}{1119744}$	0.00491122	0.01264064
β_7	$-\frac{38671}{5038848}$	0.00377364	0.00757889
β_8	$-\frac{1231807}{322486272}$	0.00281096	0.00489401
β_9	$-\frac{21367}{10077696}$	0.00211235	0.00334024
β_{10}	$-\frac{89062753}{69657034752}$	0.00161697	0.00237987
β_{11}	$-\frac{28651973}{34828517376}$	0.00126267	0.00175481
β_{12}		0.00100476	0.00133070
β_{13}		0.00081329	0.00103290
β_{14}			0.00081770

$$6 + \sum_{n=1}^{\infty} \beta_n = -f_1, \tag{A.6}$$

where f_1 was defined and evaluated in Section 4 (cf. Eqs. (65) and (72)). Notice that for $d = 3$ the coefficients β_n for $n \leqslant 5$ can also be extracted from the literature [21], and Eq. (A.6) is already satisfied within 1% precision by the six-loop β-function. We mention that, in the case $d = 1$, $\beta^{(1)}(\bar{g})$ may actually be computed analytically, and the result is

$$\beta^{(1)}(\bar{g}) = \frac{3\bar{g}^2(1-\bar{g})^{3/2}}{4(3+\bar{g})^4}\left(648 - 3732\bar{g} + 5512\bar{g}^2 - 2183\bar{g}^3 - 330\bar{g}^4 - 27\bar{g}^5\right) - \frac{6\bar{g}^2}{(3+\bar{g})^4}\left(81 - 6\bar{g} + 1750\bar{g}^2 - 1598\bar{g}^3 + 509\bar{g}^4\right). \tag{A.7}$$

The definitions (A.4) and (A.5) are obviously inappropriate in the limit $d \to 4$, in which case one may verify that

$$\frac{\beta^{(1)}(\widehat{g}_r)}{\widehat{g}_r^2} \longrightarrow 8\beta_0\widehat{g}_r^2 - 9\beta_0^2\widehat{g}_r^3 + O(\widehat{g}_r^4), \tag{A.8}$$

where $\beta_0 \to 1/(16\pi^2)$. Eq. (A.8) in turn can be compared to the known perturbative evaluation around $d = 4$:

$$\beta(\widehat{g}_r) = (d-4)\widehat{g}_r + \frac{N+8}{N}\beta_0\widehat{g}_r^2 - \frac{3(3N+14)}{N^2}\beta_0^2\widehat{g}_r^3 + O(\widehat{g}_r^4), \tag{A.9}$$

finding complete agreement to $O(1/N)$.

It is conceivable to reinterpret the $d \to 4$ limit of Eq. (A.1) in a non-perturbative sense by a principal-part prescription for the singularity occurring at the Landau pole $\bar{u}$ identified by the condition

$$\Pi(\bar{u}) = -\frac{1}{\widehat{g}_r}. \tag{A.10}$$

Work in this direction is in progress.

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