



ELSEVIER

Nuclear Physics B 554 [FS] (1999) 552–606



www.elsevier.nl/locate/npe

Mean-field expansion for spin models with medium-range interactions

Andrea Pelissetto^{a,1}, Paolo Rossi^{b,2}, Ettore Vicari^{b,3}

^a *Dipartimento di Fisica and INFN – Sezione di Roma I, Università degli Studi di Roma “La Sapienza”, I-00185 Rome, Italy*

^b *Dipartimento di Fisica and INFN – Sezione di Pisa, Università degli Studi di Pisa, I-56100 Pisa, Italy*

Received 30 March 1999; accepted 28 May 1999

Abstract

We study the critical crossover between the Gaussian and the Wilson–Fisher fixed point for general $O(N)$ -invariant spin models with medium-range interactions. We perform a systematic expansion around the mean-field solution, obtaining the universal crossover curves and their leading corrections. In particular we show that, in three dimensions, the leading correction scales as R^{-3} , R being the range of the interactions. We compare our results with the existing numerical ones obtained by Monte Carlo simulations and present a critical discussion of other approaches. © 1999 Elsevier Science B.V. All rights reserved.

PACS: 05.70.Fh; 64.60.Fr; 75.40.Cx; 75.10.Hk

Keywords: Field theory; Critical phenomena; Crossover phenomena; $O(N)$ vector models; Ising model; Mean-field expansion

1. Introduction

Every physical situation of experimental relevance has at least two scales: one scale is intrinsic to the system, while the second one is related to experimental conditions. In statistical mechanics the correlation length ξ is related to experimental conditions (it depends on the temperature), while the interaction length (Ginzburg parameter) is intrinsic. The opposite is true in quantum field theory: here the correlation length (inverse

¹ E-mail: Andrea.Pelissetto@roma1.infn.it

² E-mail: rossi@mailbox.difi.unipi.it

³ E-mail: vicari@mailbox.difi.unipi.it

mass gap) is intrinsic, while the interaction scale (inverse momentum) depends on the experiment. Physical predictions are functions of ratios of these two scales and describe the crossover from the correlation-dominated (ξ/G or p/m large) to the interaction-dominated (ξ/G or p/m small) regime. In a properly defined limit they are universal and define the unique flow between two different fixed points. This universal limit is obtained when two scales become very large with respect to any other (microscopic) scale. Their ratio becomes the (universal) control parameter of the system, whose transition from 0 to ∞ describes the critical crossover.

In this paper we will consider the crossover between the Gaussian fixed point where mean-field predictions hold (interaction-dominated regime) to the standard Wilson–Fisher fixed point (correlation-dominated regime). In recent years a lot of work has been devoted to understanding this crossover, either experimentally [1–5] or theoretically [6–20]. The traditional approach to the crossover between the Gaussian and the Wilson–Fisher fixed point starts from the standard Landau–Ginzburg Hamiltonian. On a d -dimensional lattice, it can be written as

$$H = \sum_{x_1, x_2} \frac{1}{2} J(x_1 - x_2) (\phi_{x_1} - \phi_{x_2})^2 + \sum_x \left[\frac{1}{2} r \phi_x^2 + \frac{u}{4!} \phi_x^4 - h_x \cdot \phi_x \right], \tag{1.1}$$

where ϕ_x are N -dimensional vectors, and $J(x)$ is the standard nearest-neighbour coupling. For this model the interaction scale is controlled by the coupling u and the relevant parameters are the (thermal) Ginzburg number G [21] and its magnetic counterpart G_h [17,20] defined by

$$G = u^{2/(4-d)}, \quad G_h = u^{(d+2)/[2(4-d)]}. \tag{1.2}$$

Under a renormalization-group (RG) transformation G scales like the (reduced) temperature, while G_h scales as the magnetic field. For $t \equiv r - r_c \ll G$ and $h \ll G_h$ one observes the standard critical behaviour, while in the opposite case the behaviour is classical. The critical crossover limit corresponds to considering $t, h, u \rightarrow 0$ keeping $\tilde{t} = t/G$ and $\tilde{h} = h/G_h$ fixed. This limit is universal, i.e. independent of the detailed structure of the model: any Hamiltonian of the form (1.1) shows the same universal behaviour as long as the interaction is short-ranged, i.e. for any $J(x)$ such that $\sum_x x^2 J(x) < +\infty$. The crossover functions can be related to the RG functions of the standard continuum ϕ^4 theory if one expresses them in terms of the zero-momentum four-point renormalized coupling g [6–8,11]. For the observables that are traditionally studied in statistical mechanics, for instance the susceptibility or the correlation length, the crossover functions can be computed to high precision in the fixed-dimension expansion in $d = 3$ [6–8].

Let us now consider the medium-range case. Following Refs. [16,17] we assume that $J(x)$ has the following form:

$$J(x) = \begin{cases} J & \text{for } x \in D, \\ 0 & \text{for } x \notin D, \end{cases} \tag{1.3}$$

where D is a lattice domain characterized by some scale R . Explicitly we define R and the corresponding domain volume V_R by

$$V_R \equiv \sum_{x \in D} 1, \quad R^2 \equiv \frac{1}{2d V_R} \sum_{x \in D} x^2. \quad (1.4)$$

The shape of D is irrelevant for our purposes as long as $V_R \sim R^d$ for $R \rightarrow \infty$. The constant J defines the normalization of the fields. Here we assume $J = 1/V_R$, since this choice simplifies the discussion of the limit $R \rightarrow \infty$. To understand the connection between the theory with medium-range interactions and the short-range model let us consider the continuum Hamiltonian that is obtained replacing in Eq. (1.1) the lattice sums with the corresponding integrals. Then let us perform a scale transformation [18]. We define new (“blocked”) coordinates $y = x/R$ and rescale the fields according to

$$\widehat{\phi}_y = R^{d/2} \phi_{Ry}, \quad \widehat{h}_y = R^{d/2} h_{Ry}. \quad (1.5)$$

The rescaled Hamiltonian becomes

$$\begin{aligned} \widehat{H} = & \int d^d y_1 d^d y_2 \frac{1}{2} \widehat{J}(y_1 - y_2) \left(\widehat{\phi}_{y_1} - \widehat{\phi}_{y_2} \right)^2 \\ & + \int d^d y \left[\frac{1}{2} r \widehat{\phi}_y^2 + \frac{1}{4!} \frac{u}{R^d} \widehat{\phi}_y^4 - \widehat{h}_y \cdot \widehat{\phi}_y \right], \end{aligned} \quad (1.6)$$

where now the coupling $\widehat{J}(x)$ is of short-range type in the limit $R \rightarrow \infty$. Being short-ranged, we can apply the previous arguments and define Ginzburg parameters:

$$G = (uR^{-d})^{2/(4-d)} = u^{2/(4-d)} R^{-2d/(4-d)}, \quad (1.7)$$

$$G_h = R^{-d/2} (uR^{-d})^{(d+2)/[2(4-d)]} = u^{(d+2)/[2(4-d)]} R^{-3d/(4-d)}. \quad (1.8)$$

Therefore, in the medium-range model, the critical crossover limit can be defined as $R \rightarrow \infty$, $t, h \rightarrow 0$, with $\tilde{t} \equiv t/G$, $\tilde{h} \equiv t/G_h$ fixed. The variables that are kept fixed are the same, but a different mechanism is responsible for the change of the Ginzburg parameters: in short-range models we vary u keeping the range R fixed and finite, while here we keep the interaction strength u fixed and vary the range R .

In this paper we will study a generalization of the model (1.1) in the presence of medium-range interactions. We will show explicitly in perturbation theory the equivalence between the crossover functions computed starting from the continuum ϕ^4 model and the results obtained in the medium-range model. As a byproduct we will also compute the non-universal constants relating the two cases so that we can compare the field-theory predictions with the numerical results of Refs. [17–19,22] without any free parameter. The calculation will also give us analytic predictions for the large- R behaviour of the critical point.

In numerical simulations (and also in experiments) the range R is always finite. It is therefore important to understand the behaviour of the corrections one should expect. We will show that for $d > 2$ the deviations from the universal behaviour are of order R^{-d} , provided one chooses the scale R as in Eq. (1.4). These corrections are non-universal and depend on all the details of the microscopic interaction: as a consequence they cannot be computed in the continuum field-theory framework. In two dimensions

the behaviour of the corrections is not computable in the perturbative expansion around the mean-field solution. Indeed the perturbative limit we consider – first we expand in $1/R$ at t fixed and positive and then we take the limit $t \rightarrow 0$ – does not commute with the crossover limit. We conjecture that the corrections are of order $\log R^2/R^2$ as already indicated by the numerical work of Ref. [18]. This behaviour has been explicitly checked in the large- N limit.

The paper is organized as follows. In Section 2 we review the computation of the crossover functions in the field-theory framework, extending the calculations of Refs. [6–8]. In Section 3 we introduce our model with medium-range interactions and in Section 4 we show that the expansion around the mean-field solution is equivalent to the perturbative expansion of the ϕ^4 field theory apart from non-universal computable renormalization constants. In Section 5 we discuss the corrections to the universal critical behaviour. In Section 6 we compare our theoretical results with the Monte Carlo data of Refs. [17–19,22] and we present a critical discussion of the crossover model of Refs. [12,23]. A detailed comparison with Monte Carlo results for the self-avoiding walk will appear in a separate paper [24]. Appendix A contains some details about the computation of integrals with medium-range propagators, while Appendix B discusses the medium-range model in the large- N limit verifying explicitly various results presented in the text.

Preliminary results were presented in Ref. [25].

2. Critical crossover functions from field theory

In this Section we report the computation of the various crossover functions in the continuum theory. As we described in the Introduction, the idea is the following: consider the continuum ϕ^4 theory

$$H = \int d^d x \left[\frac{1}{2} (\partial_\mu \phi)^2 + \frac{r}{2} \phi^2 + \frac{u}{4!} \phi^4 \right], \quad (2.1)$$

where ϕ is an N -dimensional vector, and consider the limit $u \rightarrow 0$, $t \equiv r - r_c \rightarrow 0$, with $\tilde{t} \equiv t/G = tu^{-2/(4-d)}$ fixed. In this limit we have

$$\tilde{\chi} \equiv \chi G \rightarrow F_\chi(\tilde{t}), \quad (2.2)$$

$$\tilde{\xi}^2 \equiv \xi^2 G \rightarrow F_\xi(\tilde{t}), \quad (2.3)$$

where

$$\chi = \sum_x \langle \phi_0 \cdot \phi_x \rangle, \quad (2.4)$$

$$\xi^2 = \frac{1}{2d\chi} \sum_x x^2 \langle \phi_0 \cdot \phi_x \rangle \quad (2.5)$$

are respectively the susceptibility and the (second-moment) correlation length. The functions $F_\chi(\tilde{t})$ and $F_\xi(\tilde{t})$ can be accurately computed by means of perturbative field-

theory calculations. There are essentially two methods: (a) the fixed-dimension expansion [26,6,7], which is at present the most precise one since seven-loop series are available [27,28]; (b) the so-called minimal renormalization without ϵ -expansion [29,11,30] which uses five-loop ϵ -expansion results [31,32]. In these two schemes the crossover functions are expressed in terms of various RG functions whose perturbative series can be resummed with high accuracy using standard methods [33,34]. Here we will consider the first approach although essentially equivalent results can be obtained using the second method. Explicitly we have for $F_\chi(\tilde{t})$ and $F_\xi(\tilde{t})$

$$F_\chi(\tilde{t}) = \chi^* \exp \left[- \int_{y_0}^g dx \frac{\gamma(x)}{\nu(x)W(x)} \right], \quad (2.6)$$

$$F_\xi(\tilde{t}) = (\xi^*)^2 \exp \left[-2 \int_{y_0}^g dx \frac{1}{W(x)} \right], \quad (2.7)$$

where \tilde{t} is related to the zero-momentum four-point renormalized coupling g by

$$\tilde{t} = -t_0 \int_g^{g^*} dx \frac{\gamma(x)}{\nu(x)W(x)} \exp \left[\int_{y_0}^x dz \frac{1}{\nu(z)W(z)} \right], \quad (2.8)$$

$\gamma(x)$, $\nu(x)$, and $W(x)$ are the standard RG functions (see Refs. [27,28] for the corresponding perturbative expressions), g^* is the critical value⁴ of g defined by $W(g^*) = 0$, and χ^* , ξ^* , t_0 and y_0 are normalization constants.

The expressions (2.6), (2.7) and (2.8) are valid for any dimension $d < 4$. The first two equations are always well defined, while Eq. (2.8) has been obtained with the additional hypothesis that the integral over x is convergent when the integration is extended up to g^* . This hypothesis is verified when the system becomes critical at a finite value of β and shows a standard critical behaviour. Therefore Eq. (2.8) is always well defined for $d > 2$, and, in two dimensions, for $N \leq 2$. For $N > 2$, one can still define \tilde{t} by integrating up to an arbitrary point g_0 . For these values of N , \tilde{t} varies between $-\infty$ and $+\infty$.

We will normalize the coupling g as in Refs. [27,33,28] so that in the perturbative limit $g \rightarrow 0$, $\tilde{t} \rightarrow \infty$, we have

$$g \approx \frac{1}{2(4\pi)^{d/2}} \frac{N+8}{3} \Gamma \left(2 - \frac{d}{2} \right) \tilde{t}^{(d-4)/2} \equiv \lambda_d \tilde{t}^{(d-4)/2}. \quad (2.9)$$

This implies that for $y_0 \rightarrow 0$ we have $t_0 \approx (y_0/\lambda_d)^{2/(d-4)}$, and $(\xi^*)^2 t_0 \approx \chi^* t_0 \approx 1$.

For future purposes we will be interested in computing the expansion of $F_\chi(\tilde{t})$ for $\tilde{t} \rightarrow \infty$. In two dimensions we have

⁴ A review of present estimates of g^* can be found in Refs. [34–37].

$$F_\chi(\tilde{t}) = \frac{1}{\tilde{t}} \left[1 + \frac{N+2}{24\pi\tilde{t}} \log \left(\frac{24\pi\tilde{t}}{N+8} \right) + \frac{N+8}{24\pi\tilde{t}} + \frac{D_2(N)}{\tilde{t}} + O(\tilde{t}^{-2} \log^2 \tilde{t}) \right], \tag{2.10}$$

where

$$D_2(N) = \frac{N+8}{24\pi} \left(-\frac{1}{g^*} + \frac{N+2}{N+8} \log g^* \right) - \frac{N+8}{24\pi} \int_0^{g^*} \frac{dx}{x} \left\{ \frac{\gamma(x)}{\nu(x)W(x)} \exp \left[\int_0^x dz \left(\frac{1}{\nu(z)W(z)} + \frac{1}{z} \right) \right] + \frac{1}{x} + \frac{N+2}{N+8} \right\}. \tag{2.11}$$

For $N \geq 2$, g^* should be replaced in $D_2(N)$ with the arbitrary point g_0 that has been used to define \tilde{t} .

Analogously in three dimensions we have

$$F_\chi(\tilde{t}) = \frac{1}{\tilde{t}} \left[1 + \frac{N+2}{24\pi\tilde{t}^{1/2}} - \frac{N+2}{288\pi^2\tilde{t}} \log \left(\frac{48\pi\sqrt{\tilde{t}}}{N+8} \right) + \frac{27N^2 + 52N - 472}{20736\pi^2\tilde{t}} + \frac{D_3(N)}{\tilde{t}} + O(\tilde{t}^{-3/2} \log \tilde{t}) \right], \tag{2.12}$$

where

$$D_3(N) = - \left(\frac{N+8}{48\pi} \right)^2 \left[\frac{1}{g^{*2}} - \frac{12}{N+8} \frac{1}{g^*} + \frac{8(N+2)}{(N+8)^2} \log g^* \right] - \left(\frac{N+8}{48\pi} \right)^2 \int_0^{g^*} \frac{dx}{x^2} \left\{ \frac{\gamma(x)}{\nu(x)W(x)} \exp \left[\int_0^x dz \left(\frac{1}{\nu(z)W(z)} + \frac{2}{z} \right) \right] + \frac{2}{x} - \frac{12}{N+8} - \frac{8(N+2)}{(N+8)^2} x \right\}. \tag{2.13}$$

In the large- N limit we have simply

$$D_3(N) = -\frac{N^2}{(48\pi)^2} + O(N). \tag{2.14}$$

The expansions (2.10) and (2.12) are nothing but the standard perturbative expansions. For generic values of d we have

$$F_\chi(\tilde{t}) = \frac{1}{\tilde{t}} \sum_n a_n \tilde{t}^{-\Delta_{mf}}, \tag{2.15}$$

where $\Delta_{mf} = (4-d)/2$. For $d = 4 - 2/n$ additional logarithms appear in the expansion. The reason of this phenomenon is well known [38-40]: the critical crossover limit

corresponds to the massless limit of the standard ϕ^4 model which is known to have logarithmic singularities for these values of the dimension.

The functions $F_\chi(\tilde{t})$ and $F_\xi(\tilde{t})$ can be computed using the perturbative results of Refs. [27,28], a Borel–Leroy transform that takes into account the large-order behaviour of the perturbative series [41,42], and a standard resummation technique [33]. Explicit expressions can be found for $N = 1, 2, 3$ and $d = 3$ in Refs. [6,7]. Here we compute $F_\chi(\tilde{t})$ and $F_\xi(\tilde{t})$ for the Ising model in two dimensions using the four-loop results of Ref. [27]. In order to improve the precision of the results, by means of appropriate subtractions, we have forced the resummed expressions to have the correct asymptotic behaviour for $\tilde{t} \rightarrow 0$, i.e. $F_\chi(\tilde{t}) \approx \tilde{t}^{-\gamma}$, $F_\xi(\tilde{t}) \approx \tilde{t}^{-2\nu}$, with $\gamma = 7/4$, $\nu = 1$. The resummed expressions are well fitted by

$$F_\chi(\tilde{t}) = \frac{1}{\tilde{t}} \left[1 + \frac{2 \log \tilde{t}}{3\pi\tilde{t}} + \frac{0.9705}{\tilde{t}} + \frac{0.3513}{\tilde{t}^2} + \frac{0.01712}{\tilde{t}^3} + \frac{0.001822}{\tilde{t}^4} \right]^{3/16}, \quad (2.16)$$

$$F_\xi(\tilde{t}) = \frac{1}{\tilde{t}} \left[1 + \frac{\log \tilde{t}}{2\pi\tilde{t}} + \frac{0.7377}{\tilde{t}} + \frac{0.1635}{\tilde{t}^2} + \frac{0.00390}{\tilde{t}^3} + \frac{0.000275}{\tilde{t}^4} \right]^{1/4}, \quad (2.17)$$

The resummation errors on $F_\chi(\tilde{t})$ ($F_\xi(\tilde{t})$) are less than 0.1% for $\tilde{t} \gtrsim 0.2$ ($\tilde{t} \gtrsim 0.1$), at most 3% (4%) for smaller values of \tilde{t} . The fitted expression introduces an additional uncertainty which is less than 0.3%.

The constants $D_d(N)$ are non-perturbative constants since they require the knowledge of the RG functions up to the critical point. By means of a Borel–Leroy transform, working as before, we obtain in three dimensions

$$D_3(0) = 0.002473(6), \quad (2.18)$$

$$D_3(1) = 0.002391(6), \quad (2.19)$$

$$D_3(2) = 0.002204(5), \quad (2.20)$$

$$D_3(3) = 0.001920(3). \quad (2.21)$$

In two dimensions we have

$$D_2(1) = -0.0524(2). \quad (2.22)$$

The results we have described above apply to the high-temperature phase of the model. For $N = 1$ the critical crossover can also be defined in the low-temperature phase [8] and the crossover functions can be computed in terms of (resummed) perturbative quantities. Using the perturbative results⁵ of Ref. [8], we can compute $F_\chi(\tilde{t})$ in the low-temperature phase of the three-dimensional Ising model. For $|\tilde{t}| > 10^{-3}$ the numerical results are well fitted by

⁵ Some perturbative series contained small errors, see footnote 27 of Ref. [20].

$$F_\chi(\tilde{t}) = \frac{1}{2|\tilde{t}|} \left(1 + \frac{0.00019}{|\tilde{t}|} \right)^{0.2372} \left[1 - \frac{0.0140674}{\tilde{t}^{1/2}} + \frac{0.0021871}{\tilde{t}} - \frac{0.000150048}{\tilde{t}^{3/2}} + \frac{4.82764 \times 10^{-6}}{\tilde{t}^2} - \frac{5.78079 \times 10^{-8}}{\tilde{t}^{5/2}} \right]. \quad (2.23)$$

The error due to the approximate form given above is at most 0.3%, while the resummation error is less than 0.1%. For $|\tilde{t}| \lesssim 10^{-3}$ we can use

$$F_\chi(\tilde{t}) = |\tilde{t}|^{-1.2372} \left(0.06125 + 1.3455\tilde{t}^{1/2} + 1.51525\tilde{t} - 124.395\tilde{t}^{3/2} \right), \quad (2.24)$$

with errors of order 0.2%. The resummation error varies approximately from 0.1% to 3%.

In the low-temperature phase we can also study the magnetization. Using the results of Ref. [8] we have

$$F_M(\tilde{t}) \equiv \langle \sigma \rangle u^{-1/2} = \sqrt{3|\tilde{t}|} \left(1 + \frac{0.3241}{\tilde{t}^{1/2}} + \frac{0.02751}{\tilde{t}} + \frac{0.001247}{\tilde{t}^{3/2}} + \frac{0.0000128}{\tilde{t}^2} \right)^{0.0868}. \quad (2.25)$$

For $|\tilde{t}| > 10^{-4}$, the error on this function is at most of order 0.5%.

We have chosen the expressions (2.23) and (2.25) so that they reproduce the exact large- \tilde{t} behaviour of the crossover functions:

$$F_\chi(\tilde{t}) = \frac{1}{2|\tilde{t}|} \left(1 - \frac{1}{16\pi\sqrt{2}} \frac{1}{\tilde{t}^{1/2}} + O(\tilde{t}^{-1} \log \tilde{t}) \right), \quad (2.26)$$

$$F_M(\tilde{t}) = \sqrt{3|\tilde{t}|} \left(1 + \frac{1}{8\pi\sqrt{2}} \frac{1}{\tilde{t}^{1/2}} + O(\tilde{t}^{-1} \log \tilde{t}) \right). \quad (2.27)$$

Notice that the leading correction to $F_\chi(\tilde{t})$ is negative so that $F_\chi(\tilde{t})$ is non-monotonic. Such a behaviour has also been observed numerically [18] and predicted analytically [20] in two dimensions.

3. The models and the critical crossover limit

In this section we will study the critical crossover limit in the presence of medium-range interactions by performing a systematic expansion around the mean-field solution. In particular we will show how to compute the critical crossover functions as a perturbative expansion in powers of $\tilde{t}^{(d-4)/2}$. A general discussion of the mean-field limit can be found e.g. in Refs. [43,34].

Generalizing the discussion of the Introduction, we assume we have a family of couplings $J_\rho(x)$ that are defined on a d -dimensional cubic lattice and that are parametrized by ρ . For each coupling $J_\rho(x)$, we define, in analogy with Eq. (1.4), the following quantities:

$$V_\rho \equiv \sum_x J_\rho(x), \quad (3.1)$$

$$R^2 \equiv \frac{1}{2dV_R} \sum_x x^2 J_\rho(x), \quad (3.2)$$

and

$$\bar{\Pi}_\rho(q) \equiv 1 - \frac{1}{V_\rho} J_\rho(q), \quad (3.3)$$

where $J_\rho(q)$ is the Fourier transform of $J_\rho(x)$. In the following we assume that there is a one-to-one correspondence between R and ρ , so that we will interchangeably think of the various observables as functions either of ρ or of R .

Notice that because of the definition (3.2) we have for $q \rightarrow 0$

$$\bar{\Pi}_R(q) \approx R^2 q^2. \quad (3.4)$$

We assume that

- (i) $J_R(x)$ is uniformly bounded in x and R , i.e. $|J_R(x)| < C$, independently of x and R ;
- (ii) V_R and R^2 are finite for finite values of ρ . For $\rho \rightarrow \infty$, $R^2 \rightarrow +\infty$ and $V_R \sim R^d$;
- (iii) the system is ferromagnetic, i.e. $\bar{\Pi}_R(q) > 0$ for all $q \neq 0$ in the first Brillouin zone;
- (iv) $\bar{\Pi}_R(q/R)$ has a finite limit for $R \rightarrow \infty$ at fixed q , i.e. $\bar{\Pi}_R(q/R) \rightarrow \Pi(q)$. This assumption is equivalent to requiring that $J_R(Rx)$ has a finite limit $J_\infty(x)$ for $R \rightarrow \infty$ at fixed x . Furthermore we assume that $J_\infty(x)$ is infinitely differentiable in $x = 0$ so that the integral $\int d^d q q^{2n} (1 - \Pi(q))$ exists for any n . Notice the following asymptotic property of $\Pi(q)$: because of Eq. (3.4) we have $\Pi(q) \approx q^2$ for $q \rightarrow 0$.

We will investigate the properties of a class of models that generalize the theory defined in Eq. (1.1). We will consider a Hamiltonian of the form

$$H = -\frac{N}{2} \sum_{x_1, x_2} J_R(x_1 - x_2) \varphi_{x_1} \cdot \varphi_{x_2} + N \sum_x h_x \cdot \varphi_x, \quad (3.5)$$

where φ_x are N -dimensional vectors, and a single-site measure $d\mu(\varphi_x)$ which we will assume of the form

$$d\mu(\varphi_x) = e^{-V(\varphi_x)} d^N \varphi_x, \quad (3.6)$$

where $V(\varphi_x)$ is an even function (often a polynomial) of φ_x which is bounded from below and that satisfies $V(x) \sim |x|^p$, $p > 2$, for $|x| \rightarrow \infty$. The partition function is defined by

$$Z[h] \equiv \int \prod_x d\mu(\varphi_x) e^{-\beta H}. \quad (3.7)$$

The Hamiltonian we defined in the Introduction is a particular case of the general model we discuss here. Indeed consider a family of domains D_R and define

$$J_R(x) \equiv \begin{cases} 1 & \text{if } x \in D_R, \\ 0 & \text{otherwise,} \end{cases} \quad (3.8)$$

and

$$V(\varphi_x) = \varphi_x^2 + \lambda (\varphi_x^2 - 1)^2, \quad (3.9)$$

with $\lambda > 0$. It is easy to see that $J_R(x)$ satisfies all the assumptions, as long as D_R satisfies some simple requirements, see Appendix A.1. To derive the relation between the two models (for simplicity assume $h_x = 0$) let us rewrite the partition function as

$$Z[0] = \int \prod_x d\varphi_x \exp \left\{ \frac{\beta N}{2} \sum_{xy} J_R(x-y) \varphi_x \cdot \varphi_y - \sum_x [\varphi_x^2 + \lambda(\varphi_x^2 - 1)^2] \right\}. \quad (3.10)$$

Then we rescale the field

$$\varphi_x = \left(\frac{\beta N V_R}{2} \right)^{-1/2} \phi_x \quad (3.11)$$

and define

$$r \equiv \frac{4(1 - 2\lambda)}{\beta N V_R} - 2, \quad u \equiv \frac{96\lambda}{(\beta N V_R)^2}. \quad (3.12)$$

In terms of ϕ we obtain again Eq. (1.1) with the coupling defined by Eq. (1.3). As we shall see in the following, the large- R limit is well defined if β goes to zero as $1/V_R$. Thus r and u remain finite as $R \rightarrow \infty$ so that the large- R limit of the Hamiltonian (1.1) and of the model defined in this section are identical.

For $\lambda \rightarrow \infty$, the model with partition function (3.10) reduces to the so-called N -vector model which can be seen as a particular case in which

$$d\mu(\varphi_x) = \delta(\varphi_x^2 - 1) d^N \varphi_x. \quad (3.13)$$

The coupling $J_R(x)$ defined in Eq. (3.5) is ambiguous for $x = 0$. Indeed one can define a new coupling and a new single-site measure

$$\widehat{J}_R(x) \equiv J_R(x) - K\delta_{x,0}, \quad (3.14)$$

$$d\widehat{\mu}(\varphi_x) \equiv d\mu(\varphi_x) e^{NBK\varphi_x^2/2}, \quad (3.15)$$

without changing the results. This step, which at first sight may appear trivial, can be interpreted as a mass renormalization needed, as we shall see, to have a scaling theory. It is also needed from a mathematical point of view: in order to make the following derivation mathematically rigorous, we will require K to be such that the Fourier transform $\widehat{J}_R(q)$ satisfies $\widehat{J}_R(q) > 0$ for all values of q .

Let us now discuss the critical limit of the model. If we consider the critical limit with R fixed, Eq. (3.7) defines a generalized $O(N)$ -symmetric model with short-range

interactions. If $d > 2$, for each value of R there is a critical point⁶ $\beta_{c,R}$; for $\beta \rightarrow \beta_{c,R}$ the susceptibility and the correlation length have the standard behaviour

$$\chi_R(\beta) \approx A_\chi(R)t^{-\gamma}(1 + B_\chi(R)t^\Delta), \quad (3.16)$$

$$\xi_R^2(\beta) \approx A_\xi(R)t^{-2\nu}(1 + B_\xi(R)t^\Delta), \quad (3.17)$$

where $t \equiv (\beta_{c,R} - \beta)/\beta_{c,R}$ and we have neglected additional subleading corrections. The exponents γ , ν and Δ do not depend on R . On the other hand, the amplitudes are non-universal.⁷ For $R \rightarrow \infty$, they behave as [17,18]

$$\begin{aligned} A_\chi(R) &\approx A_\chi^\infty R^{2d(1-\gamma)/(4-d)}, & A_\xi(R) &\approx A_\xi^\infty R^{4(2-d\nu)/(4-d)}, \\ B_\chi(R) &\approx B_\chi^\infty R^{2d\Delta/(4-d)}, & B_\xi(R) &\approx B_\xi^\infty R^{2d\Delta/(4-d)}. \end{aligned} \quad (3.18)$$

The critical point $\beta_{c,R}$ depends explicitly on R . For $R \rightarrow \infty$ we have $\beta_{c,R} \sim 1/R^d$ with corrections that will be computed in the next section.

Let us now define the critical crossover limit. In this case we consider the limit [17,18] $R \rightarrow \infty$, $t \rightarrow 0$, with $R^{2d/(4-d)}t \equiv \tilde{t}$ fixed. We will show perturbatively in the next section that

$$\tilde{\chi}_R \equiv R^{-2d/(4-d)}\chi_R(\beta) \rightarrow f_\chi(\tilde{t}), \quad (3.19)$$

$$\tilde{\xi}_R^2 \equiv R^{-8/(4-d)}\xi_R^2(\beta) \rightarrow f_\xi(\tilde{t}), \quad (3.20)$$

where the functions $f_\chi(\tilde{t})$ and $f_\xi(\tilde{t})$ are universal apart from an overall rescaling of \tilde{t} and a constant factor, in agreement with the argument presented in the Introduction.

There exists an equivalent way to define the crossover limit which is due to Thouless [15]. Let $\beta_{c,R}^{(\text{exp})}$ be the expansion of $\beta_{c,R}$ for $R \rightarrow \infty$ up to terms of order $R^{-2d/(4-d)}/V_R$, i.e. such that

$$\lim_{R \rightarrow \infty} R^{2d/(4-d)}\beta_{c,R}^{-1}(\beta_{c,R} - \beta_{c,R}^{(\text{exp})}) = b_c, \quad (3.21)$$

with $|b_c| < +\infty$. Then introduce

$$\hat{t} = R^{2d/(4-d)}\beta_{c,R}^{(\text{exp})-1}(\beta_{c,R}^{(\text{exp})} - \beta). \quad (3.22)$$

It is trivial to see that in the standard crossover limit $\tilde{t} = \hat{t} + b_c$. Therefore the crossover limit can be defined considering the limit $R \rightarrow \infty$, $\beta \rightarrow \beta_{c,R}^{(\text{exp})}$ with \hat{t} fixed. The crossover functions will be identical to the previous ones apart from a shift. Thouless' definition of critical crossover has an important advantage. It allows the definition of the critical crossover limit in models that do not have a critical point for finite values of R : indeed, even if $\beta_{c,R}$ does not exist, one can define a quantity $\beta_{c,R}^{(\text{exp})}$ and a variable

⁶ In two dimensions a critical point exists only for $N \leq 2$. Theories with $N \geq 3$ are asymptotically free and become critical only in the limit $\beta \rightarrow \infty$.

⁷ However, some ratios of correction-to-scaling amplitudes are universal. For instance $B_\xi(R)/B_\chi(R)$ is universal and therefore independent of R .

\hat{t} such that the limit $R \rightarrow \infty$ with \hat{t} fixed exists.⁸ Moreover, as we shall see, $\beta_{c,R}^{(\text{exp})}$ is known analytically: therefore, in the analysis of Monte Carlo data, no computation of $\beta_{c,R}$ is needed and a source of errors is eliminated.

4. Mean-field perturbative expansion

4.1. General framework

The starting point of our expansion is the identity [44,45] – we use matrix notation and drop the subscript R from $J_R(x)$ to simplify the notation –

$$\exp \left[\frac{N\beta}{2} \varphi \hat{J} \varphi \right] = (\det N\beta \hat{J})^{-N/2} \int \frac{d^N \phi}{(2\pi)^{N/2}} \exp \left\{ N \left[-\frac{1}{2\beta} \phi \hat{J}^{-1} \phi + \phi \varphi \right] \right\}, \tag{4.1}$$

where ϕ is another N -dimensional vector field. The second ingredient is the single-site integral that defines the function $A(\phi)$:

$$ze^{NA(\phi)} \equiv \int d\hat{\mu}(\varphi) e^{N\phi\varphi}, \tag{4.2}$$

where z is a normalization factor ensuring $A(0) = 0$. If we choose the single-site measure given in Eq. (3.6), we obtain the explicit formula

$$ze^{NA(\phi)} = 2\pi^{N/2} \int_0^\infty dx x^{N-1} e^{-V(x)+N\beta Kx^2/2} \left(\frac{2}{Nx|\phi|} \right)^{N/2-1} I_{N/2-1}(Nx|\phi|). \tag{4.3}$$

Notice that Eq. (4.3) has a regular expansion in powers of ϕ^2 , giving finally

$$A(\phi) = \sum_{k=1}^\infty \frac{a_{2k}}{(2k)!} (\phi^2)^k. \tag{4.4}$$

The first few coefficients are given by

$$a_2 = f_2, \tag{4.5}$$

$$a_4 = \frac{3N}{N+2} [Nf_4 - (N+2)f_2^2], \tag{4.6}$$

$$a_6 = \frac{15N^2}{(N+2)(N+4)} [N^2f_6 - 3N(N+4)f_4f_2 + 2(N+2)(N+4)f_2^3], \tag{4.7}$$

where

⁸ This is the case of two-dimensional models with $N \geq 3$, see the discussion in Section 4.2, and of one-dimensional models with $N \geq 1$. In the latter case we can take $\beta_{c,R}^{(\text{exp})} = 1/(\bar{a}_2 V_R)$ where \bar{a}_2 is defined in Section 4.1.

$$f_k = \frac{\int_0^\infty dx x^{N+k-1} e^{-V(x)+N\beta Kx^2/2}}{\int_0^\infty dx x^{N-1} e^{-V(x)+N\beta Kx^2/2}}. \tag{4.8}$$

The results for the N -vector model are obtained setting $f_k = 1$ in the previous formulae.

The coefficients a_{2k} depend on the various parameters that appear in the single-site measure, and on βK . In the following we will assume $V(x)$ to be independent of R , although considering R -dependent potentials does not introduce any significant change in the discussion as long as $\lim_{R \rightarrow \infty} V(x)$ exists and is finite. Under this assumption a_{2k} depends on R only through the combination βK . In the following we will always consider the limit $R \rightarrow \infty$ with $\beta K \rightarrow 0$, and therefore we will introduce

$$\bar{a}_{2k} = \lim_{\beta K \rightarrow 0} a_{2k}. \tag{4.9}$$

We will discuss the generic case⁹ in which $\bar{a}_4 < 0$. If one tunes the parameters appropriately one can obtain $\bar{a}_4 = 0$ and a different critical limit. We will not consider these cases here. For a discussion in the large- N limit, see Appendix B.1. In our expansion around the mean-field solution we will need the expansion of a_2 in powers of βK . Explicitly we have

$$a_2 = \bar{a}_2 + \beta K \left[\frac{N+2}{6N} \bar{a}_4 + \bar{a}_2^2 \right] + \beta^2 K^2 \left[\frac{(N+2)(N+4)}{120N^2} \bar{a}_6 + \frac{N+2}{2N} \bar{a}_4 \bar{a}_2 + \bar{a}_2^3 \right] + O(\beta^3 K^3). \tag{4.10}$$

Using Eqs. (4.1) and (4.2) we have

$$Z[h] \propto \int \prod_x d\phi_x \exp \left\{ N \left[-\frac{1}{2\beta} \phi \hat{\mathcal{J}}^{-1} \phi + \sum_x A(\phi_x + h_x) \right] \right\}. \tag{4.11}$$

The correlation functions of the φ -fields are obtained by taking derivatives with respect to h_x . Using the equations of motion, it is easy to relate them to correlations of the ϕ -fields. For instance, for $h_x = 0$, we have

$$\langle \varphi_x \cdot \varphi_y \rangle = -\frac{1}{\beta} \left(\hat{\mathcal{J}}^{-1} \right)_{xy} + \frac{1}{\beta^2} \sum_{wz} \left(\hat{\mathcal{J}}^{-1} \right)_{xw} \left(\hat{\mathcal{J}}^{-1} \right)_{yz} \langle \phi_w \cdot \phi_z \rangle, \tag{4.12}$$

where the expectation value $\langle \varphi_x \cdot \varphi_y \rangle$ is obtained using the Hamiltonian (3.5), while $\langle \phi_w \cdot \phi_z \rangle$ is computed in the model (4.11).

Now let us consider the (formal) perturbative expansion of the theory (4.11) with $h_x = 0$. It corresponds to a scalar model with propagator given by

⁹ Formally the discussion we will present requires only $\bar{a}_4 \neq 0$. However, for $\bar{a}_4 > 0$ the expansion we obtain would correspond to a ϕ^4 model with negative coupling. We therefore expect that potentials $V(x)$ such that $\bar{a}_4 > 0$ correspond to non-critical models. This can be checked explicitly in the large- N limit for a potential containing a ϕ^4 and a ϕ^6 coupling. In the large- N limit one can check explicitly that $\bar{a}_4 < 0$ is a necessary condition in order to obtain the critical crossover limit, see Appendix B.1.

$$\widehat{\Delta}(q) = \frac{1}{N} \frac{\beta \widehat{J}(q)}{1 - a_2 \beta \widehat{J}(q)} \tag{4.13}$$

and vertices ϕ^4, ϕ^6, \dots , that can be read from the expansion of the function $A(\phi)$, see Eq. (4.4). Let us now define

$$\bar{t} \equiv \frac{1}{a_2 V_R \beta} - 1 + \frac{K}{V_R}, \tag{4.14}$$

and consider the limit $\beta \rightarrow 0, R \rightarrow \infty$ with K and \bar{t} fixed. If $\bar{t} > 0$ this formal perturbative expansion defines an expansion in powers of R^{-d} . To prove this result let us first rewrite the propagator as

$$\widehat{\Delta}(q) \equiv \frac{1}{N} \left[-\frac{\beta K}{1 + a_2 \beta K} + \frac{1}{a_2} \frac{1}{1 + a_2 \beta K} \frac{1 - \overline{\Pi}(q)}{\overline{\Pi}(q) + \bar{t}} \right] = \widehat{\Delta}_1 + \widehat{\Delta}_2(q). \tag{4.15}$$

Now let us consider a generic l -loop graph. It has the generic form

$$\left(\prod_i a_{2k_i} \right) Q(N) \int \prod_{i < j} \frac{dq_{ij}}{(2\pi)^d} \prod_{i < j} \widehat{\Delta}(q_{ij}) \prod_i (2\pi)^d \delta \left(\sum_j q_{ij} \right), \tag{4.16}$$

where $Q(N)$ is an N -dependent constant. Let us now expand $\widehat{\Delta}(q)$ in the graph using Eq. (4.15). We obtain a sum of terms that can be represented as graphs in which each line is associated to a propagator $\widehat{\Delta}_2(q)$; these graphs are obtained from the original one contracting the lines corresponding to $\widehat{\Delta}_1$. A generic term has the form

$$(\widehat{\Delta}_1)^n \int \prod_{i < j} \frac{dq_{ij}}{(2\pi)^d} \prod_{i < j} \widehat{\Delta}_2(q_{ij}) \prod_i (2\pi)^d \delta \left(\sum_j q_{ij} \right), \tag{4.17}$$

corresponding to an m -loop subgraph. Now notice that, because of the assumptions we have made at the beginning, $\widehat{\Delta}_2(q/R)$ converges, for $R \rightarrow \infty$ at fixed q , to a function $\Delta_2(q)$ given by

$$\Delta_2(q) = \frac{1}{a_2 N} \frac{1 - \Pi(q)}{\Pi(q) + \bar{t}}, \tag{4.18}$$

that is integrable for all positive \bar{t} . Then change variables in Eq. (4.17), setting $q_{ij} = p_{ij} R$, and then take the limit $R \rightarrow \infty$ in the integrand, keeping \bar{t} fixed. We obtain

$$\frac{(\widehat{\Delta}_1)^n}{R^{dm}} \int \prod_{i < j} \frac{dq_{ij}}{(2\pi)^d} \prod_{i < j} \Delta_2(q_{ij}) \prod_i (2\pi)^d \delta \left(\sum_j q_{ij} \right), \tag{4.19}$$

where the integration is extended over \mathbb{R}^{dm} . The integral is R -independent, and, as long as \bar{t} is positive, it is finite. Since $\widehat{\Delta}_1 \sim R^{-d}$, the leading contribution of this graph behaves as $R^{-d(m+n)}$. Obviously $m + n \geq l$ (the equality corresponds to those cases in which all lines with $\widehat{\Delta}_1$ belong to one-loop tadpoles), thus proving that the leading contribution of an l -loop graph behaves as R^{-dl} .

It is immediate to see that the perturbative expansion is not uniform as $\bar{t} \rightarrow 0$. Indeed for $d \leq 4$ infrared divergences appear, so that the coefficients of the expansion diverge as $\bar{t} \rightarrow 0$. If one expands these coefficients in the limit $\bar{t} \rightarrow 0$ and chooses K appropriately – this corresponds to a mass renormalization and fixes the expression of $\beta_{c,R}^{(\text{exp})}$ – one obtains a new series which is a perturbative expansion in powers of $\bar{t}^{(d-4)/2}$ with additional logarithms if $d = 4 - 2/n$, $n \in \mathbb{N}$. The resulting expression can then be interpreted as the expansion of the critical crossover functions for $\tilde{t} \rightarrow \infty$, *provided* that the limit $R \rightarrow \infty$ at \bar{t} fixed followed by $\bar{t} \rightarrow 0$ is identical to the crossover limit. It is important to stress that this commutativity is not an obvious fact and indeed it is not true at the level of the corrections to the universal behaviour. In the following we will also show that graphs containing 6-leg (or higher-order) vertices can be neglected in the critical crossover limit. In other words, one can simply consider the ϕ^4 theory obtained from Eq. (4.11). This will explicitly give the relation between the crossover functions computed in the medium-range model and those obtained in the field-theory framework of Section 2.

We will first discuss the two-dimensional case, in which all these features can be understood easily, then we will present the general case.

4.2. Two-dimensional crossover limit

We wish now to discuss the critical crossover limit in two dimensions using the perturbative expansion of the model (4.11). Let us consider the zero-momentum correlation function

$$G^{(m,n)} \equiv \sum_{x_2, \dots, x_m, y_1, \dots, y_n} \langle (\phi^2)_{x_1} (\phi^2)_{x_2} \dots (\phi^2)_{x_m} \phi_{y_1} \dots \phi_{y_n} \rangle, \quad (4.20)$$

which contains m insertions of ϕ^2 and n fields ϕ , and its one-particle irreducible counterpart $\Gamma^{(m,n)}$.

Let us consider a generic l -loop graph contributing to $\Gamma^{(m,n)}$, and, to begin with, let us suppose that it does not contain tadpoles. We will be interested in the crossover limit $\bar{t} \rightarrow 0$, $R \rightarrow \infty$ with $R^2 \bar{t}$ fixed. After expanding $\widehat{\Delta}(q) = \widehat{\Delta}_1 + \widehat{\Delta}_2(q)$, we will obtain, apart from numerical factors, an expression which is a sum of terms of the form (4.17). We wish now to show that, if K increases with R at most logarithmically, we can disregard all terms containing $\widehat{\Delta}_1$. In other words, we can simply substitute $\widehat{\Delta}(q)$ with $\widehat{\Delta}_2(q)$ in the original graph. To prove this result, we begin by taking the limit $R \rightarrow \infty$ keeping \bar{t} fixed and rewriting the integral of Eq. (4.17) in the form of Eq. (4.19). We should now consider the behaviour of this integral for $\bar{t} \rightarrow 0$. Simple power counting indicates that the integral is infrared divergent in this limit. Let us suppose that the graph associated to Eq. (4.19) does not contain tadpoles. In this case, in order to compute the leading infrared contribution, we can substitute $\Delta_2(q)$ with its small- q expansion $1/(a_2 N(q^2 + \bar{t}))$ and then rescale $q^2 = \bar{t} p^2$, obtaining

$$\frac{(\widehat{\Delta}_1)^n (R^2 \bar{t})^{m-M_{\text{int}}}}{R^{2(2m-M_{\text{int}})}} \int \prod_{i < j} \frac{dp_{ij}}{(2\pi)^d} \prod_{i < j} \frac{1}{p_{ij}^2 + 1} \prod_i (2\pi)^d \delta \left(\sum_j p_{ij} \right), \quad (4.21)$$

where m is the number of loops and M_{int} the number of internal lines. Since, by hypothesis, the graph associated to Eq. (4.21) does not contain tadpoles, the integral is finite and thus the prefactor gives its behaviour in the crossover limit. If the integral in Eq. (4.19) is associated to a graph with tadpoles, it can be written as the product of an integral associated to a graph without tadpoles, and therefore behaving as in Eq. (4.21), and a power of the one-loop tadpole integral. Now, using Eq. (A.56), we have

$$N \int \frac{d^2q}{(2\pi)^2} \mathcal{A}_2(q) \approx \frac{R^2}{a_2(1+a_2\beta K)} I_{1,R}(\bar{t}) \approx -\frac{1}{4\pi a_2} \log \bar{t} + \frac{C_2}{a_2}. \quad (4.22)$$

Therefore, neglecting logarithms all contributions behave as $R^{-2(2m-M_{\text{int}})} \widehat{\Delta}_1^n$ in the crossover limit. Now, if K increases at most logarithmically, $\widehat{\Delta}_1$ behaves as $1/R^2$ modulo logarithms. Therefore all terms of the form (4.19) behave as

$$\frac{(\log R^2)^p}{R^{2(2m-M_{\text{int}}+n)}}, \quad (4.23)$$

for some p . Now, if N_{int} is the number of internal lines of the original l -loop graph, $N_{\text{int}} = M_{\text{int}} + n$. Moreover if the original graph does not contain tadpoles $m + n > l$ for $n \geq 1$. Thus, for $n \geq 1$,

$$2m + n - M_{\text{int}} = 2(m + n) - N_{\text{int}} > 2l - N_{\text{int}}. \quad (4.24)$$

Therefore contributions with $\widehat{\Delta}_1$ decrease faster and can be neglected. Thus the original l -loop graph we started from can be computed replacing $\widehat{\Delta}(q)$ with $\widehat{\Delta}_2(q)$ and scales as

$$\frac{1}{R^{2l}} \bar{t}^{l-N_{\text{int}}} = \frac{(R^2 \bar{t})^{l-N_{\text{int}}}}{R^{2(2l-N_{\text{int}})}}, \quad (4.25)$$

without any logarithm.

We should now consider graphs with tadpoles. As we already discussed these contributions can be written as the product of an integral associated to a graph without tadpoles, and therefore behaving according to Eq. (4.25), and a power of the one-loop tadpole diagram.

Now, using Eq. (4.22), we have

$$N \int \frac{d^2k}{(2\pi)^2} \widehat{\Delta}(k) = -\frac{\beta K}{1+a_2\beta K} + \frac{1}{R^2} \left[-\frac{1}{4\pi a_2} \log \bar{t} + \frac{C_2}{a_2} \right] + o(R^{-2}). \quad (4.26)$$

The parameter K is free and we can take it R -dependent at our will. We should also define $\beta_{c,R}^{(\text{exp})}$, i.e. the scaling behaviour of the temperature β . To fix these two variables we require that the tadpole scales as R^{-2} for $R \rightarrow \infty$ without logarithms and that $\bar{t} \sim R^{-2}$. This can be achieved by taking

$$K = \frac{V_R}{4\pi R^2} \log R^2 + \frac{c_0}{R^2} V_R, \tag{4.27}$$

$$\hat{t} = R^2 \left[1 - \frac{N + 2 \bar{a}_4}{24\pi N \bar{a}_2^2} \frac{1}{R^2} \log R^2 - \bar{a}_2 \beta V_R \right], \tag{4.28}$$

where c_0 is an arbitrary constant. In the derivation we have used the expansion of a_2 in powers of βK , see Eq. (4.10). With this choice we have

$$\bar{t} = \frac{1}{R^2} (\hat{t} + \hat{c}_0) + o(R^{-2}), \tag{4.29}$$

$$N \int \frac{d^2 k}{(2\pi)^2} \hat{\Delta}(k) = \frac{1}{\bar{a}_2 R^2} \left[-\frac{1}{4\pi} \log(\hat{t} + \hat{c}_0) + C_2 - c_0 \right] + o(R^{-2}), \tag{4.30}$$

where

$$\hat{c}_0 = -\frac{N + 2 \bar{a}_4}{6N \bar{a}_2^2} c_0. \tag{4.31}$$

Therefore, with this choice of K , in the limit $R \rightarrow \infty$, $\beta \rightarrow 0$, with \hat{t} fixed, the tadpole scales as R^{-2} , without logarithms of R^2 . It follows that all graphs scale as in Eq. (4.25) with additional logarithms of $(\bar{t}R^2)$.

Now, a simple topological argument gives

$$2l - N_{\text{int}} = 2 - m - \frac{n}{2} + \frac{1}{2} \sum_k (k - 4) V_k, \tag{4.32}$$

where V_k is the number of k -leg vertices. Therefore, the graph scales as

$$\frac{(R^2 \bar{t})^{l - N_{\text{int}}}}{R^{4 - 2m - n}} R^{-1/2 \sum_k (k - 4) V_k}, \tag{4.33}$$

with additional powers of $\log(R^2 \bar{t})$. In the limit $R \rightarrow \infty$, $\bar{t} \rightarrow 0$, with \hat{t} fixed, the previous formula shows that graphs that have one or more vertices with six or more legs vanish faster for $R \rightarrow \infty$ than graphs containing only 4-leg vertices. This means that in this limit we can simply ignore the ϕ^6, ϕ^8, \dots , terms in Eq. (4.11). In conclusion we obtain

$$\tilde{\Gamma}^{(m,n)} \equiv \Gamma^{(m,n)} R^{4 - 2m - n} = (\hat{t} + \hat{c}_0)^{2 - m - n/2} \sum_l \tilde{\Gamma}_l^{(m,n)}, \tag{4.34}$$

where $\tilde{\Gamma}_l^{(m,n)}$ behaves as \hat{t}^{-l} times logarithms of \hat{t} . Therefore the loop expansion provides the expansion of the critical crossover functions in the limit $\hat{t} \rightarrow \infty$. For instance, if one considers the susceptibility, one obtains at two loops,

$$\begin{aligned} f_\chi(\hat{t}) &\equiv \lim_{R \rightarrow \infty} \chi R^{-2} \\ &= \frac{\bar{a}_2}{\hat{t} + \hat{c}_0} - \frac{N + 2 \bar{a}_4}{6N \bar{a}_2} \frac{1}{(\hat{t} + \hat{c}_0)^2} \left[\frac{1}{4\pi} \log(\hat{t} + \hat{c}_0) + c_0 - C_2 \right] \end{aligned}$$

$$\begin{aligned}
 & + \frac{(N+2)^2 \bar{a}_4^2}{36N^2} \frac{1}{\bar{a}_2^3 (\hat{t} + \hat{c}_0)^3} \left[\left(\frac{1}{4\pi} \log(\hat{t} + \hat{c}_0) + c_0 - C_2 \right)^2 \right. \\
 & \left. - \frac{1}{4\pi} \left(\frac{1}{4\pi} \log(\hat{t} + \hat{c}_0) + c_0 - C_2 \right) + \frac{2H}{N+2} \right] + O((\hat{t} + \hat{c}_0)^{-4}), \quad (4.35)
 \end{aligned}$$

where $H = \frac{1}{24\pi^2} \psi'(\frac{1}{3}) - \frac{1}{36}$.

This result should not depend on c_0 . Expanding in \hat{t} for $\hat{t} \rightarrow \infty$ it is easy to check that no dependence remains and we can simply set $c_0 = 0$.

From the discussion we have presented, it is clear that the crossover functions can be obtained directly in the standard ϕ^4 theory with Hamiltonian

$$\int d^d x \left[\frac{1}{2} \sum_{\mu} \partial_{\mu} \phi \cdot \partial_{\mu} \phi + \frac{1}{2} r \phi^2 + \frac{1}{4!} u \phi^4 \right]. \quad (4.36)$$

Indeed all graphs, except the tadpole, have been computed using the propagator of this theory. The tadpole has been dealt with differently as it should be expected: indeed this is the only ultraviolet divergent diagram. Therefore we have proved that we can rewrite

$$f_{\chi}(\hat{t}) = \mu_{\chi} F_{\chi} [s(\hat{t} + \tau)], \quad (4.37)$$

where F_{χ} is the crossover curve computed in the short-range theory and that was explicitly expressed in terms of RG functions in Section 2. In field-theoretic terms μ_{χ} is the field renormalization, s is related to the difference in the normalization of the fields and of the coupling constant and τ is the additive shift due to the mass renormalization. Comparing the expansion (2.10) with Eq. (4.35) we obtain

$$\mu_{\chi} = -\frac{N\bar{a}_2^3}{\bar{a}_4}, \quad (4.38)$$

$$s = -\frac{N\bar{a}_2^2}{\bar{a}_4}, \quad (4.39)$$

$$\tau = -\frac{\bar{a}_4}{N\bar{a}_2^2} \left[\frac{N+2}{24\pi} \log \left(\frac{24\pi N\bar{a}_2^2}{(N+8)|\bar{a}_4|} \right) + D_2(N) + \frac{N+8}{24\pi} + \frac{N+2}{6} C_2 \right], \quad (4.40)$$

where $D_2(N)$ is defined in Eq. (2.11).

For the N -vector model we obtain

$$\mu_{\chi} = \frac{N+2}{6}, \quad (4.41)$$

$$s = \frac{N+2}{6}, \quad (4.42)$$

$$\tau = \frac{1}{4\pi} \log \left(\frac{4\pi(N+2)}{N+8} \right) + C_2 + \frac{6D_2(N)}{N+2} + \frac{1}{4\pi} \frac{N+8}{N+2}. \quad (4.43)$$

As a final remark we wish to notice that we have followed here Thouless' approach to the critical crossover. For $N \geq 3$ this is the only possibility since no critical point exists.

For $N < 2$ the crossover functions defined in this way differ by a simple shift given by the constant τ computed above.

For $N < 2$ our results give also the large- R behaviour of the critical point. Since the critical theory corresponds to $\tilde{t} = 0$ where \tilde{t} is the parameter appearing in the field-theory crossover functions, we have

$$\beta_{c,R} = \frac{1}{\bar{a}_2 V_R} \left[1 - \frac{N+2}{24\pi N} \frac{\bar{a}_4}{\bar{a}_2^2} \frac{1}{R^2} \log R^2 + \frac{\tau}{R^2} + o(R^{-2}) \right]. \quad (4.44)$$

Notice that this result depends explicitly on the single-site measure through \bar{a}_2 and \bar{a}_4 , while it does not depend on the hopping coupling $J(x)$. Indeed all the dependence on $J(x)$ is encoded in the expansion variable R^2 . For the N -vector model we obtain the simpler and N -independent expression

$$\beta_{c,R} = \frac{1}{V_R} \left[1 + \frac{1}{4\pi R^2} \log R^2 + \frac{\tau}{R^2} + o(R^{-2}) \right]. \quad (4.45)$$

The presence of a logarithmic factor in $\beta_{c,R}$ was already predicted by Thouless [15] in a modified Ising model.

4.3. Three-dimensional crossover limit

The ideas of the previous paragraphs can be generalized to any dimension $d < 4$. For generic values of d one works as follows: first one considers the graphs with l loops contributing to the one-particle irreducible two-point function, where l satisfies

$$dl \leq \frac{2d}{4-d}, \quad (4.46)$$

computing their contribution in the limit $R \rightarrow \infty$, $\beta \rightarrow 0$ with $\bar{t}R^{2d/(4-d)}$ fixed. Then one fixes the scaling behaviour of K in order to cancel all the terms that scale faster than $R^{-2d/(4-d)}$. The expression of $\beta_{c,R}^{(\text{exp})}$ is obtained requiring $\bar{t}R^{2d/(4-d)}$ to be constant for $\beta \rightarrow 0$ and $R \rightarrow \infty$. The perturbative expansion becomes an expansion in powers of $\bar{t}^{(4-d)/2}$ with additional logarithms when $2/(4-d)$ is an integer, i.e. for $d = 4 - 2/n$. The reason for the appearance of these singular terms is well known [38–40]: the critical crossover limit corresponds to the massless limit of the standard ϕ^4 theory which is known to have logarithmic singularities for these values of d .

Let us now discuss in more detail the three-dimensional case. For $d = 3$ we should consider the one-loop and two-loop graphs in the expansion of the two-point function $\langle \phi^a \phi^b \rangle$. If $\Gamma^{(0,2)}$ is the irreducible two-point function at zero external momentum, we have at two loops

$$\begin{aligned} \Gamma^{(0,2)} \approx & \frac{\bar{t}}{1 - K/V_R} - \frac{N+2}{6N} a_4 T_1 - \frac{(N+2)^2}{36N^2} a_4^2 T_1 T_2 \\ & - \frac{N+2}{18N^2} a_4^2 T_3 - \frac{(N+2)(N+4)}{120N^2} a_6 T_1^2, \end{aligned} \quad (4.47)$$

where

$$T_1 = N \int \frac{d^3q}{(2\pi)^3} \widehat{\Delta}(q), \tag{4.48}$$

$$T_2 = N^2 \int \frac{d^3q}{(2\pi)^3} \widehat{\Delta}(q)^2, \tag{4.49}$$

$$T_3 = N^3 \int \frac{d^3q}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \widehat{\Delta}(q) \widehat{\Delta}(k) \widehat{\Delta}(q+k). \tag{4.50}$$

Now, using Eq. (A.46), for $\beta K \sim O(R^{-3})$ – we will show in the following this is the correct asymptotic behaviour – we have

$$\begin{aligned} T_1 &= -\frac{\beta K}{1+a_2\beta K} + \frac{1}{a_2(1+a_2\beta K)} I_{1,R}(\bar{t}) \\ &= \frac{1}{a_2} (1-a_2\beta K) (\bar{I}_{1,R} - a_2\beta K) - \frac{1}{4\pi a_2} \frac{1}{R^6} (\bar{t}R^6)^{1/2} + o(R^{-6}). \end{aligned} \tag{4.51}$$

The estimate of T_2 is easy and we find, cf. Eq. (A.49),

$$T_2 = \frac{1}{8\pi a_2^2} (\bar{t}R^6)^{-1/2} + o(R^0). \tag{4.52}$$

Finally, using Eqs. (A.46) and (A.72) we obtain for T_3 – we will call the associated graph “two-loop watermelon” –

$$T_3 = \frac{1}{a_2^3 R^6} \left[-\frac{1}{32\pi^2} \log \bar{t} + C_3 \right] + o(R^{-6}), \tag{4.53}$$

where C_3 is a constant given in Eq. (A.73).

Now let us consider the terms that scale slower than R^{-6} for $R \rightarrow \infty$ at $\bar{t}R^6$ fixed. Using the previous results we have

$$\begin{aligned} &-\frac{N+2}{6N} \frac{a_4}{a_2} (1-a_2\beta K) (\bar{I}_{1,R} - a_2\beta K) - \frac{N+2}{192\pi^2 N^2} \frac{a_4^2}{a_2^3 R^6} \log R^2 \\ &-\frac{(N+2)^2}{36N^2} a_4^2 T_1 T_2. \end{aligned} \tag{4.54}$$

At first we neglect the last term proportional to $T_1 T_2$. If K has a finite limit for $R \rightarrow \infty$ and $\bar{t} \sim R^{-6}$ in the same limit, using Eq. (4.14), we obtain $\beta = 1/(\bar{a}_2 V_R) (1+O(R^{-3}))$. We can then determine K by requiring the expression (4.54) to be of order R^{-6} . This gives

$$K = V_R \left[\bar{I}_{1,R} + \frac{1}{32\pi^2 N} \frac{\bar{a}_4}{\bar{a}_2^2} \frac{1}{R^6} \log R^2 + \frac{c_0}{R^6} \right], \tag{4.55}$$

where c_0 is arbitrary. The scaling behaviour of β is obtained requiring $\bar{t} \sim O(R^{-6})$. If we introduce a variable \widehat{t} related to β by

$$\beta = \frac{1}{\bar{a}_2 V_R} \left\{ 1 - \frac{N+2}{6N} \frac{\bar{a}_4}{\bar{a}_2^2} \left[\bar{I}_{1,R} + \frac{1}{32\pi^2 N} \frac{\bar{a}_4}{\bar{a}_2^2} \frac{1}{R^6} \log R^2 \right] - \frac{\widehat{t}}{R^6} \right\}, \tag{4.56}$$

and define the critical crossover limit as the limit $\beta \rightarrow 0$, $R \rightarrow \infty$ with \hat{t} fixed, we find that indeed $\bar{t} \sim R^{-6}$. More precisely, using Eqs. (4.14) and (4.10), we have

$$\bar{t}R^6 = \hat{t} + \hat{c}_0 + c_1 + o(R^0), \tag{4.57}$$

where

$$c_1 = -\sigma^2 \left[\frac{(N+2)(N+4)\bar{a}_6}{120N^2\bar{a}_2^3} - \frac{(N+2)^2\bar{a}_4^2}{18N^2\bar{a}_2^4} + \frac{N+2\bar{a}_4}{6N\bar{a}_2^2} \right], \tag{4.58}$$

σ is defined in Eq. (A.40), and \hat{c}_0 is related to c_0 by Eq. (4.31). Notice that for $R \rightarrow \infty$, K converges to a constant, cf. Eq. (A.39), and $\beta \approx 1/(\bar{a}_2 V_R)$ as it was claimed before.

Using Eqs. (4.56) and (4.55) we can rewrite T_1 as

$$T_1 = -\frac{1}{32\pi^2 N} \frac{\bar{a}_4}{\bar{a}_2^3} \frac{1}{R^6} \log R^2 + O(R^{-6}) = -\frac{\bar{a}_4}{3N} T_3 + O(R^{-6}). \tag{4.59}$$

Let us now go back to Eq. (4.54). We should now deal with the term proportional to $T_1 T_2$ that was neglected in the previous treatment. We have

$$\Gamma^{(0,2)} = \frac{(N+2)^2 \bar{a}_4^3}{9216\pi^3 N^3 \bar{a}_2^5} (\bar{t}R^6)^{-1/2} \frac{1}{R^6} \log R^2 + O(R^6). \tag{4.60}$$

Because of the presence of the logarithm, this term does not scale correctly. However, it is of order $1/\hat{t}^{1/2}$, and terms of this order appear also at three loops. We will now show that this contribution is cancelled exactly by the contribution of the three-loop graph in which the tadpole has been replaced by the two-loop watermelon. This graph is associated to the integral

$$T_4 = N^5 \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{d^3r}{(2\pi)^3} \hat{\Delta}(p)^2 \hat{\Delta}(q) \hat{\Delta}(r) \hat{\Delta}(p+q+r). \tag{4.61}$$

It can be rewritten as

$$T_4 = T_3 T_2 + N^5 \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{d^3r}{(2\pi)^3} \times \hat{\Delta}(p)^2 \hat{\Delta}(q) \hat{\Delta}(r) [\hat{\Delta}(p+q+r) - \hat{\Delta}(q+r)]. \tag{4.62}$$

In the crossover limit the last term can be easily computed using the technique presented in Appendix A.3. It is easy to see that we can neglect the contributions due to $\hat{\Delta}_1$, and that we can replace

$$\hat{\Delta}_2(k) \rightarrow \frac{1}{\bar{a}_2 N} \frac{1}{k^2 R^2 + \bar{t}}. \tag{4.63}$$

Thus, neglecting terms $o(R^{-6})$, we have

$$T_4 = T_3 T_2 + \frac{(\bar{t}R^6)^{-1/2}}{\bar{a}_2^5 R^6} \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{d^3r}{(2\pi)^3} \times \frac{1}{(p^2+1)^2 (q^2+1) (r^2+1)} \left[\frac{1}{(p+q+r)^2+1} - \frac{1}{(q+r)^2+1} \right]. \tag{4.64}$$

Thus, if we keep only terms of order $R^{-6} \log R^2$, we have simply $T_4 \approx T_3 T_2$. Then including the combinatorial and group factors we find the total contribution

$$\begin{aligned}
 -\frac{(N+2)^2}{108N^3} \bar{a}_4^3 T_4 &\approx -\frac{(N+2)^2}{108N^3} \bar{a}_4^3 T_3 T_2 + O(R^{-6}) \\
 &\approx \frac{(N+2)^2}{36N^2} \bar{a}_4^2 T_1 T_2 + O(R^{-6}),
 \end{aligned}
 \tag{4.65}$$

where we have used Eq. (4.59). Comparing with Eq. (4.54) we see that the logarithms cancel, as claimed at the beginning.

This calculation illustrates the general mechanism in three dimensions. Consider a graph with only ϕ^4 vertices that does not have tadpoles or two-loop watermelons as subgraphs. In the crossover limit the contribution of this graph can be computed neglecting $\hat{\Delta}_1$ and substituting $\hat{\Delta}_2(q)$ with its small- q expression. Following the argument presented in two dimensions it is easy to see that an l -loop contribution scales exactly as $\hat{t}^{1-l/2}$. The same argument we have presented in two dimensions proves also that the contributions of graphs with ϕ^6, ϕ^8, \dots vertices are suppressed and can be neglected in the critical crossover limit. Finally consider a graph with only ϕ^4 vertices that has tadpoles or two-loop watermelons as subgraphs. This graph generates non-scaling terms involving powers of $\log R^2$. However, the logarithmic contribution associated to each tadpole is exactly cancelled, using Eq. (4.59), by the analogous term which is associated to the graph in which the tadpole is replaced by the two-loop watermelon. The sum of all contributions scales as R^{-6} without logarithms of R^2 . Our discussion proves therefore that the perturbative expansion corresponds to an expansion in powers of $1/\hat{t}^{1/2}$ of the critical crossover functions with additional logarithms of \hat{t} . Explicitly for the susceptibility we obtain at two loops

$$\begin{aligned}
 (f_\chi(\hat{t}))^{-1} &\equiv \lim_{R \rightarrow \infty} \chi^{-1} R^6 \\
 &= \bar{a}_2 \hat{t} + \frac{N+2}{24\pi N} \frac{\bar{a}_4}{\bar{a}_2} \hat{t}^{1/2} + \bar{a}_2 c_1 + \frac{N+2}{6N} \frac{\bar{a}_4}{\bar{a}_2} \sigma^2 \\
 &\quad - \frac{N+2}{18N^2} \frac{\bar{a}_4^2}{\bar{a}_2^3} \left[-\frac{1}{32\pi^2} \log \hat{t} + C_3 \right] + \frac{(N+2)^2}{1152\pi^2 N^2} \frac{\bar{a}_4^2}{\bar{a}_2^3} + O(\hat{t}^{-1/2} \log \hat{t}).
 \end{aligned}
 \tag{4.66}$$

As expected, the contribution proportional to a_6 at two-loops disappears in the crossover limit: again only the ϕ^4 -vertex is relevant. The result is also independent on c_0 .

The discussion we have presented shows also that the critical crossover functions can be computed in the standard continuum ϕ^4 theory. Therefore, as we did in two dimensions, we can write

$$f_\chi(\hat{t}) = \mu_\chi F_\chi [s(\hat{t} + \tau)], \tag{4.67}$$

where F_χ is the crossover curve computed in the short-range theory and that was explicitly expressed in terms of RG functions in Section 2. Comparing the expansion (2.12) with Eq. (4.66), we obtain

$$\mu_\chi = \frac{N^2 \bar{a}_2^3}{\bar{a}_4^2}, \tag{4.68}$$

$$s = \frac{N^2 \bar{a}_2^4}{\bar{a}_4^4}, \tag{4.69}$$

$$\begin{aligned} \tau = & -\frac{N+2}{288\pi^2 N^2} \frac{\bar{a}_4^2}{\bar{a}_2^4} \log\left(\frac{48\pi N \bar{a}_2^2}{(N+8)|\bar{a}_4|}\right) + \frac{\bar{a}_4^2}{\bar{a}_2^4} \frac{D_3(N)}{N^2} + \frac{\bar{a}_4^2}{\bar{a}_2^4} \frac{9N^2 - 20N - 544}{20736\pi^2 N^2} \\ & + c_1 + \frac{N+2}{6N} \frac{\bar{a}_4}{\bar{a}_2^2} \sigma^2 - \frac{N+2}{18N^2} \frac{\bar{a}_4^2}{\bar{a}_2^4} C_3, \end{aligned} \tag{4.70}$$

where $D_3(N)$ is defined in Eq. (2.13). For the N -vector model these equations become

$$\mu_\chi = \frac{(N+2)^2}{36}, \tag{4.71}$$

$$s = \frac{(N+2)^2}{36}, \tag{4.72}$$

$$\begin{aligned} \tau = & -\frac{1}{8\pi^2(N+2)} \log\left(\frac{8\pi(N+2)}{(N+8)}\right) + \frac{36D_3(N)}{(N+2)^2} \\ & + \frac{9N^2 - 20N - 544}{576\pi^2(N+2)^2} - \frac{2C_3}{N+2}. \end{aligned} \tag{4.73}$$

As remarked in the previous section, Eq. (4.56) gives us the behaviour of the critical point $\beta_{c,R}$ up to terms of order R^{-6} . We have

$$\beta_{c,R} = \frac{1}{\bar{a}_2 V_R} \left\{ 1 - \frac{N+2}{6N} \frac{\bar{a}_4}{\bar{a}_2^2} \left[\bar{I}_{1,R} + \frac{1}{32\pi^2 N} \frac{\bar{a}_4}{\bar{a}_2^2} \frac{1}{R^6} \log R^2 \right] + \frac{\tau}{R^6} + o(R^{-6}) \right\}. \tag{4.74}$$

In the N -vector case, this expression simplifies becoming

$$\beta_{c,R} = \frac{1}{V_R} \left[1 + \bar{I}_{1,R} - \frac{3}{16\pi^2} \frac{1}{N+2} \frac{1}{R^6} \log R^2 + \frac{\tau}{R^6} + o(R^{-6}) \right]. \tag{4.75}$$

The first correction to $\beta_{c,R}$ was derived in Refs. [46–48], while the presence of the logarithmic correction was predicted in a modified Ising model with long-range interactions by Thouless [15]. Notice that for $N \rightarrow \infty$ the logarithmic term disappears and that, using Eq. (2.14), $\tau \sim O(1/N)$ so that the resulting formula gives the exact large- N prediction for $\beta_{c,R}$.

5. Corrections to the critical crossover functions

In this section we will study the corrections to the critical crossover functions. First we will present the results for the three-dimensional case, which is the easiest one, then we will discuss the corrections in two dimensions. To derive the behaviour of these corrections we must introduce some additional hypotheses on the function $J_R(x)$. We will thus assume:

(v) $\sum_x (x^2)^2 J_R(x)$ is finite. It follows $\Pi(q) \approx q^2 + O(q^4)$.

(vi) In the limit $R \rightarrow \infty$ at fixed q , $\bar{\Pi}_R(q/R)$ has an expansion in powers of $1/R^2$. Explicitly

$$\bar{\Pi}_R(q/R) = \Pi(q) + \sum_{n=1}^{\infty} \frac{1}{R^{2n}} \Pi_n(q). \tag{5.1}$$

Notice that, because of property (v), $\Pi_n(q) \sim q^4$ for $q \rightarrow 0$.

5.1. Corrections in three dimensions

We begin by discussing the three-dimensional case. We will show at two loops – but we conjecture that this is true to all orders of perturbation theory – that the leading correction to the scaling behaviour is of order R^{-3} , provided one appropriately defines \hat{t} . In other words, we will show that

$$\tilde{\chi} = f_\chi(\hat{t}) + \frac{1}{R^d} g_\chi(\hat{t}) + \dots \tag{5.2}$$

in the crossover limit for a suitable definition of \hat{t} . This type of behaviour should be true for any dimension $2 < d < 4$. It is obvious that an expansion of the form (5.2) cannot be valid generically. Indeed if $\beta_{c,R}^{(\text{exp})}$ is such that Eq. (5.2) holds – \hat{t} is defined in Eq. (3.22) – consider $\beta_{\text{New},c,R}^{(\text{exp})}$ defined by

$$\beta_{\text{New},c,R}^{(\text{exp})} \equiv \beta_{c,R}^{(\text{exp})} (1 + AR^{-6-\alpha}), \tag{5.3}$$

with $0 < \alpha < 3$. If \hat{t}_{New} is the corresponding scaling variable we have $\hat{t}_{\text{New}} = \hat{t} + AR^{-\alpha}$. Therefore the two definitions are identical in the critical crossover limit. However, in the variable \hat{t}_{New} the corrections are of order $R^{-\alpha}$.

Let us now go back to Eq. (4.47), and again let us neglect at first the contribution proportional to $T_1 T_2$. The expression for T_1 appearing in Eq. (4.51) is valid up to terms of order R^{-9} as it can be seen from the results of Appendix A.2. Using the expressions for K and β , cf. Eqs. (4.55) and (4.56), one finds that the leading correction in T_1 is of order $\log R^2/R^9$. Let us now consider T_3 . Using the results of Appendix A.3 we find

$$T_3 = \frac{1}{a_2^3} I_{2,R}(\bar{t}) + O(R^{-9}) = \frac{1}{a_2^3 R^6} \left[-\frac{1}{32\pi^2} \log \bar{t} + C_3 + \frac{F_3}{R^2} \right] + O(R^{-9}). \tag{5.4}$$

Therefore with the definition of $\beta_{c,R}^{(\text{exp})}$ appearing in Eq. (4.56) we would obtain corrections of order R^{-2} and $R^{-3} \log R^2$. We will now show that these corrections can be eliminated with a proper redefinition of the expressions of K and $\beta_{c,R}^{(\text{exp})}$. Considering for simplicity the N -vector case [we have $a_2 = 1$, and $a_4 = -6N/(N+2)$] we assume

$$K = V_R \left[\bar{I}_{1,R} - \frac{3}{16\pi^2(N+2)} \frac{1}{R^6} \log R^2 + \frac{K_1}{R^8} + \frac{K_2}{R^9} \log R^2 \right], \tag{5.5}$$

$$\beta = \frac{1}{V_R} \left[1 + \bar{I}_{1,R} - \frac{3}{16\pi^2(N+2)} \frac{1}{R^6} \log R^2 - \frac{\hat{t}}{R^6} + \frac{b_1}{R^8} + \frac{b_2}{R^9} \log R^2 \right], \tag{5.6}$$

where $K_1, K_2, b_1,$ and b_2 are constants to be determined. Then the corrections of order $1/R^8$ and $\log R^2/R^9$ in T_1 and T_2 are the following:

$$\begin{aligned}
 T_1 = \dots &+ \left[-K_1 + \frac{b_1 - K_1}{8\pi} (\hat{t} + \sigma^2)^{-1/2} \right] \frac{1}{R^8} \\
 &+ \left[-K_2 + \frac{3}{16\pi^2} \frac{\sigma}{N+2} + \frac{1}{8\pi} (\hat{t} + \sigma^2)^{-1/2} \left(b_2 - K_2 + \frac{3}{8\pi^2} \frac{\sigma}{N+2} \right) \right] \frac{\log R^2}{R^9} \\
 &+ O(R^{-9}), \tag{5.7}
 \end{aligned}$$

$$\begin{aligned}
 T_3 = \dots &+ \left[F_3 + \frac{b_1 - K_1}{32\pi^2} (\hat{t} + \sigma^2)^{-1} \right] \frac{1}{R^8} \\
 &+ \left[-\frac{9\sigma}{32\pi^2} + \frac{1}{32\pi^2} (\hat{t} + \sigma^2)^{-1} \left(b_2 - K_2 + \frac{3}{8\pi^2} \frac{\sigma}{N+2} \right) \right] \frac{\log R^2}{R^9} + O(R^{-9}), \tag{5.8}
 \end{aligned}$$

where the dots indicate terms that scale as R^{-6} and $R^{-6} \log R^2$. To cancel the unwanted corrections we must require that the combination $T_1 - 2T_3/(N+2)$ is free of terms that scale as R^{-8} and $R^{-9} \log R^2$. In this way we determine the constants $K_1, K_2, b_1,$ and b_2 . Explicitly

$$K_1 = b_1 = -\frac{2F_3}{N+2}, \quad K_2 = 2b_2 = \frac{3}{4\pi^2} \frac{\sigma}{N+2}. \tag{5.9}$$

We must now consider the terms proportional to $T_1 T_2$. As we already discussed before we must consider at the same time the diagram associated to T_4 . A simple analysis shows that Eq. (4.64) has corrections of order $O(R^{-9})$ which are therefore negligible in the present discussion. Therefore, including the combinatorial and group factors we must show that

$$T_1 T_2 - \frac{2T_2 T_3}{N+2} = T_2 \left(T_1 - \frac{2T_3}{N+2} \right) \tag{5.10}$$

is free of terms that scale as R^{-8} and $R^{-9} \log R^2$. We have already shown that the term in parenthesis has this property. For T_2 , using Eq. (A.49), we can show that Eq. (4.52) is valid up to terms of order $O(R^{-3})$. Therefore the previous expression is free of the unwanted corrections. In conclusion we have proved at two loops Eq. (5.2). We conjecture this is true to all orders: graphs without tadpoles or insertions of the two-loop watermelon should have corrections of order R^{-9} , while terms of order R^{-8} and $R^{-9} \log R^2$ that appear in graphs with tadpoles or two-loop watermelon insertions should cancel with the mechanism we presented above. At the order we are considering graphs containing vertices with more than four legs should still be negligible: at two loops the contribution proportional to a_6 in Eq. (4.47) scales as $\log^2 R^2/R^{12}$.

Using the perturbative expansion we can compute the function $g_\chi(\hat{t})$ in the limit $\hat{t} \rightarrow \infty$. We have

$$g_\chi(\hat{t}) = -\frac{E_3}{\hat{t}} + O(\hat{t}^{-3/2}), \tag{5.11}$$

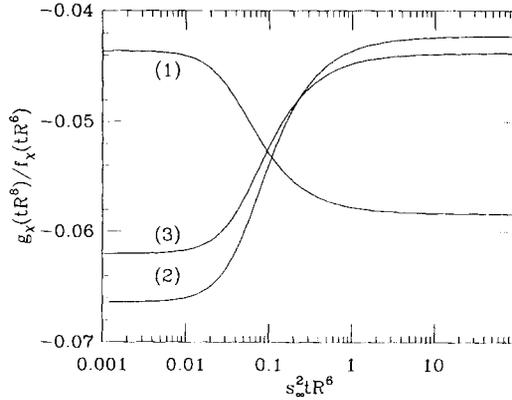


Fig. 1. Ratio $g_X(\tilde{t})/f_X(\tilde{t})$ in the large- N limit for three different models: (1) N -vector model; (2) potential $W(\varphi^2) = \varphi^2 + (\varphi^2 - 1)^2$; (3) potential $W(\varphi^2) = \varphi^2 + (\varphi^2 - 1)^2 + (\varphi^2)^3$. In all cases $J(x)$ is given in Eq. (3.8) with domain (A.2). s_∞ is a constant defined in Eq. (B.12).

where E_3 is defined in Eq. (A.47). Notice that this behaviour cannot be changed by modifying the definition of $\beta_{c,R}^{(exp)}$.

Let us now show that if one uses \tilde{t} defined using the exact $\beta_{c,R}$ one directly obtains the expansion (5.2). To prove this fact, assume the opposite and write

$$\tilde{\chi} = f_X(\tilde{t}) + \frac{1}{R^\alpha} h_X(\tilde{t}) + o(R^{-\alpha}), \tag{5.12}$$

with $\alpha < 3$. For $\tilde{t} \rightarrow 0$ and any value of R , $\tilde{\chi} \sim \tilde{t}^{-\gamma}$. Therefore $f_X(\tilde{t}) \sim \tilde{t}^{-\gamma}$ and $h_X(\tilde{t}) \sim \tilde{t}^{-\gamma}$ in this limit. Now, it follows from our discussion that the term of order $R^{-\alpha}$ can be eliminated if we introduce a new variable $\hat{t} = \tilde{t} + AR^{-\alpha}$. Substituting in Eq. (5.12) and expanding in $R^{-\alpha}$ we have

$$\tilde{\chi} = f_X(\hat{t}) + \frac{A}{R^\alpha} f'_X(\hat{t}) + \frac{1}{R^\alpha} h_X(\hat{t}) + o(R^{-\alpha}). \tag{5.13}$$

Cancellation of the terms of order $R^{-\alpha}$ requires $A f'_X(\hat{t}) + h_X(\hat{t}) = 0$. However, this relation cannot be true since $f'_X(\hat{t}) \sim \hat{t}^{-\gamma-1}$ for $\hat{t} \rightarrow 0$. Therefore $h_X(\hat{t}) = 0$. We have therefore shown that the variable \tilde{t} is a particularly good one, since it automatically eliminates a whole class of corrections to the leading behaviour. Another consequence of these results is that we can now estimate the order of the neglected terms in Eqs. (4.74), (4.75): the terms $o(R^{-6})$ are of order R^{-8} .

In Appendix B.1 we compute the function $g_X(\tilde{t})$ for our general model in the large- N limit. The graph of $g_X(\tilde{t})/f_X(\tilde{t})$ for some particular cases is reported in Fig. 1. We consider: (1) the N -vector model, (2) the potential $V(\varphi) = N(\varphi^4 - \varphi^2)$, and (3) the potential $V(\varphi) = N(\varphi^6 + \varphi^4 - \varphi^2)$. Although the function is not universal since it depends explicitly on various constants whose value is specific of the model one uses, the qualitative features are similar in all cases: $g_X(\tilde{t})/f_X(\tilde{t})$ interpolates smoothly between the values for $\tilde{t} = 0$ and $\tilde{t} = \infty$. Notice however that the function is decreasing in the N -vector model, while it is increasing in the other two cases.

Finally let us notice that the result (5.2) depends crucially on our use of R as scale and on the specific field normalizations used in the definition of our model. In general with an arbitrary scale ρ and arbitrarily normalized fields we have

$$\tilde{\chi} = A(\rho) f_{\chi}(B(\rho)\tilde{t}) + \frac{1}{\rho^d} g_{\chi}(\tilde{t}) + \dots, \quad (5.14)$$

where $A(\infty)$ and $B(\infty)$ are non-vanishing constants.

5.2. Corrections in two dimensions

We wish now to discuss the corrections to the universal crossover curves in two dimensions. If we repeat the perturbative analysis we have performed in the previous section we face immediately a difficulty. Working at one loop and using the results of Appendix A.2 we find corrections to the crossover functions of order $\log^2 R^2/R^2$ and $\log R^2/R^2$. However, at variance with the three-dimensional case, only the former terms can be cancelled with a redefinition of K and \hat{t} : the terms proportional to $\log R^2/R^2$ cannot be cancelled. Indeed let us suppose that

$$K = \frac{V_R}{R^2} \left[\frac{1}{4\pi} \log R^2 + \frac{K_0}{R^2} \log^2 R^2 + \frac{K_1}{R^2} \log R^2 \right], \quad (5.15)$$

$$\beta = \frac{1}{V_R} \left[1 + \frac{1}{4\pi R^2} \log R^2 + \frac{b_0}{R^4} \log^2 R^2 + \frac{b_1}{R^4} \log R^2 - \frac{\hat{t}}{R^2} \right]. \quad (5.16)$$

At one loop we obtain

$$\Gamma^{(0,2)} = \dots + \Gamma_2(\hat{t}) \frac{\log^2 R^2}{R^4} + \Gamma_1(\hat{t}) \frac{\log^2 R}{R^4} + \Gamma_0(\hat{t}) \frac{1}{R^4} + o(R^{-4}), \quad (5.17)$$

where the dots indicate terms that scale as R^{-2} . The coefficient $\Gamma_2(\hat{t})$ is given by

$$\Gamma_2(\hat{t}) = -b_0 - \frac{1}{4\pi\hat{t}} \left[K_0 - b_0 + \frac{1}{16\pi^2} \right]. \quad (5.18)$$

This term can be cancelled setting

$$b_0 = 0, \quad K_0 = -\frac{1}{16\pi^2}. \quad (5.19)$$

Let us now consider $\Gamma_1(\hat{t})$. We have

$$\Gamma_1(\hat{t}) = \frac{1}{8\pi} (4\alpha_1 + 3\alpha_2 + 2)\hat{t} + O(\log \hat{t}), \quad (5.20)$$

where α_1 and α_2 are determined by the low-momentum expansion of $\Pi(q)$, cf. Eq. (A.53). This term does not depend on K_1 or b_1 and it is therefore impossible to eliminate it. Therefore at one loop we obtain correction terms of order $\log R^2/R^2$. At two loops terms proportional to $\log^2 R^2/R^2$ pop in and in general we have

$$\tilde{\chi} f_{\chi}(\hat{t})^{-1} \approx 1 + \frac{1}{R^2} \sum_{n=0}^{\infty} \log^n R^2 g_n(\hat{t}). \tag{5.21}$$

The presence of this infinite series of logarithms may indicate that perturbation theory does not provide us with the correct corrections and that a resummation of the perturbative series is needed. In other words, the perturbative limit, $R \rightarrow \infty$ at \bar{t} fixed followed by $\bar{t} \rightarrow 0$ may not commute with the crossover limit $R \rightarrow \infty$, $\bar{t} \rightarrow 0$ at $\bar{t}R^2$ fixed at the level of the corrections to the universal behaviour. This phenomenon is not new in two-dimensional models. Indeed a similar non-commutativity appears in the corrections to the finite-size scaling functions [49,50].

In the large- N limit, cf. Appendix B.2, the corrections can be computed exactly and in this case one finds

$$\tilde{\chi} = f_{\chi}(\hat{t}) \left[1 + A(\hat{t}) \frac{\log R^2}{R^2} + B(\hat{t}) \frac{1}{R^2} \right] + o(R^{-2}). \tag{5.22}$$

However, this simple behaviour may be due to the large- N limit. In general, as long as $N \geq 3$, we do not expect a change in the exponent, but a more complicated behaviour of the logarithmic corrections would not be surprising. By analogy with what has been found for the finite-size scaling corrections in Refs. [49,50], we could have a behaviour of the form

$$\tilde{\chi} f_{\chi}(\hat{t})^{-1} \approx 1 + \frac{\log R^2}{R^2} \sum_{n=0}^{\infty} \frac{g_n(\hat{t})}{(\log R^2)^n}. \tag{5.23}$$

Also for the Ising model it is unlikely that a new exponent appears. The numerical work of Refs. [17,18] confirms this expectation: indeed they find that the corrections to scaling are well described in terms of a behaviour of the form (5.22). In these works $A(\hat{t})$ and $B(\hat{t})$ are assumed independent of \hat{t} . Of course this is an approximation, but it is not surprising it works well, since these two functions should be slowly varying, as indicated by the large- N solution.

One may wonder if the non-commutativity we have discussed above is peculiar of two-dimensional models. A simple analysis indicates that a similar problem should also appear in three dimensions if one considers the corrections of order R^{-6} . Indeed at this order T_3 gives rise to terms $\log R^2$ that cannot be eliminated by changing the scaling of K and β .

6. Discussion

In this section we wish to compare the analytic results obtained in the previous sections with the numerical ones presented in Refs. [17–19,22] and discuss other approaches to the crossover problem.

Let us first compare our results for $\beta_{c,R}$ with the numerical determinations of Refs. [17,22]. These simulations are performed in the Ising model with coupling given in Eq. (3.8) and domain family

$$D_\rho = \left\{ x : \sum_{i=1}^d x_i^2 \leq \rho^2 \right\}. \quad (6.1)$$

Using the numerical results of Appendix A for the constants C_2 and C_3 and the numerical estimates of Section 2 for the non-perturbative constants $D_2(N)$ and $D_3(N)$, we obtain in two dimensions the asymptotic expression

$$\begin{aligned} \beta_{c,R} V_R &\approx 1 + \frac{1}{4\pi R^2} \log R^2 + \frac{0.1975(5)}{R^2} \\ &\approx 1 + \frac{1}{R^2} (0.0796 \log R^2 + 0.1975), \end{aligned} \quad (6.2)$$

while in three dimensions

$$\beta_{c,R} V_R \approx 1 + \bar{T}_{1,R} - \frac{1}{16\pi^2 R^6} \log R^2 - \frac{0.0017(1)}{R^6}, \quad (6.3)$$

where $\bar{T}_{1,R}$ is defined in Eq. (A.37). Numerical estimates for selected values of ρ are reported in Table A.1. If one is interested in the expression of $\beta_{c,R} V_R$ up to terms of order $o(R^{-3})$, one can replace $\bar{T}_{1,R}$ with the asymptotic expression (A.39). In two dimensions Eq. (6.2) agrees approximately with the fit of the numerical data of Ref. [18]. They quote¹⁰

$$\beta_{c,R} V_R \approx 1 + \frac{1}{R^2} (0.076(3) \log R^2 + 0.172(7)). \quad (6.4)$$

A slightly different analysis gives [51]

$$\beta_{c,R} V_R \approx 1 + \frac{1}{R^2} (0.0781(9) \log R^2 + 0.166(6)). \quad (6.5)$$

To understand better the discrepancies we have considered

$$\Delta(R) = \left(\frac{\beta_{c,R,\text{approx}}}{\beta_{c,R,\text{exact}}} - 1 \right) R^2, \quad (6.6)$$

where $\beta_{c,R,\text{approx}}$ is the asymptotic form (6.2), while $\beta_{c,R,\text{exact}}$ is the exact value determined in the Monte Carlo simulation. Asymptotically we should observe $\Delta(R) \rightarrow 0$. However, for the values of ρ used in the simulation $\Delta(R)$ shows a somewhat erratic behaviour. For $\rho^2 = 32, 50, 72, 100, 140$, we have $\Delta(R) = -0.0812, 0.1649, 0.1660, -0.1943, -0.1124$ with an error of approximately 5×10^{-4} due mainly to the uncertainty in Eq. (6.2) (the error on $\beta_{c,R,\text{exact}}$ is much smaller). Clearly these values of ρ are too small for the asymptotic expansion to be valid. Similar discrepancies are observed in three dimensions. The non-monotonic behaviour of the corrections appears

¹⁰ Notice the different normalization of R^2 : R^2 in Ref. [18] is four times our definition of R^2 .

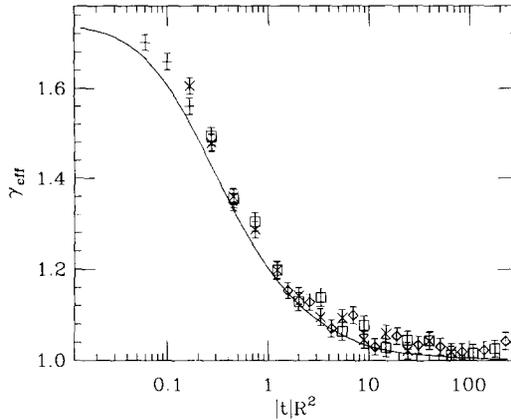


Fig. 2. Effective susceptibility exponent as a function of \tilde{t} in the high-temperature phase of the two-dimensional Ising model. The points are the numerical results of Ref. [18]; pluses, crosses, squares, and diamonds correspond to data with $\rho^2 = 10, 72, 140,$ and 1000 respectively. In the mean-field limit $\gamma_{\text{eff}} = 1$, while for $\tilde{t} \rightarrow 0$, $\gamma_{\text{eff}} = 7/4$.

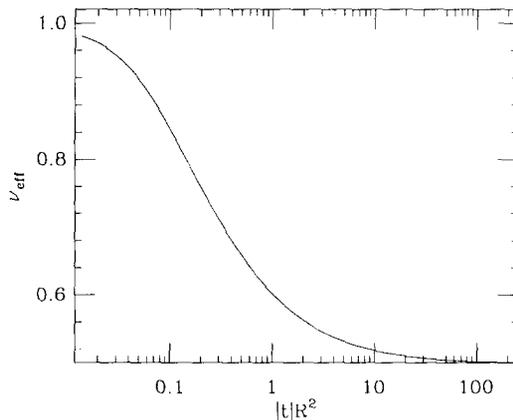


Fig. 3. Effective correlation-length exponent as a function of \tilde{t} for the high-temperature phase of the two-dimensional Ising model. In the mean-field limit $\nu_{\text{eff}} = 1/2$, while for $\tilde{t} \rightarrow 0$, $\nu_{\text{eff}} = 1$.

to be a general phenomenon for the family of domains used in the simulations, and it is probably connected with the fact that the shape is not natural on a cubic lattice. Similar oscillation with R are observed in lattice integrals. For instance, from the results of Table A.1 in Appendix A.2, one can see that the integral $\bar{I}_{1,R}$ does not have a monotonic behaviour even for $\rho^2 \approx 10^3$.

Let us now compare the results for the crossover curves. In Figs. 2, 3, 4, and 5, we report the graph of the effective exponents γ_{eff} , ν_{eff} and β_{eff} defined by

$$\gamma_{\text{eff}}(\tilde{t}) = -\frac{\tilde{t}}{f_\chi(\tilde{t})} \frac{df_\chi(\tilde{t})}{d\tilde{t}}, \tag{6.7}$$

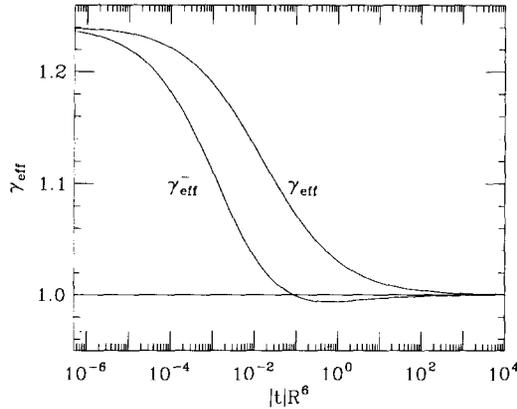


Fig. 4. Effective susceptibility exponent as a function of \tilde{t} for the high- (γ_{eff}^+) and low- (γ_{eff}^-) temperature phase of the three-dimensional Ising model. In the mean-field limit $\gamma_{\text{eff}} = 1$, while for $|\tilde{t}| \rightarrow 0$, $\gamma_{\text{eff}} \approx 1.237$.

$$\nu_{\text{eff}}(\tilde{t}) = -\frac{\tilde{t}}{2f_{\xi}(\tilde{t})} \frac{df_{\xi}(\tilde{t})}{d\tilde{t}}, \tag{6.8}$$

$$\beta_{\text{eff}}(\tilde{t}) = \frac{\tilde{t}}{f_M(\tilde{t})} \frac{df_M(\tilde{t})}{d\tilde{t}}, \tag{6.9}$$

for the Ising model in two and three dimensions, using the field-theory results presented in Section 2 and the rescalings (4.41), (4.42), (4.71), and (4.72). In two dimensions we can compare our results for $\gamma_{\text{eff}}(\tilde{t})$ with the numerical ones of Ref. [18]. In Fig. 2 we report our result for $\gamma_{\text{eff}}(\tilde{t})$ together with the Monte Carlo data of Ref. [18]. The agreement is good, showing nicely the equivalence of medium-range and field-theory calculations in the large- R limit.

In Figs. 4 and 5 we report the results for $\gamma_{\text{eff}}(\tilde{t})$ and $\beta_{\text{eff}}(\tilde{t})$ in three dimensions. As already discussed in Ref. [19], in the high-temperature phase, $\gamma_{\text{eff}}(\tilde{t})$ agrees nicely with the Monte Carlo data in the mean-field region while discrepancies appear in the neighbourhood of the Wilson–Fisher point. However, for $\tilde{t} \rightarrow 0$, only data with small values of ρ are present, so that the differences that are observed should be due to the corrections to the universal behaviour. The low-temperature phase shows a similar behaviour: good agreement in the mean-field region, and a difference near the Wilson–Fisher point where again only point with small ρ are available [51]. We can also compare the results for the magnetization. In Fig. 6 we report the combination $2 - \gamma_{\text{eff}}^- - 2\beta_{\text{eff}}$ which should be compared with the analogous figure appearing in Ref. [19]: the behaviour of the two curves is completely analogous.

We wish now to discuss a different approach to the crossover that has been developed in Refs. [12,52,13,4] following the so-called RG matching [53,54] and that has been applied successfully to many different experimental situations [12,52,3,4,55,5]. These papers consider phenomenological parametrizations which are able to describe the crossover even outside the universal critical regime. Let us now introduce this model in the formulation of Ref. [23], which is intended to apply directly to our class of

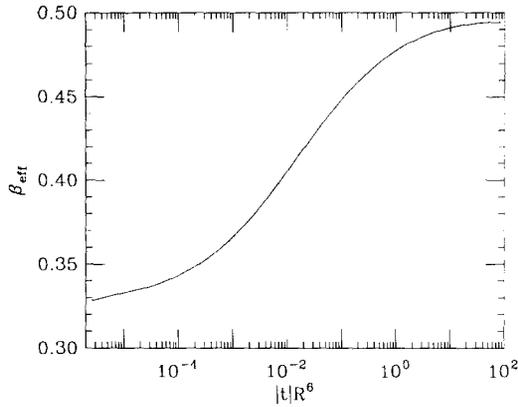


Fig. 5. Effective magnetization exponent as a function of \tilde{t} in the low-temperature phase of the three-dimensional Ising model. In the mean-field limit $\beta_{\text{eff}} = 1/2$, while for $|\tilde{t}| \rightarrow 0$, $\beta_{\text{eff}} \approx 0.327$.

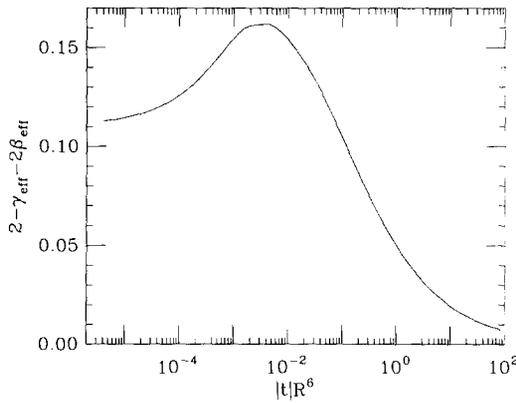


Fig. 6. Combination $2 - \gamma_{\text{eff}} - 2\beta_{\text{eff}}$ as a function of \tilde{t} in the low-temperature phase of the three-dimensional Ising model. In the mean-field limit this combination vanishes, while for $\tilde{t} \rightarrow 0$ it is equal to the specific-heat exponent $\alpha \approx 0.110$.

Hamiltonians. If t is the reduced temperature, one introduces two functions $\kappa(t)$ and $Y(t)$ defined by the set of equations

$$\kappa(t)^2 = c_t t Y(t)^{(2\nu-1)/\Delta}, \tag{6.10}$$

$$1 - (1 - \bar{u})Y(t) = \bar{u} \left[1 + \left(\frac{A}{\kappa(t)} \right)^2 \right]^{1/2} Y(t)^{\nu/\Delta}. \tag{6.11}$$

Notice that, although three non-universal constants c_t , A and \bar{u} appear in these equations, $Y(t)$ and $\kappa^2(t)/c_t$ depend only on \bar{u} and on the combination $\sqrt{c_t}/A$. The susceptibility is given by

$$\chi^{-1} = c_\rho^2 c_t \frac{t}{t+1} Y(t)^{(\gamma-1)/\Delta} (1 + y), \tag{6.12}$$

where

$$y = \frac{u^* \nu}{2\Delta} \left\{ 2 \left(\frac{\kappa(t)}{\Lambda} \right)^2 \left[1 + \left(\frac{\Lambda}{\kappa(t)} \right)^2 \right] \times \left(\frac{\nu}{\Delta} + \frac{(1-\bar{u})Y(t)}{1-(1-\bar{u})Y(t)} \right) - \frac{2\nu-1}{\Delta} \right\}^{-1}, \quad (6.13)$$

where u^* is a numerical constant, $u^* = 0.472$, and c_ρ is another normalization non-universal parameter. Notice that $\chi^{-1}/(c_\rho^2 c_t)$ depends only on \bar{u} and $\sqrt{c_t}/\Lambda$, so that Λ or c_t could be fixed to any value without loss of generality. In order to interpret the Monte Carlo results of Ref. [19], the authors of Ref. [23] further assume that c_t and \bar{u} scale as

$$c_t = \frac{c_{t0}}{R^2}, \quad \bar{u} = \frac{\bar{u}_0}{R^4}. \quad (6.14)$$

In order to have the correct scaling of χ , one should also set $c_\rho = c_{\rho 0} R$. Then in the critical crossover limit $t \rightarrow 0$, $R \rightarrow \infty$, with $\tilde{t} \equiv tR^6$ fixed, we obtain

$$\tilde{\chi}^{-1} = c_{\rho 0}^2 c_{t0} \tilde{t} Y_0(\tilde{t})^{(\gamma-1)/\Delta} \left[1 + \frac{u^* \nu}{2} \frac{1 - Y_0(\tilde{t})}{1 + (2\Delta - 1) Y_0(\tilde{t})} \right], \quad (6.15)$$

where $Y_0(\tilde{t})$ satisfies the equation

$$1 - Y_0(\tilde{t}) = \frac{1}{\sqrt{\alpha^2 \tilde{t}}} Y_0(\tilde{t})^{1/2\Delta}, \quad (6.16)$$

with

$$\alpha \equiv \frac{\sqrt{c_{t0}}}{\bar{u}_0 \Lambda}. \quad (6.17)$$

Eq. (6.15) defines the universal crossover function in this approach, the model dependence being included in the constants α and $c_{\rho 0}^2 c_{t0}$. In order to understand the accuracy of this approach we can compare $\tilde{\chi}$ obtained from Eq. (6.15) with the very precise results of Bagnuls and Bervillier [6,7]. First of all let us compare the asymptotic behaviour for $\tilde{t} \rightarrow 0$ and $\tilde{t} \rightarrow \infty$. In the mean-field limit $\tilde{t} \rightarrow \infty$ we have

$$\tilde{\chi}^{-1} = c_{\rho 0}^2 c_{t0} \tilde{t} \left[1 - \frac{g_2}{\sqrt{\alpha^2 \tilde{t}}} + O(\tilde{t}^{-1}) \right], \quad (6.18)$$

where $g_2 \approx 0.311$. Using the results¹¹ of the fit of Ref. [23], $\bar{u}_0 = 1.22/36$, $c_{t0} = 1.72/6$, $\Lambda = \pi$, we have

$$\tilde{\chi}^{-1} = c_{\rho 0}^2 c_{t0} \tilde{t} \left[1 - \frac{a}{\sqrt{\tilde{t}}} + O(\tilde{t}^{-1}) \right], \quad (6.19)$$

with $a \approx 0.062$, to be compared with the exact result, cf. Eq. (4.66), $a = 1/(4\pi) \approx 0.0796$. Notice that if we wish to reproduce the correct behaviour for $\tilde{t} \rightarrow \infty$, we should also require $c_{\rho 0}^2 c_{t0} = 1$. Analogously for $\tilde{t} \rightarrow 0$ we have

¹¹ Notice that we use a different normalization for R^2 : our R^2 is $1/6$ of R^2 used in Ref. [19].

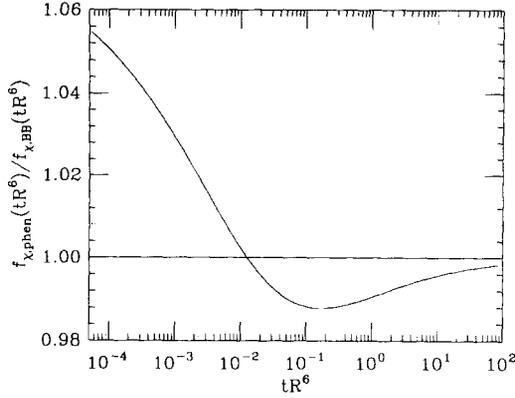


Fig. 7. Ratio $f_{\chi,\text{phen}}(\tilde{t})/f_{\chi,\text{BB}}(\tilde{t})$ as a function of \tilde{t} .

$$\tilde{\chi}^{-1} = c_{\rho 0}^2 c_{t 0} \left(1 + \frac{u^* \nu}{2} \right) \alpha^{2(\gamma-1)} \tilde{t}^\gamma \left(1 - g_1 \alpha^{2\Delta} \tilde{t}^\Delta + O(\tilde{t}^{2\Delta}) \right), \tag{6.20}$$

where $g_1 \approx 0.618$. Using $c_{\rho 0}^2 c_{t 0} = 1$, we obtain numerically

$$\tilde{\chi}^{-1} = 2.49 \tilde{t}^\gamma \left(1 - 3.42 \tilde{t}^\Delta + O(\tilde{t}^{2\Delta}) \right), \tag{6.21}$$

to be compared with

$$\tilde{\chi}^{-1} = (2.70 \pm 0.04) \tilde{t}^\gamma \left(1 - (4.0 \pm 0.1) \tilde{t}^\Delta + O(\tilde{t}^{2\Delta}) \right), \tag{6.22}$$

obtained using the results of Ref. [7] and Eqs. (4.71), (4.72). Finally we report in Fig. 7

$$\Delta_\chi(\tilde{t}) = \frac{f_{\chi,\text{phen}}(\tilde{t})}{f_{\chi,\text{BB}}(\tilde{t})}, \tag{6.23}$$

where $f_{\chi,\text{phen}}(\tilde{t})$ is given by Eq. (6.15) with the numerical values of Ref. [23], and $f_{\chi,\text{BB}}(\tilde{t})$ is obtained using the expressions of Ref. [7] and fixing the non-universal constants with the help of Eqs. (4.71), (4.72) (therefore $f_{\chi,\text{BB}}(\tilde{t})$ does not have any free parameter). The agreement is overall good – the difference is less than 1.5% – except in a small neighbourhood of the Wilson–Fisher point where the difference increases to 8% as it can be seen comparing Eqs. (6.21) and (6.22). Notice however that the region where the discrepancies are large is outside the domain investigated in the Monte Carlo simulation of Ref. [19]. Similar discrepancies were already observed in Ref. [13].

Let us now consider the corrections to the leading behaviour. If we use the expressions (6.14) we find corrections of order R^{-4} , in contrast with the theoretical analysis we have presented. However, there is a simple modification that gives the correct corrections and that does not change the leading behaviour we have discussed before. It is enough to assume that, for $R \rightarrow \infty$,

$$\bar{u} \rightarrow \frac{\bar{u}_0}{R^3}, \quad c_t \rightarrow c_{t0}, \quad c_\rho \rightarrow c_{\rho 0}. \tag{6.24}$$

Notice that this scaling of \bar{u} is more natural, since, as we discussed in the Introduction and in Ref. [20], any coupling constant u should scale as R^{-3} in the crossover limit. Observe also that if one wishes to keep the interpretation of $\kappa(t)$ as an inverse correlation length, then $\kappa(t) \sim (R/\xi)$, i.e. $1/\kappa(t)$ is a correlation measured in units of the interaction range. Using these rescalings, we can write

$$\tilde{\chi} = f_\chi(\tilde{t}) + \frac{\bar{u}_0}{R^3} g_\chi(\tilde{t}) + O(R^{-6}), \quad (6.25)$$

where $g_\chi(\tilde{t})$ depends only on $\alpha^2 \tilde{t}$ apart from a multiplicative constant. By means of an explicit computation we obtain

$$\begin{aligned} \frac{g_\chi(\tilde{t})}{f_\chi(\tilde{t})} = & -\frac{2(\gamma-1)Y_0(\tilde{t})}{(2\Delta-1)Y_0(\tilde{t})+1} \\ & - \left[(2\Delta-1)Y_0(\tilde{t})+1 + \frac{u^* \nu}{2}(1-Y_0(\tilde{t})) \right]^{-1} \frac{\Delta u^* \nu Y_0(\tilde{t})(1-Y_0(\tilde{t}))}{[(2\Delta-1)Y_0(\tilde{t})+1]^2}. \end{aligned} \quad (6.26)$$

For $\tilde{t} \rightarrow \infty$ we have

$$\frac{g_\chi(\tilde{t})}{f_\chi(\tilde{t})} \rightarrow -\frac{\gamma-1}{\Delta} \approx -0.45, \quad (6.27)$$

while, for $\tilde{t} \rightarrow 0$, we have

$$\frac{g_\chi(\tilde{t})}{f_\chi(\tilde{t})} \approx -g_1 \alpha^{2\Delta} \tilde{t}^\Delta. \quad (6.28)$$

The behaviour $g_\chi(\tilde{t})/f_\chi(\tilde{t}) \sim \tilde{t}^\Delta$ for $\tilde{t} \rightarrow 0$ is not what one should expect in general, see Fig. 1 for an example in the large- N limit, and it is related to our assumptions on c_ρ and c_t . If we include $1/R^3$ corrections, i.e. assume

$$c_\rho = c_{\rho 0} + \frac{c_{\rho 1}}{R^3}, \quad c_t = c_{t 0} + \frac{c_{t 1}}{R^3}, \quad (6.29)$$

then $g_\chi(\tilde{t})/f_\chi(\tilde{t})$ would tend to a non-vanishing constant for $\tilde{t} \rightarrow 0$. For $\tilde{t} \rightarrow \infty$, we should compare Eq. (6.27) with the exact result $g_\chi(\tilde{t})/f_\chi(\tilde{t}) = -E_3/\bar{u}_0$ derived in Section 5.1.

A graph of $g_\chi(\tilde{t})/f_\chi(\tilde{t})$ as a function of $\alpha^2 \tilde{t}$ is reported in Fig. 8. It shows a behaviour analogous to that found in the large- N limit, and also the numerical size of the corrections is similar.

It should be emphasized that the function $g_\chi(\tilde{t})$ is non-universal and that it cannot be determined in continuum field theory. Therefore the expression (6.26) cannot be justified and represents some natural – but nonetheless totally arbitrary – generalization of the field-theory results. For the model at hand it provides a reasonable qualitative approximation, but this is not true for any model one can consider. For instance, in the large- N limit, the ratio $g_\chi(\tilde{t})/f_\chi(\tilde{t})$ can be either decreasing or increasing, see Fig. 1,

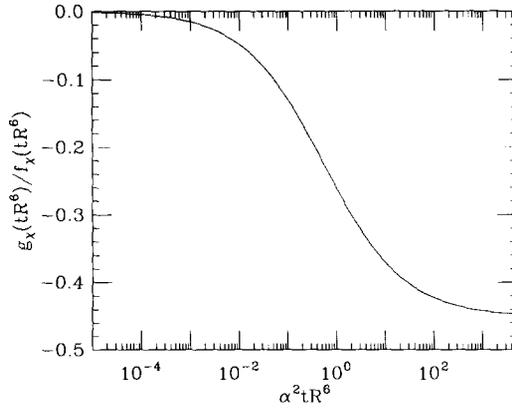


Fig. 8. $g_\chi(\tilde{t})/f_\chi(\tilde{t})$ in the model of Ref. [23] as a function of $\alpha^2\tilde{t}$.

depending on the Hamiltonian. On the other hand, the function in Eq. (6.26) is decreasing for any choice of the parameters. Therefore this phenomenological extension is not even guaranteed to be qualitatively correct. If one is interested in phenomenological interpolations that can describe the crossover even outside the universal regime, one could proceed in a more straightforward way, distinguishing clearly what can be predicted using the field-theory approach (the limiting universal curve) and what is introduced phenomenologically (the corrections to the universal behaviour). For instance one could use the essentially exact $f_\chi(\tilde{t})$ derived from perturbative field theory and any arbitrary reasonable definition for the corrections depending on some parameters that could be fitted to obtain the best agreement between data and model. In this way one could also obtain good phenomenological interpolations of the numerical (or experimental) data.

Finally we wish to comment on the role played by the terms $[1 + \Lambda^2/\kappa(t)^2]$ appearing in Eqs. (6.11), (6.13). In our large- R expansion they can be simply replaced by $\Lambda^2/\kappa(t)^2$ with corrections of order R^{-6} (of order R^{-8} with the original scalings). Therefore, these terms that were introduced in Refs. [12,52] in order to improve the behaviour in the mean-field region, represents a way to introduce additional corrections of order R^{-6} . In the analysis of the numerical results of Ref. [19] they play little role, since

$$\frac{\kappa(t)^2}{\Lambda^2} = \frac{c_t}{\Lambda^2} t Y(t)^{(2\nu-1)/\Delta} \approx 0.17 \frac{t}{6R^2} Y(t)^{(2\nu-1)/\Delta} \tag{6.30}$$

and $t \lesssim 0.05$, $Y(t) \lesssim 1$, $6R^2 \gtrsim 1$. In practice the leading term and the first correction $g_\chi(\tilde{t})$ already provide a good interpolation of the data of Ref. [19].

Acknowledgments

We thank Erik Luijten for useful correspondence and for sending us the numerical results of Ref. [18].

Appendix A. Integrals with medium-range propagators

A.1. Lattice propagators

In this appendix we will compute the quantity

$$\sum_{x \in D} e^{ik \cdot x}, \tag{A.1}$$

for two choices of interaction domain $D \subset \mathbb{Z}^d$. For integer ρ we define

$$D_\rho^{(1)} \equiv \{x \in \mathbb{Z}^d : |x_i| \leq \rho \text{ for } i: 1, \dots, d\}, \tag{A.2}$$

$$D_\rho^{(2)} \equiv \left\{ x \in \mathbb{Z}^d : \sum_{i=1}^d |x_i| \leq \rho \right\}. \tag{A.3}$$

In order to compare with the numerical results of Refs. [17–19,22] we will be also interested in the following family of domains

$$D_\rho^{(3)} \equiv \left\{ x \in \mathbb{Z}^d : \sum_{i=1}^d x_i^2 \leq \rho^2 \right\}. \tag{A.4}$$

We will not be able to compute (A.1) for this class of domains. However, we will obtain some numerical results that will be used in the main text.

Let us compute (A.1) for $D_\rho^{(1)}$. The computation is trivial and we obtain

$$\Omega_{\rho,d}^{(1)} \equiv \sum_{x \in D^{(1)}} e^{ik \cdot x} = \prod_{i=1}^d \frac{\sin k_i L}{\sin(k_i/2)}, \tag{A.5}$$

where $L = \rho + 1/2$. Correspondingly we find

$$V_\rho = (2\rho + 1)^d = (2L)^d, \quad R^2 = \frac{1}{6} \rho(\rho + 1) = \frac{4L^2 - 1}{24}. \tag{A.6}$$

Let us now consider the second case. The computation is now much more involved. The result can be expressed in terms of the determinant of two d -dimensional matrices. Define

$$A_{ij} \equiv \begin{cases} (\cos k_j)^{i-1} & \text{for } i = 1, \dots, d-1, j = 1, \dots, d; \\ f_d(k_j) & \text{for } i = d, j = 1, \dots, d; \end{cases} \tag{A.7}$$

$$B_{ij} \equiv (\cos k_j)^{i-1}. \tag{A.8}$$

Then

$$\Omega_{\rho,d}^{(2)} \equiv \sum_{x \in D^{(2)}} e^{ik \cdot x} = \frac{\det A}{\det B}. \tag{A.9}$$

The result depends on the function of a single variable $f_d(k)$ given by

$$f_d(k) = -2 \cos \frac{k}{2} (\sin k)^{d-2} \cos \left(kL + \frac{d\pi}{2} \right) \tag{A.10}$$

where, as before, $L = \rho + 1/2$. Explicitly in two and three dimensions we have

$$f_2(k) = 2 \cos \frac{k}{2} \cos kL, \tag{A.11}$$

$$f_3(k) = -2 \cos \frac{k}{2} \sin k \sin kL. \tag{A.12}$$

Expanding in powers of k it is possible to compute V_ρ and R . In two dimensions we obtain

$$V_\rho = 2\rho^2 + 2\rho + 1 = \frac{1}{2}(4L^2 + 1), \tag{A.13}$$

$$V_\rho R^2 = \frac{1}{6}\rho(1 + \rho)(1 + \rho + \rho^2) = \frac{1}{96}(4L^2 + 3)(4L^2 - 1). \tag{A.14}$$

In three dimensions we have

$$V_\rho = \frac{1}{3}(2\rho + 1)(2\rho^2 + 2\rho + 3) = \frac{L}{3}(4L^2 + 5), \tag{A.15}$$

$$V_\rho R^2 = \frac{1}{30}\rho(\rho + 1)(2\rho + 1)(\rho^2 + \rho + 3) = \frac{1}{240}L(4L^2 - 1)(4L^2 + 11). \tag{A.16}$$

For large values of ρ one finds

$$V_\rho \rightarrow \frac{(2L)^d}{d!}, \quad R^2 \rightarrow \frac{L^2}{(d + 1)(d + 2)}. \tag{A.17}$$

To prove Eq. (A.9), let us suppose that the result has the form

$$\Omega_{\rho,d}^{(2)}(k) = i^d \sum_{i=1}^d \alpha_{i,d} (e^{ik_i L} + \beta_{i,d} e^{-ik_i L}), \tag{A.18}$$

where $\alpha_{i,d}$ and $\beta_{i,d}$ depend on k but not on L . This Ansatz is a natural generalization of the result that can be obtained in two and three dimensions by direct computation. Using the fact that

$$\Omega_{\rho,d}^{(2)}(k_1, \dots, k_d) = \sum_{n=-\rho}^{\rho} e^{ik_d n} \Omega_{\rho-|n|,d-1}^{(2)}(k_1, \dots, k_{d-1}), \tag{A.19}$$

we obtain $\beta_{i,d} = (-1)^d$ and the following recursion relations:

$$\alpha_{i,d} = \alpha_{i,d-1} \frac{\sin k_i}{\cos k_i - \cos k_d}, \tag{A.20}$$

$$\alpha_{d,d} = \sum_{i=1}^{d-1} \alpha_{i,d-1} \frac{1}{\cos k_i - \cos k_d} \left[\sin \left(\frac{k_i - k_d}{2} \right) - (-1)^d \sin \left(\frac{k_i + k_d}{2} \right) \right], \tag{A.21}$$

where in Eq. (A.20) $i = 1, \dots, d - 1$. Using $\alpha_{1,1} = (2 \sin(k/2))^{-1}$, Eq. (A.20), and the obvious symmetry under permutation of the labels of the coordinates, we obtain, for $1 \leq i \leq d$,

$$\alpha_{i,d} = \frac{(\sin k_i)^{d-2}}{\prod_{j=1, j \neq i}^d (\cos k_i - \cos k_j)} \cos \frac{k_i}{2}. \tag{A.22}$$

We should now prove that this expression solves Eq. (A.21) which is the consistency condition of the Ansatz (A.18). Assuming d even, we can rewrite Eq. (A.21) as

$$\sum_{i=1}^d \frac{(\sin k_i)^{d-2}}{\prod_{j=1, j \neq i}^d (\cos k_i - \cos k_j)} = 0. \tag{A.23}$$

Let us now use the following result: given x_1, \dots, x_n , consider the n -dimensional matrix $M_{ij} = x_i^{j-1}$. Then it is easy to see that (in the mathematical literature this determinant is known as Vandermonde determinant)

$$\det M = (-1)^{n(n-1)/2} \prod_{i=1}^{n-1} \prod_{j=i+1}^n (x_i - x_j). \tag{A.24}$$

Now define the matrix

$$C_{ij} \equiv \begin{cases} (\cos k_j)^{i-1} & \text{for } i = 1, \dots, d - 1; \\ (\sin k_j)^{d-2} & \text{for } i = d. \end{cases} \tag{A.25}$$

Using Eq. (A.24), it is easy to convince oneself that Eq. (A.23) can be written as

$$\frac{\det C}{\det B} = 0. \tag{A.26}$$

Since d is even, one can express $(\sin k_j)^{d-2}$ as a sum of $\cos^{2i} k_j$, with $0 \leq i \leq d - 1$. Thus the last row of C is a linear combination of the previous rows and therefore $\det C = 0$. When d is odd the discussion is analogous. We have thus proved that the consistency condition (A.21) is satisfied. Therefore the Ansatz (A.18) with α_{id} given by Eq. (A.22) is the solution of the recurrence relation (A.19) that uniquely defines $\Omega_{\rho,d}^{(2)}(k)$. Using again Eq. (A.24) we obtain the result (A.9).

If $J_\rho(x)$ defined in Section 3 is given by Eq. (3.8), then $J_\rho(q) = \Omega_{\rho,d}(q)$. We wish now to prove that $\Omega_{\rho,d}(q)$ satisfies the properties mentioned at the beginning of Section 3. Property (i) is obvious, while properties (ii) and (iii) depend on D_ρ : they are satisfied if $V_\rho \sim R^d$ and if, for any $x \in \mathbb{Z}^d$, there exist¹² $x_1, \dots, x_d \in D_\rho$ such that $x = \sum_i \alpha_i x_i$, $\alpha_i \in \mathbb{Z}$. To check the fourth property, define $v \equiv kR$ and consider the limit of $\Omega_{\rho,d}(v/R)$ at fixed v . An easy computation for the domains $D^{(1)}$ and $D^{(2)}$ gives

$$\frac{\Omega_{\rho,d}(v/R)}{V_R} \rightarrow \Omega_0(v) \tag{A.27}$$

¹² Notice that it is not sufficient that $V_\rho \sim R^d$ to ensure property (iii). For instance consider in one dimension the set $D_\rho = \{x = 2n, n \in \mathbb{Z}, |n| \leq \rho\}$.

with

$$\Omega_0(v) = \prod_{i=1}^d \frac{\sin u_i}{u_i}, \tag{A.28}$$

$$\Omega_0(v) = -d! \sum_{i=1}^d \frac{u_i^{d-2} \cos(u_i + d\pi/2)}{\prod_{j \neq i} (u_j^2 - u_i^2)}, \tag{A.29}$$

where

$$u = v \lim_{R \rightarrow \infty} \frac{L}{R}. \tag{A.30}$$

Notice that in two dimensions there is a simple relation for $\Omega_0(v)$ for the two domains $D^{(1)}$ and $D^{(2)}$. Indeed

$$\Omega_0^{(2)}(u_x, u_y) = \Omega_0^{(1)}\left(\frac{u_x + u_y}{2}, \frac{u_x - u_y}{2}\right). \tag{A.31}$$

For the family of domains $D^{(3)}$, $\Omega_0(v)$ was computed in Appendix A of Ref. [17] finding

$$\Omega_0(v) = \Gamma\left(\frac{d}{2} + 1\right) \left(\frac{|u|}{2}\right)^{-d/2} J_{d/2}(|u|), \tag{A.32}$$

where

$$u_i = \sqrt{2(d+2)} v_i. \tag{A.33}$$

From these expressions it is easy to see that $\Pi(q) = 1 - \Omega_0(q)$ satisfies condition (iv).

For $D^{(1)}$ and $D^{(2)}$ it is also easy to show explicitly properties (v) and (vi) of Section 5. Indeed in the limit $R \rightarrow \infty$ at v fixed we have

$$\frac{1}{V_\rho} \Omega_{\rho,d}(k) = \sum_{n=0}^{\infty} \frac{1}{R^{2n}} \Omega_n(v), \tag{A.34}$$

without odd powers of $1/R$. Indeed, from the explicit results we immediately see that $\Omega_{\rho,d}(k)$ is even under the transformations $k \rightarrow -k$ and $L \rightarrow -L$. Moreover L^2 is an analytic function of R^2 . Therefore for v fixed, $\Omega_{\rho,d}(k)$ is even in R , proving Eq. (A.34). Since we used the explicit expressions we computed before this proof applies only to the two cases we have studied.

A.2. One-loop integrals

Let us now consider, for $d < 4$, the following class of one-loop integrals:

$$I_{1,R}(m^2) \equiv \int \frac{d^d k}{(2\pi)^d} \frac{1 - \overline{\Pi}_R(k)}{\overline{\Pi}_R(k) + m^2}, \tag{A.35}$$

$$J_{1,R}(m^2) \equiv \int \frac{d^d k}{(2\pi)^d} \frac{1 - \overline{\Pi}_R(k)}{(\overline{\Pi}_R(k) + m^2)^2}, \quad (\text{A.36})$$

where $\overline{\Pi}_R(q)$ is a function with the properties mentioned at the beginning of Section 3 and 5. We will be interested in computing these integrals in the crossover limit that corresponds to $R \rightarrow \infty$, $m \rightarrow 0$, with $m^2 R^{2d/(4-d)}$ fixed. The integral (A.35) exists for $m = 0$ only for $d > 2$, while the second one is always infrared divergent. In order to analyze the asymptotic behaviour of these integrals we will distinguish three cases: (a) $d > 2$; (b) $d < 2$; (c) $d = 2$.

A.2.1. Case (a): $2 < d < 4$

For $d > 2$ the integral (A.35) is well defined for $m^2 \rightarrow 0$ and thus we begin by studying

$$\overline{I}_{1,R} \equiv \lim_{m^2 \rightarrow 0} I_{1,R}(m^2). \quad (\text{A.37})$$

We wish to compute its asymptotic behaviour for $R \rightarrow \infty$. Defining $p = kR$, we rewrite

$$\overline{I}_{1,R} = \frac{1}{R^d} \int \frac{d^d p}{(2\pi)^d} \frac{1 - \overline{\Pi}_R(p/R)}{\overline{\Pi}_R(p/R) + m^2} \rightarrow \frac{1}{R^d} \int \frac{d^d p}{(2\pi)^d} \frac{1 - \Pi(p)}{\Pi(p)}, \quad (\text{A.38})$$

using property (iv). The last integral can be extended over all \mathbb{R}^d . Thus we obtain

$$\overline{I}_{1,R} \approx \frac{\sigma}{R^d} \quad (\text{A.39})$$

with

$$\sigma \equiv \int \frac{d^d p}{(2\pi)^d} \frac{1 - \Pi(p)}{\Pi(p)}. \quad (\text{A.40})$$

If additionally we assume that $\overline{\Pi}_R(q)$ satisfies properties (v) and (vi) of Section 5, we can easily prove that $\overline{I}_{1,R}$ admits an expansion of the form

$$\overline{I}_{1,R} = \frac{\sigma}{R^d} + \frac{1}{R^d} \sum_{n=1}^{\infty} \frac{\sigma_n}{R^{2n}}. \quad (\text{A.41})$$

In three dimensions, for $D^{(1)}$, $D^{(2)}$, $D^{(3)}$, we have respectively¹³

$$\begin{aligned} \overline{I}_{1,R} &= \frac{0.0435562069}{R^3} + O(R^{-5}), \\ \overline{I}_{1,R} &= \frac{0.04336529}{R^3} + O(R^{-5}), \\ \overline{I}_{1,R} &= \frac{0.04139}{R^3} + o(R^{-3}). \end{aligned} \quad (\text{A.42})$$

Estimates of $\overline{I}_{1,R}$ for various values of R are reported in Table A.1.

¹³ Luijten [22] has noticed that an approximate expression of σ can be obtained using the numerical results of Ref. [47]. He writes $\sigma \approx 4.46 \lim_{R \rightarrow \infty} R^3/V_R$. For the three domains one obtains 0.0379, 0.0374, 0.0337 respectively; these estimates are not very far from the exact values.

Table A.1

Estimates of $R^3 \bar{I}_{1,R}$ for various values of ρ for the three domains introduced in the text

ρ	$D^{(1)}$	$D^{(2)}$	$D^{(3)}$
3	0.042971778	0.043960387	0.041600702
4	0.043202728	0.043921767	0.041279504
5	0.043319601	0.043713672	0.041423800
6	0.043386811	0.043664053	0.041387988
7	0.043428975	0.043574469	0.041394901
8	0.043457153	0.043547206	0.041384933
10	0.043491295	0.043486698	0.041398502
12	0.043510406	0.043451767	0.041386965
14	0.043522170	0.043429899	0.041392394
16	0.043529921	0.043415345	0.041389669
18	0.043535297	0.043405187	0.041392740
20	0.043539178	0.043397824	0.041391612

Let us now go back to $I_{1,R}(m^2)$ and let us compute the leading correction depending on m^2 in the crossover limit. We rewrite

$$I_{1,R}(m^2) = \bar{I}_{1,R} - m^2 \int \frac{d^d q}{(2\pi)^d} \frac{1 - \bar{\Pi}_R(q)}{\bar{\Pi}_R(q)(\bar{\Pi}_R(q) + m^2)}. \tag{A.43}$$

Setting $q = v/R$ and using property (iv) of $\bar{\Pi}_R(q)$ we find the leading term as $R \rightarrow \infty$

$$I_{1,R}(m^2) \approx \bar{I}_{1,R} - \frac{m^2}{R^d} \int \frac{d^d v}{(2\pi)^d} \frac{1 - \Pi(v)}{\Pi(v)(\Pi(v) + m^2)}, \tag{A.44}$$

where the integration is extended over \mathbb{R}^d . If $\bar{\Pi}_R(q)$ satisfies properties (v) and (vi) the neglected terms are of order m^2/R^{d+2} [in three dimensions they are of order R^{-11}]. Now, for $d < 4$, the last integral is infrared divergent for $m \rightarrow 0$. Thus the leading contribution in the limit $m \rightarrow 0$ is obtained replacing $\Pi(v)$ with v^2 in the denominator and with 1 in the numerator. We have therefore

$$I_{1,R}(m^2) \approx \bar{I}_{1,R} - \frac{m^2}{R^d} \int \frac{d^d v}{(2\pi)^d} \frac{1}{v^2(v^2 + m^2)} - \frac{m^2}{R^d} \int \frac{d^d v}{(2\pi)^d} \left[\frac{1 - \Pi(v)}{\Pi(v)(\Pi(v) + m^2)} - \frac{1}{v^2(v^2 + m^2)} \right]. \tag{A.45}$$

The first integral can be computed exactly, while in the second one we can simply take the limit $m \rightarrow 0$. We have finally

$$I_{1,R}(m^2) \approx \bar{I}_{1,R} + (4\pi)^{-d/2} \Gamma\left(1 - \frac{d}{2}\right) \frac{m^{d-2}}{R^d} + E_d \frac{m^2}{R^d} + o(R^{-d(6-d)/(4-d)}), \tag{A.46}$$

where

$$E_d \equiv - \int \frac{d^d v}{(2\pi)^d} \left[\frac{1 - \Pi(v)}{\Pi(v)^2} - \frac{1}{(v^2)^2} \right]. \tag{A.47}$$

Numerically, in three dimensions, we have for the interaction (3.8) and the domains $D^{(1)}$, $D^{(2)}$ and $D^{(3)}$

$$\begin{aligned} E_3 &= 0.058391 \quad \text{for } D^{(1)}, \\ E_3 &= 0.058545 \quad \text{for } D^{(2)}, \\ E_3 &= 0.0635 \quad \text{for } D^{(3)}. \end{aligned} \tag{A.48}$$

The computation of $J_{1,R}(m^2)$ is analogous. We obtain

$$J_{1,R}(m^2) \approx (4\pi)^{-d/2} \Gamma\left(2 - \frac{d}{2}\right) \frac{m^{d-4}}{R^d} - \frac{E_d}{R^d} + o(R^{-d}). \tag{A.49}$$

A.2.2. Case (b): $d < 2$

Let us now consider $I_{1,R}(m^2)$ for $d < 2$. In this case the integral is infrared divergent. Following the previous steps, we have

$$\begin{aligned} I_{1,R}(m^2) &\approx \frac{1}{R^d} \int \frac{d^d v}{(2\pi)^d} \frac{1 - \Pi(v)}{\Pi(v) + m^2} + O(R^{-d-2}) \\ &= \frac{1}{R^d} \int \frac{d^d v}{(2\pi)^d} \frac{1}{v^2 + m^2} + \frac{1}{R^d} \int \frac{d^d v}{(2\pi)^d} \left[\frac{1 - \Pi(v)}{\Pi(v) + m^2} - \frac{1}{v^2 + m^2} \right] \\ &\approx (4\pi)^{-d/2} \Gamma\left(1 - \frac{d}{2}\right) \frac{m^{d-2}}{R^d} + \frac{1}{R^d} \int \frac{d^d v}{(2\pi)^d} \left[\frac{1 - \Pi(v)}{\Pi(v)} - \frac{1}{v^2} \right] \\ &\quad + o(R^{-d}). \end{aligned} \tag{A.50}$$

Analogously

$$J_{1,R}(m^2) \approx (4\pi)^{-d/2} \Gamma\left(2 - \frac{d}{2}\right) \frac{m^{d-4}}{R^d} + O(m^{d-2} R^{-d}). \tag{A.51}$$

A.2.3. Case (c): $d = 2$

Let us now consider the case $d = 2$. Also in this case the integral is infrared divergent for $m^2 \rightarrow 0$. However, we cannot proceed as in the case $d < 2$, since the subtracted integral in Eq. (A.50) is ultraviolet divergent. First we set $q = v/R$ and expand $\overline{\Pi}_R(v/R)$ obtaining

$$\begin{aligned} I_{1,R}(m^2) &\approx \frac{1}{R^2} \int \frac{d^2 v}{(2\pi)^2} \frac{1 - \Pi(v)}{\Pi(v) + m^2} - \frac{1 + m^2}{R^4} \int \frac{d^2 v}{(2\pi)^2} \frac{\Pi_1(v)}{(\Pi(v) + m^2)^2} \\ &\quad + O(R^{-6}). \end{aligned} \tag{A.52}$$

Since $\Pi_1(v) \sim v^4$, we can simply take the limit $m \rightarrow 0$ in the last term. Let us now deal with the first one. Because of the lattice symmetry and of property (v) we have, for $v \rightarrow 0$,

$$\Pi(v) = v^2 + \alpha_1(v^2)^2 + \alpha_2 v^4 + o(v^4), \tag{A.53}$$

where $v^4 = v_1^4 + v_2^4$. Then we rewrite

$$\int \frac{d^2v}{(2\pi)^2} \frac{1 - \Pi(v)}{\Pi(v) + m^2} = \int \frac{d^2v}{(2\pi)^2} \left\{ \frac{1 - \Pi(v)}{\Pi(v) + m^2} - \frac{e^{-v^2}}{v^2 + m^2} + \frac{e^{-v^2} [\alpha_1(v^2)^2 + \alpha_2v^4]}{(v^2 + m^2)^2} \right\} + \int \frac{d^2v}{(2\pi)^2} \left\{ \frac{e^{-v^2}}{v^2 + m^2} - \frac{e^{-v^2} [\alpha_1(v^2)^2 + \alpha_2v^4]}{(v^2 + m^2)^2} \right\}. \tag{A.54}$$

The first integral can be expanded in powers of m^2 neglecting terms of order $m^4 \log m^2$, while the second one can be computed exactly. We obtain finally

$$\int \frac{d^2v}{(2\pi)^2} \frac{1 - \Pi(v)}{\Pi(v) + m^2} = -\frac{1}{4\pi} (\log m^2 + \gamma_E) - \frac{m^2}{4\pi} (\log m^2 + \gamma_E - 1) - \frac{m^2}{16\pi} (4\alpha_1 + 3\alpha_2) (2 \log m^2 + 2\gamma_E + 1) + \int \frac{d^2v}{(2\pi)^2} \left[\frac{1 - \Pi(v)}{\Pi(v)} - \frac{e^{-v^2}}{v^2} \right] - m^2 \int \frac{d^2v}{(2\pi)^2} \left\{ \frac{1 - \Pi(v)}{\Pi(v)^2} - \frac{e^{-v^2}}{(v^2)^2} + \frac{2e^{-v^2} [\alpha_1(v^2)^2 + \alpha_2v^4]}{(v^2)^3} \right\} + O(R^{-4} \log R^2) \tag{A.55}$$

We obtain finally, up to terms $o(R^{-4})$,

$$I_{1,R}(m^2) \approx -\frac{1}{4\pi R^2} \log m^2 + \frac{C_2}{R^2} - \frac{m^2}{8\pi R^2} (4\alpha_1 + 3\alpha_2 + 2) \log m^2 + \frac{m^2 G_1}{R^2} + \frac{G_2}{R^4}, \tag{A.56}$$

where

$$C_2 \equiv -\frac{\gamma_E}{4\pi} + \int \frac{d^2v}{(2\pi)^2} \left[\frac{1 - \Pi(v)}{\Pi(v)} - \frac{e^{-v^2}}{v^2} \right], \tag{A.57}$$

and G_1 and G_2 can be computed from Eqs. (A.52) and (A.55). For the interaction defined in Eq. (3.8) and for the domains $D^{(1)}$, $D^{(2)}$, $D^{(3)}$ we have respectively

$$C_2 = -0.04578786 \quad \text{for } D^{(1)} \text{ and } D^{(2)}, \\ C_2 = -0.05045 \quad \text{for } D^{(3)}. \tag{A.58}$$

The equality of C_2 for the domains $D^{(1)}$ and $D^{(2)}$ follows from the identity (A.31). Analogously

$$J_{1,R}(m^2) = \frac{1}{4\pi m^2 R^2} + \frac{1}{8\pi R^2} (4\alpha_1 + 3\alpha_2 + 2) (\log m^2 + 1) - \frac{G_1}{R^2} + o(R^{-2}). \tag{A.59}$$

A.3. The two-loop integral

We wish now to discuss the two-loop integral

$$I_{2,R}(m^2) \equiv \int \frac{d^d q}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{(1 - \overline{\Pi}_R(q))(1 - \overline{\Pi}_R(k))(1 - \overline{\Pi}_R(q+k))}{(\overline{\Pi}_R(q) + m^2)(\overline{\Pi}_R(k) + m^2)(\overline{\Pi}_R(q+k) + m^2)}, \quad (\text{A.60})$$

in the crossover limit $R \rightarrow \infty$, $m \rightarrow 0$ with $m^2 R^{2d/(4-d)}$ fixed.

Since we wish to compute the integral in the large- R limit, we can rescale the internal momenta and use properties (iv), (vi) in order to rewrite $I_{2,R}(m^2)$ in the form

$$I_{2,R}(m^2) \approx \frac{1}{R^{2d}} \int \frac{d^d q}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{(1 - \Pi(q))(1 - \Pi(k))(1 - \Pi(q+k))}{(\Pi(q) + m^2)(\Pi(k) + m^2)(\Pi(q+k) + m^2)} - \frac{3}{R^{2d+2}} \int \frac{d^d q}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{\Pi_1(q)(1 - \Pi(k))(1 - \Pi(q+k))}{(\Pi(q) + m^2)^2(\Pi(k) + m^2)(\Pi(q+k) + m^2)}, \quad (\text{A.61})$$

where the integrals are extended over \mathbb{R}^{2d} . The neglected terms are of order $O(R^{2d+4})$ and $O(R^{2d+4}m^{2d-4})$. $I_{2,R}(m^2)$ is infrared divergent for $d \leq 3$ and therefore we will distinguish three cases: (a) $d > 3$; (b) $d < 3$; (c) $d = 3$.

A.3.1. Case (a): $3 < d < 4$

For $d > 3$ the integral is finite for $m^2 \rightarrow 0$. In analogy with the discussion of Appendix A.2.1 we have

$$I_{2,R}(m^2) \approx \bar{I}_{2,R} + \frac{m^{2d-6}}{R^{2d}} \int \frac{d^d q}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \times \left[\frac{1}{(q^2+1)(k^2+1)((q+k)^2+1)} - \frac{1}{q^2 k^2 (q+k)^2} \right] + o(R^{-2d}), \quad (\text{A.62})$$

where

$$\begin{aligned} \bar{I}_{2,R} &= \int \frac{d^d q}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{(1 - \overline{\Pi}_R(q))(1 - \overline{\Pi}_R(k))(1 - \overline{\Pi}_R(q+k))}{\overline{\Pi}_R(q)\overline{\Pi}_R(k)\overline{\Pi}_R(q+k)} \\ &\approx \frac{1}{R^{2d}} \int \frac{d^d q}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{(1 - \Pi(q))(1 - \Pi(k))(1 - \Pi(q+k))}{\Pi(q)\Pi(k)\Pi(q+k)} \\ &\quad + O(R^{-2d-2}). \end{aligned} \quad (\text{A.63})$$

A.3.2. Case (b): $d < 3$

For $d < 3$ the integral is infrared divergent for $m^2 \rightarrow 0$. As we did in Section A.2.2 we have

$$I_{2,R}(m^2) \approx \frac{m^{2d-6}}{R^{2d}} \int \frac{d^d q}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \frac{1}{(q^2+1)(k^2+1)((q+k)^2+1)}$$

$$\begin{aligned}
 & + \frac{1}{R^{2d}} \int \frac{d^d q}{(2\pi)^d} \frac{d^d k}{(2\pi)^d} \left[\frac{(1 - \Pi(q))(1 - \Pi(k))(1 - \Pi(q+k))}{\Pi(q)\Pi(k)\Pi(q+k)} \right. \\
 & \left. - \frac{1}{q^2 k^2 (q+k)^2} \right] + o(R^{-2d}). \tag{A.64}
 \end{aligned}$$

In two dimensions we have explicitly

$$I_{2,R}(m^2) = \frac{1}{m^2 R^4} \left(\frac{1}{24\pi^2} \psi'(1/3) - \frac{1}{36} \right) + O(R^{-4}). \tag{A.65}$$

A.3.3. Case (c): $d = 3$

For $d = 3$ the integral is infrared divergent for $m^2 \rightarrow 0$. However, we cannot proceed as in the case $d < 3$ because the subtracted integral in Eq. (A.64) is ultraviolet divergent. We will obtain the asymptotic behaviour using the same method we used in two dimensions to deal with $I_{1,R}(m^2)$. We write

$$\begin{aligned}
 I_{2,R}(m^2) &= \frac{1}{R^6} I_2^{\text{cont}}(m^2) + \frac{F_3}{R^8} + \frac{1}{R^6} \int \frac{d^3 q}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} \\
 &\times \left[\frac{(1 - \Pi(q))(1 - \Pi(k))(1 - \Pi(q+k))}{\Pi(q)\Pi(k)\Pi(q+k)} - \frac{e^{-q^2 - k^2 - (q+k)^2}}{q^2 k^2 (q+k)^2} \right], \tag{A.66}
 \end{aligned}$$

where

$$I_2^{\text{cont}}(m^2) \equiv \int \frac{d^3 q}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} \frac{e^{-q^2 - k^2 - (q+k)^2}}{(q^2 + m^2)(k^2 + m^2)((q+k)^2 + m^2)}, \tag{A.67}$$

$$F_3 \equiv -3 \int \frac{d^3 q}{(2\pi)^3} \frac{d^3 k}{(2\pi)^3} \frac{\Pi_1(q)(1 - \Pi(k))(1 - \Pi(q+k))}{\Pi(q)^2 \Pi(k)\Pi(q+k)}. \tag{A.68}$$

where all integrals are extended over \mathbb{R}^6 and terms of order $O(R^{-10})$ have been neglected. In order to compute $I_2^{\text{cont}}(m^2)$, let us define

$$\begin{aligned}
 P(x, m) &\equiv \int \frac{d^3 p}{(2\pi)^3} \frac{e^{-p^2 + ip \cdot x}}{p^2 + m^2} \\
 &= -\frac{e^{m^2}}{8\pi|x|} \left[2 \sinh m|x| + e^{-m|x|} \operatorname{erf} \left(m - \frac{|x|}{2} \right) - e^{m|x|} \operatorname{erf} \left(m + \frac{|x|}{2} \right) \right], \tag{A.69}
 \end{aligned}$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt e^{-t^2}. \tag{A.70}$$

Then, for $m \rightarrow 0$,

$$\begin{aligned}
 I_2^{\text{cont}}(m^2) = & -\frac{1}{32\pi^2} [\log(9m^2) + 2\gamma_E] \\
 & + 4\pi \int_0^\infty dx \left[x^2 P^3(x, 0) - \frac{1}{64\pi^3} \frac{1}{x+1} \right] + O(m).
 \end{aligned}
 \tag{A.71}$$

Collecting all terms we have

$$I_{2,R}(m^2) = -\frac{1}{32\pi^2 R^6} \log m^2 + \frac{C_3}{R^6} + \frac{F_3}{R^8} + O(R^{-9}),
 \tag{A.72}$$

where C_3 is defined as

$$\begin{aligned}
 C_3 \equiv & -\frac{1}{16\pi^2} (\log 3 + \gamma_E) + 4\pi \int_0^\infty dx \left[x^2 P^3(x, 0) - \frac{1}{64\pi^3} \frac{1}{x+1} \right] \\
 & + \int \frac{d^3q}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \left[\frac{(1 - \Pi(q))(1 - \Pi(k))(1 - \Pi(q+k))}{\Pi(q)\Pi(k)\Pi(q+k)} \right. \\
 & \left. - \frac{e^{-q^2-k^2-(q+k)^2}}{q^2 k^2 (q+k)^2} \right].
 \end{aligned}
 \tag{A.73}$$

We have computed the constant C_3 for the three different domains introduced at the beginning. We obtain

$$\begin{aligned}
 C_3 = & -0.0127 \quad \text{for } D^{(1)}, \\
 C_3 = & -0.0127 \quad \text{for } D^{(2)}, \\
 C_3 = & -0.0129 \quad \text{for } D^{(3)}.
 \end{aligned}
 \tag{A.74}$$

Appendix B. Large- N limit

In this Appendix we will compute the crossover functions in the large- N limit. In Appendix B.1 we will discuss our general model for $d > 2$, while in Appendix B.2 we will consider the two-dimensional case for the N -vector model.

B.1. Crossover limit for $2 < d < 4$

In this Appendix we will study the model introduced in Section 3 in the large- N limit following the strategy of Refs. [56–58]. We write $V(\varphi) = NW(\varphi^2)$ and study the limit $N \rightarrow \infty$ with β and $W(x)$ fixed. The basic trick consists in rewriting

$$e^{-NW(\varphi^2)} \sim \int d\rho d\lambda \exp \left[-\frac{N}{2} \lambda(\varphi^2 - \rho) - NW(\rho) \right].
 \tag{B.1}$$

The saddle point is given by the equations

$$W'(\rho) = \frac{1}{2}\beta V_R(1+m^2), \quad \rho\beta V_R = \frac{1}{1+m^2}[1+I_{1,R}(m^2)], \tag{B.2}$$

where $I_{1,R}(m^2)$ is defined in Eq. (A.35), while the two-point function is given by

$$\langle\varphi_0\varphi_x\rangle = \frac{1}{\beta}\int\frac{d^d p}{\bar{\Pi}_R(p)+m^2}. \tag{B.3}$$

The critical point corresponds to $m^2 = 0$ and therefore the critical values of ρ and β satisfy the equations

$$W'(\rho_c) = \frac{1}{2}\beta_c V_R, \quad \rho_c\beta_c V_R = 1 + \bar{I}_{1,R}. \tag{B.4}$$

The critical value ρ_c is the solution of the equation

$$1 + \bar{I}_{1,R} = 2\rho_c W'(\rho_c). \tag{B.5}$$

We will not need to solve Eq. (B.5) explicitly, we will only assume $W(x)$ to be such that a positive solution exists. Since $\bar{I}_{1,R} \sim R^{-d}$, for $R \rightarrow \infty$ we can expand

$$\rho_c = \sum_{n=0}^{\infty} \rho_{cn} \bar{I}_{1,R}^n, \tag{B.6}$$

where

$$2\rho_{c0}W'(\rho_{c0}) = 1, \tag{B.7}$$

and $\rho_{c1}, \rho_{c2}, \dots$ can be computed iteratively in terms of the derivatives of $W(\rho)$ computed at $\rho = \rho_{c0}$. Correspondingly we obtain the expansion of β_c for $R \rightarrow \infty$. We have

$$\beta_c V_R = \frac{1}{\rho_{c0}} + \frac{2\rho_{c0}W_2}{1+2\rho_{c0}^2W_2}\bar{I}_{1,R} + \frac{\rho_{c0}^2(W_3-4\rho_{c0}W_2^2)}{(1+2\rho_{c0}^2W_2)^3}\bar{I}_{1,R}^2 + O(R^{-3d}), \tag{B.8}$$

where $W_n = W^{(n)}(\rho_{c0})$. Let us now discuss the scaling behaviour. Using Eqs. (B.2) and (A.46), we have

$$(\rho\beta - \rho_c\beta_c)V_R = A_d\frac{m^{d-2}}{R^d} - m^2 + (E_d - \sigma)\frac{m^2}{R^d} + o(m^2R^{-d}), \tag{B.9}$$

where $A_d = (4\pi)^{-d/2}\Gamma(1-d/2)$ and σ is defined in Eq. (A.40). Now let us introduce¹⁴

$$\delta \equiv \rho - \rho_c, \quad t \equiv \frac{\beta_c - \beta}{\beta_{c, MF}} = \rho_{c0}V_R(\beta_c - \beta). \tag{B.10}$$

Using the first gap equation and Eq. (B.8) we obtain

¹⁴ The normalization of t is chosen so that the results can be directly compared with those of Section 5. One could have defined $t \equiv (\beta_c - \beta)/\beta_c$, as in Section 3. This choice does not change the leading crossover curve, but changes the corrections.

$$\delta = \frac{1}{2\rho_{c0} W''(\rho_c)} \left[m^2 - t + \frac{\sigma}{s_\infty} \frac{m^2}{R^d} \right] + O(t^2, m^4, tm^2, tR^{-2d}), \quad (\text{B.11})$$

where

$$s_\infty = \frac{1 + 2\rho_{c0}^2 W_2}{2\rho_{c0}^2 W_2}, \quad (\text{B.12})$$

and we have assumed $W_2 \neq 0$. Going back to Eq. (B.9) we obtain finally

$$\begin{aligned} & \left(s_\infty + \frac{s_{1,\infty}}{R^d} \right) (m^2 - t) - A_d \frac{m^{d-2}}{R^d} - (E_d - \sigma) \frac{m^2}{R^d} \\ & = O(tm^2, tR^{-2d}, m^2 R^{-2d}), \end{aligned} \quad (\text{B.13})$$

where

$$s_{1,\infty} = \frac{\sigma}{\rho_{c0}^2 W_2 s_\infty} \left(1 - \frac{W_3}{4\rho_{c0} W_2^2} \right). \quad (\text{B.14})$$

Notice that this equation is also valid in the N -vector model with $s_\infty = 1$, $s_{1,\infty} = 0$. Let us now consider the critical crossover limit. We introduce

$$\tilde{m}^2 \equiv m^2 R^{2d/(4-d)}, \quad \tilde{t} \equiv t R^{2d/(4-d)}, \quad (\text{B.15})$$

and expand

$$\tilde{m}^2 = f_m(\tilde{t}) + \frac{1}{R^d} g_m(\tilde{t}) + o(R^{-d}). \quad (\text{B.16})$$

The universal crossover function $f_m(\tilde{t})$ is given by

$$s_\infty [f_m(\tilde{t}) - \tilde{t}] - A_d f_m(\tilde{t})^{(d-2)/2} = 0. \quad (\text{B.17})$$

This result shows clearly the universality of $f_m(\tilde{t})$. Indeed all the model dependence is included in the constant s_∞ that can be eliminated with a proper rescaling of $f_m(\tilde{t})$ and \tilde{t} .

It is important to notice that Eq. (B.17) has a solution for $\tilde{t} \rightarrow 0$ only if $s_\infty > 0$. Indeed we can rewrite the equation as

$$\tilde{t} = -\frac{A_d}{s_\infty} f_m(\tilde{t})^{(d-2)/2} + f_m(\tilde{t}). \quad (\text{B.18})$$

For $\tilde{t} \rightarrow 0$, $f_m(\tilde{t}) \rightarrow 0$, and thus we can neglect the last term in the right-hand side. Since the left-hand side is positive – we are considering the high-temperature phase – we should have $A_d/s_\infty < 0$. Since $A_d < 0$, we obtain $s_\infty > 0$. Notice that, requiring the stability of the free energy in the limit $R \rightarrow \infty$ one obtains the condition $W_2 > 0$, which, using Eq. (B.12), gives again $s_\infty > 0$. It is important to remark, as we shall show below, that this condition is equivalent to the requirement $\bar{a}_4 < 0$ that was introduced in Section 4.1.

In three dimensions Eq. (B.17) can be solved explicitly, finding

$$f_m(\tilde{t}) = \frac{1}{64\pi^2 s_\infty^2} \left[\sqrt{1 + 64\pi^2 s_\infty^2 \tilde{t}} - 1 \right]^2. \tag{B.19}$$

It is easy to compute the correction function $g_m(\tilde{t})$, obtaining

$$g_m(\tilde{t}) = f_m(\tilde{t}) \left[(E_d - \sigma) - \lambda f_m(\tilde{t})^{(d-4)/2} \right] \left[s_\infty - \frac{d-2}{2} A_d f_m(\tilde{t})^{(d-4)/2} \right]^{-1}, \tag{B.20}$$

where $\lambda = s_{1,\infty} A_d / s_\infty$.

Let us now compute the asymptotic behaviour of $f_m(\tilde{t})$ and $g_m(\tilde{t})$. For $\tilde{t} \rightarrow \infty$ we have

$$f_m(\tilde{t}) = \tilde{t} \left[1 + \frac{A_d}{s_\infty} \tilde{t}^{(d-4)/2} + O(\tilde{t}^{(d-4)}) \right], \tag{B.21}$$

$$g_m(\tilde{t}) = \frac{E_d - \sigma}{s_\infty} \tilde{t} \left[1 + O(\tilde{t}^{(d-4)/2}) \right]. \tag{B.22}$$

For $\tilde{t} \rightarrow 0$ we have

$$f_m(\tilde{t}) = \mu \tilde{t}^\gamma \left[1 - \frac{2\mu \tilde{t}^\Delta}{d-2} + O(\tilde{t}^{2\Delta}) \right], \tag{B.23}$$

$$g_m(\tilde{t}) = \frac{s_{1,\infty}}{s_\infty} \frac{2\mu \tilde{t}^\gamma}{d-2} (1 + O(\tilde{t}^\Delta)), \tag{B.24}$$

where $\gamma = 2/(d-2)$, $\Delta = (4-d)/(d-2)$, $\mu = (s_\infty/|A_d|)^\gamma$.

We can also compute the crossover function for the susceptibility. Since

$$\chi = \frac{1}{\beta V_R m^2}, \tag{B.25}$$

we have

$$f_\chi(\tilde{t}) = \frac{\rho_{c0}}{f_m(\tilde{t})}, \tag{B.26}$$

$$g_\chi(\tilde{t}) = -\frac{1}{\rho_{c0}} g_m(\tilde{t}) f_\chi(\tilde{t})^2 - \frac{\sigma}{s_\infty} f_\chi(\tilde{t}). \tag{B.27}$$

While $f_\chi(\tilde{t})$ is universal apart from normalization factors, the function $g_\chi(\tilde{t})$ is model-dependent. For this class of models, we have a three-parameter family of functions $g_\chi(\tilde{t})$ parametrized, for instance, by $s_{1,\infty}$, s_∞ , and $E_d - \sigma$. The fact that the $g_\chi(\tilde{t})$ depends only on three parameters should be due to the large- N limit. For general values of N we expect $g_\chi(\tilde{t})$ to be a non-trivial functional of the Hamiltonian. In Fig. 1 we report $g_\chi(\tilde{t})/f_\chi(\tilde{t})$ for three different models: the N -vector model and the theories corresponding to the potentials

$$W(\varphi^2) = \varphi^2 + (\varphi^2 - 1)^2, \tag{B.28}$$

$$W(\varphi^2) = \varphi^2 + (\varphi^2 - 1)^2 + (\varphi^2)^3, \tag{B.29}$$

using in all cases the coupling (3.8) and the domain (A.2). Notice that although the curves are quantitatively different, the qualitative behaviour is similar.

We wish now to compare the results presented above with the explicit calculations of Section 4.3. The basic ingredient is the large- N expansion of the integral

$$\int_0^\infty dx x^{N+k-1} e^{-NW(x^2)} = \frac{1}{2} \int_0^\infty dy y^{k/2-1} e^{N(\log y/2 - W(y))}. \quad (\text{B.30})$$

The saddle point equation is

$$\frac{1}{2y} = W'(y), \quad (\text{B.31})$$

which is exactly the equation defining ρ_{c0} , cf. Eq. (B.7). The large- N expression of Eq. (B.30) is obtained defining $y = \rho_{c0} + z/\sqrt{N}$ and expanding in powers of $1/\sqrt{N}$. In this way we obtain

$$\bar{a}_2 = \rho_{c0}, \quad (\text{B.32})$$

$$\bar{a}_4 = -\frac{6\rho_{c0}^2}{s_\infty}, \quad (\text{B.33})$$

$$\bar{a}_6 = \frac{15}{s_\infty^3 W_2^3} (12\rho_{c0} W_2^2 + 16\rho_{c0}^3 W_2^3 - W_3). \quad (\text{B.34})$$

These expressions are obtained assuming $W_2 > 0$. As it can be seen from Eq. (B.33), this condition is equivalent to requiring $\bar{a}_4 < 0$. Using the previous expressions, it is easy to verify all the formulae reported in Section 4.3. Notice that in the N -vector case $s_\infty = \rho_{c0} = 1$.

All the considerations we have presented above apply if $W_2 > 0$. If $W_2 = 0$, but $W_3 \neq 0$, the leading behaviour of δ changes and, for $R \rightarrow \infty$, we have

$$\delta \rightarrow \frac{R^d}{2\rho_{c0}^2 \sigma W_3} (m^2 - t) \equiv q R^d (m^2 - t). \quad (\text{B.35})$$

Thus, for $R \rightarrow \infty$, keeping only the leading terms, we have

$$(q + \rho_{c0})(m^2 - t) - \rho_{c0} A_d \frac{m^{d-2}}{R^{2d}} \approx 0. \quad (\text{B.36})$$

If we scale

$$\tilde{m}^2 = R^{4d/(4-d)} m^2, \quad \tilde{t} = R^{4d/(4-d)} t, \quad (\text{B.37})$$

we obtain again the same crossover scaling function. To interpret these results we should notice that $W''(\rho_c) = 0$ corresponds to the tricritical point. Therefore, here we are considering theories that for any finite R have a standard critical point, while for $R \rightarrow \infty$ converge to the mean-field tricritical point which is also a Gaussian point. We find that the scaling crossover functions are unchanged, although the scaling variables are different. In the framework of the Introduction, these theories correspond to models

in which the bare coupling u scales also with R as $1/R^d$, so that the Ginzburg parameter G becomes $G = (R^{-2d})^{2/(4-d)} = R^{-4d/(4-d)}$.

In order to have different crossover functions we must consider a family of potentials such that $W''(\rho_c) = 0$ for all R , i.e. consider theories at the tricritical point for any value of R . With our field normalization it is impossible to realize such a case, unless $W(\varphi^2)$ depends explicitly on R , i.e. $W(\varphi^2) = W(\varphi^2, R)$. Assuming therefore $\partial^2 W(\rho_c, R)/\partial \rho^2 = 0$, a simple computation gives

$$\delta^2 \approx \frac{1}{\rho_{c0} W_3} (m^2 - t), \tag{B.38}$$

where $W_n = \partial^n W(\rho, \infty)/\partial \rho^n$ evaluated at $\rho = \rho_{c0}$. Then we have

$$\left(\frac{m^2 - t}{\rho_{c0} W_3}\right)^{1/2} + \rho_{c0} A_d \frac{m^{d-2}}{R^d} = 0. \tag{B.39}$$

Rescaling¹⁵

$$\tilde{m}^2 = R^{2d/(3-d)} m^2, \quad \tilde{t} = R^{2d/(3-d)} t, \tag{B.40}$$

we obtain the equation for the crossover function

$$s^{\text{tr}} (\tilde{m}^2 - \tilde{t})^{1/2} - A_d \tilde{m}^{d-2} = 0. \tag{B.41}$$

valid, of course, for $d < 3$. One can go further and define multicritical crossover functions. If $W''(\rho_c, R) = \dots W^{(n)}(\rho_c, R) = 0$, then, by a rescaling

$$\tilde{m}^2 = R^{2nd/(2+2n-nd)} m^2, \quad \tilde{t} = R^{2nd/(2+2n-nd)} t, \tag{B.42}$$

we obtain for $R \rightarrow \infty$

$$s^{(n)} (\tilde{m}^2 - \tilde{t})^{1/n} - A_d \tilde{m}^{d-2} = 0, \tag{B.43}$$

for a suitable constant $s^{(n)}$.

B.2. Crossover limit in two dimensions

In this appendix we will discuss the critical crossover limit in two dimensions but we will consider only the N -vector model, since it already exhibits all the general features.

The gap equation is given by

$$\beta V_R = \int \frac{d^2 q}{(2\pi)^2} \frac{1}{\overline{\Pi}_R(q) + m^2} = \frac{1}{1 + m^2} (1 + I_{1,R}(m^2)). \tag{B.44}$$

The asymptotic behaviour of $I_{1,R}(m^2)$ is reported in Appendix A.2.3. Now define \hat{t} from

¹⁵ These rescalings can be derived in the formalism presented in the Introduction starting from a Hamiltonian with a ϕ^6 coupling.

$$\beta V_R = 1 + \frac{1}{4\pi R^2} \log R^2 + \frac{a_0}{R^4} \log R^2 + \frac{a_1}{R^4} - \frac{\hat{t}}{R^2}. \quad (\text{B.45})$$

Here we have introduced two free parameters a_0 and a_1 that represent the possible ambiguity in the definition of $\beta_{c,R}^{(\text{exp})}$. Then assume for $R \rightarrow \infty$ at \hat{t} fixed that

$$m^2 = \frac{1}{R^2} f_m(\hat{t}) + \frac{1}{R^4} g_m(\hat{t}, \log R) + O(R^{-6} \log R^2). \quad (\text{B.46})$$

Using the gap equation we obtain for the leading term

$$f_m(\hat{t}) + \frac{1}{4\pi} \log f_m(\hat{t}) = \hat{t} + C_2. \quad (\text{B.47})$$

Eq. (B.47) defines implicitly the crossover curve for the correlation length $f_\xi(\hat{t}) = 1/f_m(\hat{t})$ and for the susceptibility $f_\chi(\hat{t}) = f_\xi(\hat{t})$. It is easy to check that this equation has the correct limits. For $\hat{t} \rightarrow -\infty$ we have the standard asymptotic-scaling behaviour

$$f_m(\hat{t}) = e^{-4\pi|\hat{t}|+4\pi C_2} \left[1 + O(e^{-4\pi|\hat{t}|}) \right], \quad (\text{B.48})$$

while, for $\hat{t} \rightarrow +\infty$, we have

$$f_m(\hat{t}) = \hat{t} \left[1 - \frac{1}{4\pi\hat{t}} \log \hat{t} + \frac{C_2}{\hat{t}} + O(\hat{t}^{-2} \log \hat{t}) \right]. \quad (\text{B.49})$$

This expansion agrees with the results presented in Section 4.2.

For the correction term we obtain

$$\begin{aligned} g_m(\hat{t}, \log R) = & \frac{4\pi f_m(\hat{t})}{1 + 4\pi f_m(\hat{t})} \left[\frac{1}{8\pi} (4\alpha_1 + 3\alpha_2) f_m(\hat{t}) - a_0 \right] \log R^2 \\ & - \frac{4\pi f_m(\hat{t})}{1 + 4\pi f_m(\hat{t})} \left[\frac{1}{8\pi} (4\alpha_1 + 3\alpha_2) f_m(\hat{t}) \log f_m(\hat{t}) + a_1 \right. \\ & \left. - G_2 - (G_1 - C_2) f_m(\hat{t}) \right]. \end{aligned} \quad (\text{B.50})$$

Notice that in general there is no choice of a_0 which allow to cancel the logarithmic term. On the other hand, if one chooses Hamiltonians such that $\alpha_1 = \alpha_2 = 0$, the logarithmic term cancels. This class of Hamiltonians are called Symanzik tree-level improved [59].

For $\hat{t} \rightarrow -\infty$ we have

$$g_m(\hat{t}, \log R) \approx 4\pi f_m(\hat{t}) \left[-a_0 \log R^2 + a_1 - G_2 + O(e^{-4\pi|\hat{t}|}) \right], \quad (\text{B.51})$$

while for $\hat{t} \rightarrow +\infty$ we have

$$g_m(\hat{t}, \log R) \approx \frac{1}{8\pi} (4\alpha_1 + 3\alpha_2) \hat{t} \log(R^2 \hat{t}) - (G_1 - C_2) \hat{t} + O(\log \hat{t}). \quad (\text{B.52})$$

Notice that if one takes $a_0 = 0$ and $a_1 = G_2$, the corrections are strongly reduced in the limit $\hat{t} \rightarrow -\infty$. On the other hand, in the mean-field limit, the corrections do not depend on the scaling of β .

Finally we wish to compute $g_\chi(\hat{t}, \log R)$. Using Eq. (B.25) we have

$$g_\chi(\hat{t}, \log R) = -\frac{g_m(\hat{t}, \log R)}{f_m(\hat{t})^2} + \frac{4\pi\hat{t} - \log R^2}{4\pi f_m(\hat{t})}. \quad (\text{B.53})$$

For $\hat{t} \rightarrow \infty$ we have

$$g_\chi(\hat{t}, \log R) = -\frac{1}{8\pi\hat{t}}(4\alpha_1 + 3\alpha_2 + 2) \log R^2 + 1 - \frac{1}{8\pi\hat{t}}(4\alpha_1 + 3\alpha_2) \log \hat{t} + \frac{G_1 - C_2}{\hat{t}}, \quad (\text{B.54})$$

which agrees with the perturbative result (5.20).

References

- [1] M. Corti and V. Degiorgio, Phys. Rev. Lett. 55 (1985) 2005.
- [2] G. Dietler and D.S. Cannel, Phys. Rev. Lett. 60 (1988) 1852.
- [3] M.A. Anisimov, A.A. Povodyrev, V.D. Kulikov and J.V. Sengers, Phys. Rev. Lett. 75 (1995) 3146.
- [4] M.A. Anisimov, A.A. Povodyrev, V.D. Kulikov and J.V. Sengers, Phys. Rev. Lett. 76 (1996) 4095.
- [5] J. Jacob, A. Kumar, M.A. Anisimov and A.A. Povodyrev and J.V. Sengers, Phys. Rev. E 58 (1998) 2188.
- [6] C. Bagnuls and C. Bervillier, J. Phys. Lett. (Paris) 45 (1984) L-95.
- [7] C. Bagnuls and C. Bervillier, Phys. Rev. B 32 (1985) 7209.
- [8] C. Bagnuls, C. Bervillier, D.I. Meiron and B.G. Nickel, Phys. Rev. B 35 (1987) 3585.
- [9] M.E. Fisher, Phys. Rev. Lett. 57 (1986) 1911.
- [10] C. Bagnuls and C. Bervillier, Phys. Rev. Lett. 58 (1987) 435.
- [11] R. Schloms and V. Dohm, Nucl. Phys. B 328 (1989) 639.
- [12] Z.Y. Chen, P.C. Albright and J.V. Sengers, Phys. Rev. A 41 (1990) 3161.
- [13] M.A. Anisimov, S.B. Kiselev, J.V. Sengers and S. Tang, Physica A 188 (1992) 487.
- [14] M.Y. Belyakov and S.B. Kiselev, Physica A 190 (1992) 75.
- [15] D.J. Thouless, Phys. Rev. 181 (1969) 954.
- [16] K.K. Mon and K. Binder, Phys. Rev. E 48 (1993) 2498.
- [17] E. Luijten, H.W.J. Blöte and K. Binder, Phys. Rev. E 54 (1996) 4626.
- [18] E. Luijten, H.W.J. Blöte and K. Binder, Phys. Rev. Lett. 79 (1997) 561; Phys. Rev. E 56 (1997) 6540.
- [19] E. Luijten and K. Binder, Phys. Rev. E 58 (1998) R4060.
- [20] A. Pelissetto, P. Rossi and E. Vicari, Phys. Rev. E 58 (1998) 7146.
- [21] V.L. Ginzburg, Fiz. Tverd. Tela 2 (1960) 2031 [Sov. Phys. Solid State 2 (1960) 1824].
- [22] E. Luijten, Phys. Rev. E 59 (1999) 4997.
- [23] M.A. Anisimov, E. Luijten, V.A. Agayan, J.V. Sengers and K. Binder, Shape of crossover between mean-field and asymptotic critical behaviour in a three-dimensional Ising lattice, cond-mat/9810252.
- [24] S. Caracciolo, M.S. Causo, A. Pelissetto, P. Rossi and E. Vicari, in preparation.
- [25] S. Caracciolo, M.S. Causo, A. Pelissetto, P. Rossi and E. Vicari, Crossover scaling from classical to non-classical critical behaviour, Nucl. Phys. B (Proc. Suppl.) (1999), hep-lat/9809101.
- [26] G. Parisi, Cargèse Lectures (1973), J. Stat. Phys. 23 (1980) 49.
- [27] G.A. Baker Jr., B.G. Nickel, M.S. Green and D.I. Meiron, Phys. Rev. Lett. 36 (1977) 1351; G.A. Baker Jr., B.G. Nickel and D.I. Meiron, Phys. Rev. B 17 (1978) 1365.
- [28] D.B. Murray and B.G. Nickel, Revised estimates for critical exponents for the continuum n -vector model in 3 dimensions, unpublished Guelph University report (1991).
- [29] V. Dohm, Z. Phys. B 60 (1985) 61; B 61 (1985) 193.
- [30] H.J. Krause, R. Schloms and V. Dohm, Z. Phys. B 79 (1990) 287.
- [31] K.G. Chetyrkin, S.G. Gorishny, S.A. Larin and F.V. Tkachov, Phys. Lett. B 132 (1983) 351.
- [32] H. Kleinert, J. Neu, V. Schulte-Frohlinde, K.G. Chetyrkin and S.A. Larin, Phys. Lett. B 272 (1991) 39; B 319 (1993) 545 (E).

- [33] J.C. Le Guillou and J. Zinn-Justin, *Phys. Rev. B* 21 (1980) 3976.
- [34] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, third edition (Clarendon, Oxford, 1996).
- [35] A. Pelissetto and E. Vicari, *Nucl. Phys. B* 519 [FS] (1998) 605; Non-analyticity of the Callan-Symanzik β -function of $O(N)$ models, *Nucl. Phys. B (Proc. Suppl.)* (1999), hep-lat/9809041.
- [36] P. Butera and M. Comi, *Phys. Rev. B* 58 (1998) 11552.
- [37] R. Guida and J. Zinn-Justin, *J. Phys. A* 31 (1998) 8103.
- [38] K. Symanzik, *Lett. Nuovo Cimento* 8 (1973) 771.
- [39] G. Parisi, *Nucl. Phys. B* 150 (1979) 163.
- [40] M.C. Bergère and F. David, *Ann. Phys. (N.Y.)* 142 (1982) 416.
- [41] L.N. Lipatov, *Zh. Eksp. Teor. Fiz.* 72 (1977) 411 [*JETP* 45 (1977) 216].
- [42] E. Brézin, J.C. Le Guillou and J. Zinn-Justin, *Phys. Rev. D* 15 (1977) 1544.
- [43] E. Brézin, J.C. Le Guillou and J. Zinn-Justin, *in Phase Transitions and Critical Phenomena*, Vol. 6, ed. C. Domb, M.S. Green (Academic, London, 1976).
- [44] G.A. Baker, *Phys. Rev.* 126 (1962) 2071.
- [45] J. Hubbard, *Phys. Lett. A* 39 (1972) 365.
- [46] R. Brout, *Phys. Rev.* 118 (1960) 1009.
- [47] N.W. Dalton and C. Domb, *Proc. Phys. Soc. (London)* 89 (1966) 873.
- [48] V.G. Vaks, A.I. Larkin and S.A. Pikin, *Zh. Eksp. Teor. Fiz.* 51 (1966) 361 [*JETP* 24 (1967) 240].
- [49] S. Caracciolo and A. Pelissetto, *Phys. Rev. D* 58 (1998) 105007; *Nucl. Phys. B (Proc. Suppl.)* 53B (1997) 693.
- [50] S. Caracciolo, A. Montanari and A. Pelissetto, Testing the efficiency of different improvement programs, hep-lat/9812014, *Nucl. Phys. B* (to appear).
- [51] E. Luijten, private communication.
- [52] Z.Y. Chen, A. Abbaci, S. Tang and J.V. Sengers, *Phys. Rev. A* 42 (1990) 4470.
- [53] J.F. Nicoll and J.K. Bhattacharjee, *Phys. Rev. B* 23 (1981) 389.
- [54] J.F. Nicoll and P.C. Albright, *Phys. Rev. B* 31 (1985) 4576.
- [55] Y.B. Melnichenko, M.A. Anisimov, A.A. Povodyrev, G.D. Wignall, J.V. Sengers and W.A. Van Hook, *Phys. Rev. Lett.* 79 (1997) 5266.
- [56] S. Sarbach and M.E. Fisher, *Phys. Rev. B* 18 (1978) 2350.
- [57] V.J. Emery, *Phys. Rev. B* 11 (1975) 239; *B* 11 (1975) 3397.
- [58] S. Sarbach and T. Schneider, *Phys. Rev. B* 13 (1976) 464; *B* 16 (1977) 347.
- [59] K. Symanzik, *in Mathematical Problems in Theoretical Physics*, ed. R. Schrader et al. (Springer, Berlin, 1982); *Nucl. Phys. B* 226 (1983) 187, 205.