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Large-n critical behavior of $O(n) \times O(m)$ spin models

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Abstract

We consider the Landau–Ginzburg–Wilson Hamiltonian with $O(n) \times O(m)$ symmetry and compute the critical exponents at all fixed points to $O(n^{-2})$ and to $O(\epsilon^3)$ in a $\epsilon = 4 - d$ expansion. We also consider the corresponding non-linear σ-model and determine the fixed points and the critical exponents to $O(\tilde{\epsilon}^2)$ in the $\tilde{\epsilon} = d - 2$ expansion. Using these results, we draw quite general conclusions on the fixed-point structure of models with $O(n) \times O(m)$ symmetry for n large and all $2 \leqslant d \leqslant 4$.

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1. Introduction

The critical behavior of frustrated XY and Heisenberg spin systems with noncollinear order has been the subject of many recent theoretical studies, where the standard tools of renormalization-group (RG) theory have been applied to field theories which were conjectured to be appropriate for the description of the systems under investigation (see, e.g., Refs. [1,2] for reviews on this issue).

The critical behavior of these systems is rather controversial. Indeed, while experimentally there is good evidence of a second-order phase transition [1] belonging to a new (chiral)

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[1] The experimental results are reviewed in, e.g., Refs. [1,3]. Essentially, experiments with hexagonal perovskites find a clear second-order phase transition except for $CsCuCl_3$. The results for helimagnetic rare earths are instead less clear. We also mention Ref. [4] where it is shown experimentally that chiral order and spin order occur simultaneously, thereby supporting Kawamura's [5,6] conjecture that chiral transitions are different from the standard $O(n)$ transitions.

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universality class, theoretically the issue is still debated. On one side, field-theoretical studies based in approximate solutions of the RG equations (ERG) do not find any stable fixed point and favor a first-order phase transition [7–9]. On the other hand, perturbative field theory gives the opposite answer: a stable fixed point is identified with exponents in agreement with the experiments [10]. Monte Carlo simulations [11–13] do not help clarifying the issue. While simulations of the antiferromagnetic $O(n)$ model on a stacked triangular lattice find a second-order phase transition with exponents reasonably near to the experimental ones, modified spin systems which supposedly belong to the same universality class apparently favor a first-order transition [13].[2] Note that the existence of a new chiral universality class does not exclude the possibility that some systems undergo a first-order transition. Indeed, they may lie outside the attraction domain of the stable fixed point and thus belong to runaway RG trajectories. In this case, a first-order transition is expected.

The field-theoretical studies have been focusing either on the so-called Landau–Ginzburg–Wilson (LGW) Hamiltonian with $O(n) \times O(2)$ symmetry or on the corresponding nonlinear sigma (NLσ) model. In this paper we will study a generalization of these theories, by considering general $O(n) \times O(m)$ Hamiltonians and we will try to understand the nature of the fixed points of the theory. In particular, we will relate the LGW and the NLσ descriptions showing explicitly that the stable fixed points of the two models are exactly the same, as conjectured in Refs. [14,15,32], for values of m and n consistent with the existence of a second-order phase transition. Moreover, we will clarify the nature of the unstable fixed points. However, this analysis will only be valid in the large-n region, where, by using the large-n expansion, we will be able to identify nonperturbatively all fixed points of the different Hamiltonians.

Since the large-n expansion plays a major role in our discussion, our results will only be valid for $n > \bar{n}(m,d)$, i.e., in the region of large-n analyticity. Such a function is conjectured to be identified with the line $n^+(m,d)$ on which the LGW chiral and antichiral fixed points merge. The function $n^+(m,d)$ has been the object of extensive studies that tried to understand whether, in the physical case $d = 3$ and $m = 2$, $\bar{n}(m,d)$ was smaller or larger than three. In this case, studies using various approaches gave $\bar{n}(2,3) \approx 4$ (Refs. [8, 16,17]), 5 (Ref. [9]), ≈ 6 (Ref. [10]). Here, we will provide another determination, together with generalizations for other values of m, that substantially confirms previous findings, i.e., $\bar{n}(2,3) \approx 5$. Since the results that we will present are essentially adiabatic moving from the large-n and small $\epsilon \equiv 4 - d$ region, they are not expected to provide the (essentially nonperturbative) features of the models in the region below $\bar{n}(m,d)$. Therefore, the fact that $\bar{n}(2,3) > 3$ does not necessarily imply an inconsistency with the field-theoretical results of Ref. [10], where a rather robust evidence for stable chiral fixed points was found for $O(n) \times O(2)$ models with $n = 2, 3$ in fixed dimension $d = 3$. Such fixed points are not analytically connected with the large-n and small-ϵ criticalities discussed above.

[2] Note that not all modified models show a first-order phase transitions. Some of them have a behavior that is compatible with a second-order phase transition. However, the measured exponents do not satisfy the condition $\eta \geqslant 0$, which must be satisfied in unitary (reflection-positive) models as these are, so that, the measured exponents can only be effective ones. This is interpreted as a signal of a first-order phase transition. However, there is also the possibility that the results are strongly biased by corrections to scaling induced by the constraint.

We will also show that for $m > 2$ the identification $\bar{n}(m,d) = n^+(m,d)$ may not be correct for d near two dimensions. Indeed, in this case a new critical line appears which corresponds to the merging of the chiral fixed point with the NLσ antichiral fixed point.

In order to obtain quantitative predictions for all m and n, we have extended the $\epsilon \equiv 4 - d$ expansion of the LGW theory to order ϵ^3 and the $\tilde{\epsilon} \equiv d - 2$ expansion of the NLσ model to order $\tilde{\epsilon}^2$ for all n and m. Also, we present $\mathrm{O}(1/n^2)$ results for the LGW theory.

The paper is organized as follows. In Section 2 we define the general class of models with $O(n) \times O(m)$ symmetry that will be considered in the paper and find a general representation that is the starting point of the large-n expansion. In Section 3 we compute the $\mathrm{O}(\epsilon^3)$ contributions to the critical exponents η and ν^{-1} within the $\epsilon = 4 - d$ expansion of the LGW Hamiltonian. In Section 4 we analyze in detail the $1/n$ expansion of the LGW Hamiltonian with $O(n) \times O(m)$ symmetry to $\mathrm{O}(1/n^2)$, thereby extending the results of Ref. [14]. Interestingly enough, we can determine the large-n expansion of the exponents at all fixed points and show explicitly their different physical nature: at the stable fixed point both tensor and scalar excitations propagate, while at each unstable fixed point one of the degrees of freedom is suppressed. At the Heisenberg fixed point there are only scalar excitations, while at the antichiral one, there are only tensor excitations. In Section 5 we discuss the $1/n$ expansion of a more general theory in which the coupling to a (gauge) vector field is included, extending the results of Ref. [18]. In Section 6 we extend to arbitrary values of m and to $\mathrm{O}(\tilde{\epsilon}^2)$ the $\tilde{\epsilon}$-expansion of NLσ models, evaluating the unstable fixed point and the coalescence value of n under which the two fixed points actually disappear. We also identify the "gauge" criticality of the models. In Section 7 we draw some general conclusions and present a new determination of the function $\bar{n}(m,d)$.

2. Models

We will consider a non-Abelian gauge model coupled to a scalar field with gauge symmetry $O(m)$ and global symmetry $O(n)$. In particular, we consider a set of m n-dimensional vectors $\boldsymbol{\phi}_\alpha = \{\phi_{\alpha a}\}$, $\alpha = 1, \ldots, m$, $a = 1, \ldots, n$, a vector field $A_\mu^{\alpha\beta}$ antisymmetric in α and β, and the Hamiltonian density

$$\mathcal{H} = \frac{1}{2} \sum_\alpha \left(\partial_\mu \boldsymbol{\phi}_\alpha + g_0 A_\mu^{\alpha\beta} \boldsymbol{\phi}_\beta\right)^2 + \frac{1}{2} r_0 \sum_\alpha \boldsymbol{\phi}_\alpha^2 + \frac{1}{4!} u_0 \left(\sum_\alpha \boldsymbol{\phi}_\alpha^2\right)^2 + \frac{1}{4!} v_0 \sum_{\alpha\beta} \left[(\boldsymbol{\phi}_\alpha \cdot \boldsymbol{\phi}_\beta)^2 - \boldsymbol{\phi}_\alpha^2 \boldsymbol{\phi}_\beta^2\right] + \frac{1}{4} F_{\mu\nu}^2 + \frac{t_0}{2} \sum_{\alpha\beta} A_\mu^{\alpha\beta} A_\mu^{\alpha\beta}, \tag{2.1}$$

where $F_{\mu\nu}$ is the non-Abelian field strength associated with the fields $A_\mu^{\alpha\beta}$. This Hamiltonian is gauge-invariant (with local $O(m)$ invariance) for $t_0 = 0$, and in this case it has already been studied[3] in Ref. [18]. For $t_0 = 0$ and $A_\mu^{\alpha\beta} = 0$ we obtain a generic LGW Hamiltonian density with global $O(m) \times O(n)$ invariance:

$$\mathcal{H} = \frac{1}{2}\sum_\alpha (\partial_\mu \boldsymbol{\phi}_\alpha)^2 + \frac{1}{2} r_0 \sum_\alpha \boldsymbol{\phi}_\alpha^2 + \frac{1}{4!} u_0 \left(\sum_\alpha \boldsymbol{\phi}_\alpha^2\right)^2 + \frac{1}{4!} v_0 \sum_{\alpha\beta} \left[(\boldsymbol{\phi}_\alpha \cdot \boldsymbol{\phi}_\beta)^2 - \boldsymbol{\phi}_\alpha^2 \boldsymbol{\phi}_\beta^2\right]. \tag{2.2}$$

Assuming $n > m$, stability requires $u_0 > 0$ and $w_0 \equiv u_0 + (1 - \mathcal{N}) v_0/\mathcal{N} > 0$, where $\mathcal{N} = \min(m, n)$.

Other particular cases of the Hamiltonian (2.1) are interesting. If we set $u_0 = v_0$ and $r_0 = -v_0\eta_1/6$ and take the limit $v_0 \to +\infty$ keeping η_1 fixed, we obtain an $O(n) \times O(m)$ σ-model coupled to an $O(m)$ vector field. The Hamiltonian density is given by

$$\mathcal{H} = \frac{1}{2}\sum_\alpha \left(\partial_\mu \boldsymbol{\phi}_\alpha + g_0 A_\mu^{\alpha\beta} \boldsymbol{\phi}_\beta\right)^2 + \frac{1}{4} F_{\mu\nu}^2 + \frac{t_0}{2} \sum_{\alpha\beta} A_\mu^{\alpha\beta} A_\mu^{\alpha\beta}, \tag{2.3}$$

where the fields $\boldsymbol{\phi}$ satisfy the constraint

$$\boldsymbol{\phi}_\alpha \cdot \boldsymbol{\phi}_\beta = \eta_1 \delta_{\alpha\beta}. \tag{2.4}$$

This limit is well defined only if $n \geqslant m$, otherwise the constraint (2.4) cannot be satisfied. In the absence of the kinetic term for the vector field, the Hamiltonian density depends quadratically on the vector field that can then be eliminated by integration. We obtain a new Hamiltonian of the form

$$\mathcal{H} = \frac{1}{2}\sum_\alpha \left(\partial_\mu \boldsymbol{\phi}_\alpha\right)^2 - \frac{g_0^2}{8(g_0^2\eta_1 + t_0)} \sum_{\alpha\beta} \left(\boldsymbol{\phi}_\alpha \partial_\mu \boldsymbol{\phi}_\beta - \boldsymbol{\phi}_\beta \partial_\mu \boldsymbol{\phi}_\alpha\right)^2. \tag{2.5}$$

This is the σ-model studied in Refs. [14,15,32]. In order to recover the notations[4] of Ref. [15], we set $\boldsymbol{\phi}_\alpha = \sqrt{\eta_1}\, \mathbf{e}_\alpha$ and $\eta_2 \equiv 2t_0\eta_1/(g_0^2\eta_1 + t_0)$. Then

$$\widetilde{H} = \frac{1}{2}\eta_1 \sum_{\alpha=1}^{m} \partial_\mu \mathbf{e}_\alpha \cdot \partial_\mu \mathbf{e}_\alpha + \left(\frac{1}{2}\eta_2 - \eta_1\right) \sum_{\alpha>\beta}^{m} (\mathbf{e}_\alpha \cdot \partial_\mu \mathbf{e}_\beta)^2, \tag{2.6}$$

where the fields $\mathbf{e}_\alpha$ are m n-component vectors (or equivalently $m \times n$ matrices) with $n \geqslant m$ subject to the nonlinear constraint

$$\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = \delta_{\alpha\beta}. \tag{2.7}$$

In order to study the large-n behavior of these models we rewrite the general Hamiltonian (2.1) in such a way that the dependence on the field $\boldsymbol{\phi}_\alpha$ is quadratic. This is obtained

[3] Hikami's couplings [18], labelled by the subscript H, are related to ours by the correspondence: $\rho_H = (u_0 - v_0)/3$, $v_H = v_0/6$, $\lambda_H \equiv \rho_H + 2v_H = u_0/3$.

[4] Notice that our η_i are consistent with the couplings employed in Section 4 of Ref. [15], but they are twice as big as the couplings defined in Appendix B of Ref. [15], due to a slight inconsistency in the notation adopted by these authors. The couplings η_i are related to those of Ref. [14] by $\eta_1 \equiv 1/T + 1/T'$, $\eta_2 \equiv 2/T$. For easy comparison, the reader should keep in mind that, due to the constraints, $\mathbf{e}_\alpha \cdot \partial_\mu \mathbf{e}_\beta = -\mathbf{e}_\beta \cdot \partial_\mu \mathbf{e}_\alpha$.

by introducing two auxiliary fields: a scalar field S and a symmetric and traceless tensor field $T^{\alpha\beta}$, i.e., such that $T^{\alpha\beta} = T^{\beta\alpha}$, $T^{\alpha\alpha} = 0$. By means of these auxiliary fields we can rewrite the Hamiltonian (2.1) as

$$H = H_{\text{eff}} - \frac{3v_0}{2} T^2 - \frac{3w_0}{2} S^2 + \frac{t_0}{2} A^2 + \frac{1}{4} F^2, \tag{2.8}$$

where

$$H_{\text{eff}} = \frac{1}{2} \sum_{\alpha,\beta} \boldsymbol{\phi}_\alpha \cdot X^{\alpha\beta} \boldsymbol{\phi}_\beta, \tag{2.9}$$

and

$$X^{\alpha\beta} = -\partial_\mu \partial_\mu \delta^{\alpha\beta} + r_0 \delta^{\alpha\beta} - 2g_0 A_\mu^{\alpha\beta} \partial_\mu + v_0 T^{\alpha\beta} + w_0 S \delta^{\alpha\beta} + g_0^2 A_\mu^{\beta\gamma} A_\mu^{\alpha\gamma}. \tag{2.10}$$

Note that the effective action for the ϕ fields is the most general one which is $O(m)$ covariant. Therefore, the analysis of this class of models provides the critical behavior of the most general $O(m) \times O(n)$ theory.

3. ϵ-expansion for the Landau–Ginzburg–Wilson model

In this section we study the LGW Hamiltonian (2.2) and report our results for the critical exponents and the β-function to order ϵ^3, thereby extending the results of Ref. [19] to three loops. We consider the massless theory and renormalize it using the $\overline{\text{MS}}$ scheme. We set

$$\boldsymbol{\phi} = \left[Z_\phi(u, v)\right]^{1/2} \boldsymbol{\phi}_R, \tag{3.1}$$

$$u_0 = \mu^\epsilon Z_u(u, v) N_d^{-1}, \tag{3.2}$$

$$v_0 = \mu^\epsilon Z_v(u, v) N_d^{-1}, \tag{3.3}$$

where the renormalization constants are normalized so that $Z_\phi(u, v) \approx 1$, $Z_u(u, v) \approx u$, and $Z_v(u, v) \approx v$ at tree level. Here N_d is a d-dependent constant given by $N_d^{-1} = 2^{d-1} \pi^{d/2} \Gamma(d/2)$. Moreover, we introduce a mass renormalization constant $Z_t(u, v)$ by requiring $Z_t \Gamma^{(1,2)}$ to be finite when expressed in terms of u and v. Here $\Gamma^{(1,2)}$ is the two-point function with an insertion of ϕ^2. Once the renormalization constants are determined, we compute the β functions from

$$\beta_u(u, v) = \mu \frac{\partial u}{\partial \mu}\bigg|_{u_0, v_0}, \qquad \beta_v(u, v) = \mu \frac{\partial v}{\partial \mu}\bigg|_{u_0, v_0}, \tag{3.4}$$

and the critical exponents from

$$\eta = \frac{\partial \log Z_\phi}{\partial \log \mu}\bigg|_{u_0, v_0}, \tag{3.5}$$

$$\eta_t = \frac{\partial \log Z_t}{\partial \log \mu}\bigg|_{u_0, v_0}, \tag{3.6}$$

$$\nu = (2 - \eta - \eta_t)^{-1}. \tag{3.7}$$

For the β-functions we obtain:[5]

$$\begin{aligned}
\beta_u = {} & -\epsilon u + \frac{mn+8}{6}u^2 - \frac{(m-1)(n-1)}{3}v\left(u - \frac{v}{2}\right) - \frac{3mn+14}{12}u^3 \\
& + (m-1)(n-1)v\left(\frac{11}{18}u^2 - \frac{13}{24}uv + \frac{5}{36}v^2\right) \\
& + \frac{u^4}{1728}\left[33m^2n^2 + 922mn + 2960 + \zeta(3)(480mn + 2112)\right] \\
& + \frac{v}{3456}(m-1)(n-1) \\
& \times \Big\{ -4\left[79mn + 1318 + 768\zeta(3)\right]u^3 \\
& \quad + \left[555mn - 460(m+n) + 6836 + 4032\zeta(3)\right]u^2 v \\
& \quad - 2\left[213mn - 358(m+n) + 1933 + 960\zeta(3)\right]uv^2 \\
& \quad + \left[121mn - 309(m+n) + 817 + 216\zeta(3)\right]v^3 \Big\},
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\beta_v = {} & -\epsilon v + 2uv + \frac{m+n-8}{6}v^2 - \frac{5mn+82}{36}u^2 v \\
& + \frac{5mn - 11(m+n) + 53}{18}uv^2 - \frac{13mn - 35(m+n) + 99}{72}v^3 \\
& + \frac{v^4}{3456}\Big\{52m^2n^2 - 57mn(m+n) - 2206mn - 111\left(m^2+n^2\right) \\
& \qquad + 4291(m+n) - 8084 \\
& \qquad + \left[-1416mn + 3216(m+n) - 7392\right]\zeta(3)\Big\} \\
& + \frac{v^3 u}{864}\Big\{ -39m^2n^2 + 35mn(m+n) + 1302mn + 36\left(m^2+n^2\right) \\
& \qquad - 2401(m+n) + 5725 \\
& \qquad + \left[768mn - 1824(m+n) + 4896\right]\zeta(3)\Big\} \\
& + \frac{u^2v^2}{1728}\Big\{78m^2n^2 - 35mn(m+n) - 2114mn + 3182(m+n) - 12520 \\
& \qquad + \left[-1152mn + 2304(m+n) - 10368\right]\zeta(3)\Big\} \\
& + \frac{u^3 v}{864}\left[-13m^2n^2 + 368mn + 3284 + (192mn + 2688)\,\zeta(3)\right].
\end{aligned} \tag{3.9}$$

[5] We mention that for the particular case $n = m = 2$ one may derive the corresponding four-loop ϵ-expansion from the series reported in Ref. [20] for the so-called tetragonal model. Indeed an exact mapping [16] exists bringing from the LGW Hamiltonian (2.2) with $m = n = 2$ to the tetragonal model considered in Ref. [20]. Note that our renormalized couplings differ from those defined for $m = 2$ in Ref. [6]. Kawamura's couplings u_K, v_K are related to ours by $u = 12N_d u_K$, $v = 6N_d v_K$.

For the critical exponents we obtain:

$$\eta = \frac{mn+2}{72}u^2 + (m-1)(n-1)v\left(\frac{v}{48} - \frac{u}{36}\right) - \frac{(mn+2)(mn+8)}{1728}u^3$$
$$+ \frac{(m-1)(n-1)}{3456}v\big[v^2(2mn-5m-5n+26)$$
$$- 6uv(mn-m-n+10) + 6u^2(mn+8)\big], \tag{3.10}$$

$$\frac{1}{\nu} = 2 - \frac{mn+2}{6}u + \frac{(m-1)(n-1)}{6}v + \frac{5(mn+2)}{72}u^2$$
$$+ 5(m-1)(n-1)v\left(\frac{v}{48} - \frac{u}{36}\right) - \frac{(mn+2)(5mn+37)}{288}u^3$$
$$+ \frac{(m-1)(n-1)}{1152}v\big[3v^2(7mn-16m-16n+79)$$
$$- uv(61mn-58m-58n+550) + 12u^2(5mn+37)\big]. \tag{3.11}$$

As discussed at length in Refs. [1,6,19], the critical behavior of these systems depends on the values of n and m. In general, the β-functions admit four solutions: the Gaussian fixed point ($u^* = v^* = 0$), the $O(mn)$ Heisenberg fixed point ($v^* = 0$) and two new fixed points with nontrivial values of u^* and v^*, the chiral and the antichiral fixed points. These two additional fixed points do not exist for all n and m, but, at fixed m, only for $n \geqslant n^+(m)$ and $n \leqslant n^-(m)$. The functions $n^\pm(m)$ will be computed below. The critical behavior depends on the stability of the fixed points. At fixed m, for the physically relevant case $m \geqslant 1$, the ϵ-expansion predicts four regimes:

(1) For $n > n^+(m)$, there are four fixed points, and the chiral one is the stable one.
(2) For $n^-(m) < n < n^+(m)$, only the Gaussian and the Heisenberg $O(n \times m)$-symmetric fixed points are present, and none of them is stable.
(3) For $n_H(m) < n < n^-(m)$, there are again four fixed points, and the chiral one is the stable one. For small ϵ, the chiral fixed point has $v < 0$ for $m < 7$ and $v > 0$ for $m > 7$.
(4) For $n < n_H(m)$, there are again four fixed points, and the Heisenberg $O(m \times n)$-symmetric one is the stable one.

The antichiral fixed point is Gaussian for $m \to 1$ and $m = -2$ (or, equivalently for $n \to 1$ and $n = -2$). Indeed, for $m \to 1$, $u^* \to 0$ for the antichiral fixed point, so that, from Eqs. (3.10) and (3.11), we obtain $\eta = 0$ and $\nu = 1/2$ at order ϵ^3. For $m = -2$ we obtain $u^* = 3v^*/2$ and again $\eta = 0$ and $\nu = 1/2$ at order ϵ^3. We conjecture that this holds to all orders in ϵ, and in Section 4 we will provide a large-n interpretation of these results. The general behavior for n and m is better understood from Fig. 1. In particular, the two functions $n^\pm(m)$ are nothing but the two different branches of the curve that separates the region in which no fixed point is stable from the region in which the chiral fixed point is the stable one. Note that the boundaries of the different regions are symmetric under the exchange (n, m). Because of this symmetry it is more natural to consider the behavior in the variables

$$\Sigma = m + n, \qquad \Delta = m - n. \tag{3.12}$$

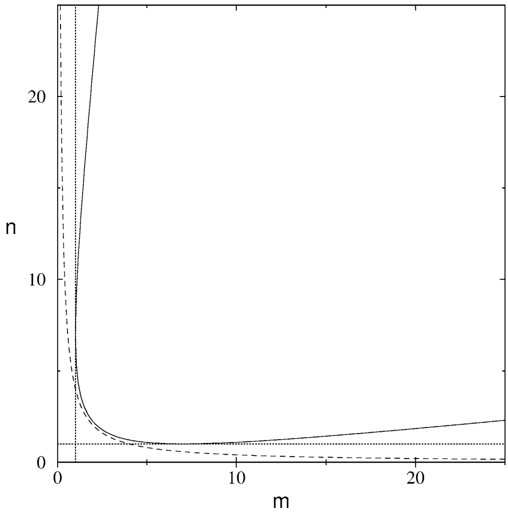

Fig. 1. The fixed-point structure in the (m,n) plane for $d=4$. The solid line represents the curves $n^{\pm}(m)$. The dashed line shows $n_H(m)$ and the symmetric curve obtained interchanging n and m.

At fixed Δ there are then three regions:

(1) For $\Sigma > \Sigma^+$ only the Gaussian and the Heisenberg $O(m \times n)$-symmetric fixed points are present and none is stable.
(2) For $\Sigma_H < \Sigma \leqslant \Sigma^+$ there are four fixed points and the chiral one is stable.
(3) For $\Sigma \leqslant \Sigma_H$, there are four fixed points and the $O(m \times n)$-symmetric one is the stable one.

Using the above results, we can compute the ϵ expansion of $n^{\pm}(m)$ and $n_H(m)$. For $n^{\pm}(m)$ we expand

$$n^{\pm}(m) = n_0^{\pm} + n_1^{\pm}\epsilon + n_2^{\pm}\epsilon^2 + \mathrm{O}(\epsilon^3). \tag{3.13}$$

Then, by requiring

$$\beta_u(u^*, v^*; n^{\pm}) = 0, \qquad \beta_v(u^*, v^*; n^{\pm}) = 0, \tag{3.14}$$

and

$$\det\left|\frac{\partial(\beta_u, \beta_v)}{\partial(u, v)}\right|(u^*, v^*; n^{\pm}) = 0, \tag{3.15}$$

we obtain

$$n_0^{\pm} = 5m + 2 \pm 2s, \tag{3.16}$$

$$n_1^{\pm} = -5m - 2 \mp \frac{1}{2s}\left(25m^2 + 22m - 32\right), \tag{3.17}$$

$$n_2^{\pm} = \frac{1}{24}\frac{R_1}{Q_1} \pm \frac{1}{32}\frac{R_2}{sQ_1} + \frac{1}{8}\frac{R_3}{Q_2}\zeta(3) \pm \frac{1}{48}\frac{sR_4}{Q_2}\zeta(3). \tag{3.18}$$

Here

$$
\begin{aligned}
s &= \sqrt{6(m-1)(m+2)},\\
R_1 &= -33024 + 18880m + 45444m^2 + 9288m^3 - 1883m^4 - 417m^5\\
&\quad + 21m^6 + 4m^7,\\
R_2 &= -253952 - 160256m + 176192m^2 + 139240m^3 + 7756m^4\\
&\quad - 5854m^5 - 389m^6 + 58m^7 + 5m^8,\\
R_3 &= 1632 + 1184m - 1376m^2 - 426m^3 + 31m^4 + 8m^5,\\
R_4 &= 6176 - 2960m - 1230m^2 + 73m^3 + 20m^4,\\
Q_1 &= (m+8)^2(m+2)(m-1)(m-7)^2,\\
Q_2 &= (m+8)(m+2)(m-1)(m-7).
\end{aligned}
\tag{3.19}
$$

For $m = 2$ this expression is in agreement with that given in Ref. [17]. The expression for $n_2^{\pm}$ are singular for $m = 7$. However, this is not the case for n_2^+, and indeed, by taking the limit we obtain

$$
n_2^+ = \frac{23871617}{9331200} + \frac{5487}{320}\zeta(3). \tag{3.20}
$$

For $n_H(m)$ we have

$$
n_H(m) = \frac{1}{m}\left[4 - 2\epsilon + \frac{5}{12}\big(6\zeta(3) - 1\big)\epsilon^2 + \mathrm{O}\big(\epsilon^3\big)\right], \tag{3.21}
$$

which is a trivial generalization of the result of Ref. [17]. The calculation of the functions $\Sigma^+(\Delta)$ and $\Sigma_H(\Delta)$ follows the same lines. In particular,

$$
\Sigma^+(\Delta) = -1 + \frac{1}{2}\tilde{s} + \frac{\epsilon}{8\tilde{s}}\big(5\Delta^2 - 24 - 2\tilde{s}\big) + \mathrm{O}\big(\epsilon^2\big), \tag{3.22}
$$

where $\tilde{s} = \sqrt{6(\Delta^2 + 18)}$.

From our calculation of the RG functions we can derive the fixed points of the theory. We expand

$$
u^* = u_1\epsilon + u_2\epsilon^2 + u_3\epsilon^3 + \mathrm{O}\big(\epsilon^4\big), \qquad v^* = v_1\epsilon + v_2\epsilon^2 + v_3\epsilon^3 + \mathrm{O}\big(\epsilon^4\big). \tag{3.23}
$$

Following Ref. [19] we define

$$
\begin{aligned}
B_{mn}^{-1} &\equiv (mn+8)(m+n-8)^2 + 24(m-1)(n-1)(m+n-2),\\
D_{mn} &\equiv mn(m+n) - 10(m+n) + 4mn - 4,\\
R_{mn} &\equiv (m+n-8)^2 - 12(m-1)(n-1).
\end{aligned}
\tag{3.24}
$$

We then easily find:

$$
\begin{aligned}
u_1^{\pm} &= \frac{1}{2} - \frac{1}{2}(m+n-8)\, B_{mn}\big[D_{mn} \mp 6R_{mn}^{1/2}\big],\\
v_1^{\pm} &= 6B_{mn}\big[D_{mn} \mp 6R_{mn}^{1/2}\big],
\end{aligned}
\tag{3.25}
$$

where we indicate by $(+)$ the stable chiral fixed point and by $(-)$ the unstable antichiral one. In order to compute $u_2^{\pm}$ and $v_2^{\pm}$, it is convenient to define two additional auxiliary functions:

$$
\begin{aligned}
S_1 &\equiv -\frac{(3mn+14)}{12}u_1^3 + (m-1)(n-1)v_1\left(\frac{11}{18}u_1^2 - \frac{13}{24}u_1v_1 + \frac{5}{36}v_1^2\right),\\
S_2 &\equiv -\frac{(5mn+82)}{36}u_1^2v_1 + \frac{5mn-11(m+n)+53}{18}u_1v_1^2\\
&\quad - \frac{13mn-35(m+n)+99}{72}v_1^3.
\end{aligned}
\tag{3.26}
$$

Then, the $O(\epsilon^2)$ coefficients of the fixed points are given by

$$
\begin{aligned}
u_2^{\pm} &= \pm 6\frac{\left(1-2u_1^{\pm} - \frac{m+n-8}{3}v_1^{\pm}\right)S_1^{\pm} - \frac{(m-1)(n-1)}{3}(u_1^{\pm}-v_1^{\pm})S_2^{\pm}}{v_1^{\pm}R_{mn}^{1/2}},\\
v_2^{\pm} &= \pm 6\frac{2v_1^{\pm}S_1^{\pm} + \left(1-\frac{mn+8}{3}u_1^{\pm} + \frac{(m-1)(n-1)}{3}v_1^{\pm}\right)S_2^{\pm}}{v_1^{\pm}R_{mn}^{1/2}}.
\end{aligned}
\tag{3.27}
$$

The expressions for $u_3^{\pm}$, $v_3^{\pm}$ are particularly cumbersome and they will not be reported here.

Once the fixed points are determined, the critical exponents are computed by expanding in power of ϵ the exponent series computed at the fixed point. Such a computation gives us the exponents only for

$$
n > 5m + 2 + 2\sqrt{6(m+2)(m-1)},
\tag{3.28}
$$

or

$$
n < 5m + 2 - 2\sqrt{6(m+2)(m-1)}.
\tag{3.29}
$$

Indeed, if these bounds are not satisfied the fixed points are complex and therefore also the exponent series. In order to obtain series for the exponents in all the relevant domain we can perform the following trick. For $n > n^+$, which is the case of physical interest, we set $n = n^+(m,\epsilon) + \Delta n$ and reexpand all series in powers of ϵ keeping Δn fixed. In particular, for $\Delta n = 0$ we obtain the critical exponents for $n = n^+$. In such a case, for $m = 2$ we obtain:

$$
\begin{aligned}
\eta &= \frac{1}{48}\epsilon^2 + \frac{5}{288}\epsilon^3,\\
\frac{1}{\nu} &= 2 - \frac{1}{2}\epsilon + \epsilon^2\left(\frac{\sqrt{6}}{50} - \frac{1}{50}\right)\\
&\quad + \epsilon^3\left(\frac{397}{15000} - \frac{37\sqrt{6}}{15000} + \frac{37}{1000}\zeta(3) + \frac{\sqrt{6}}{250}\zeta(3)\right).
\end{aligned}
\tag{3.30}
$$

4. The Landau–Ginzburg–Wilson theory in the large-n limit

In this section, we study the large n-behavior of the LGW theory (2.2) at fixed m. The starting point is the general Hamiltonian (2.8) with $A_\mu^{\alpha\beta} = 0$. In the high-temperature phase the symmetry is unbroken and thus the relevant saddle point is given by

$$
\langle S\rangle = \sigma, \qquad \langle T_{\alpha\beta}\rangle = 0.
\tag{4.1}
$$

Correspondingly, we obtain the gap equation:

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + M^2} = \frac{6\sigma}{nm}, \tag{4.2}$$

where $M^2 = r_0 + w_0\sigma$. For the σ-model (2.3) we obtain analogously [6]

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + M^2} = \frac{\eta_1}{n}. \tag{4.3}$$

From the gap equation we can obtain the scaling of the mass and thus the exponent ν. For $w_0 \neq 0$, proceeding as in the case of the ordinary $O(n)$ model, we obtain for $2 < d < 4$

$$\nu = \frac{1}{d-2}. \tag{4.4}$$

However, for $w_0 = 0$, we obtain simply $M^2 = r_0$, indicating

$$\nu = \frac{1}{2}, \tag{4.5}$$

for all values of the dimension d.

Within the large-n limit we can recover the critical behavior of the theory at all fixed points. For generic v_0 and u_0, satisfying $w > 0$, $v_0 > 0$ we obtain the critical chiral behavior. The standard Heisenberg behavior is obtained by setting $v_0 = 0$, while the antichiral critical behavior is obtained at the stability boundary, i.e., by setting $w_0 = 0$. It is easy to see the different types of excitations that appear in these cases: at the chiral fixed point both the scalar and the tensor degrees of freedom propagate, while at the Heisenberg and antichiral fixed points one observes only the scalar and the tensor degrees of freedom respectively. Note that, as a consequence of Eqs. (4.4) and (4.5), the Heisenberg and the chiral point have the same exponents for $n = \infty$, and that they differ from those of the antichiral point which shows mean-field behavior in all dimensions.

In order to perform the calculation we heavily relied upon the results obtained by Vasil'ev et al. [21,22], who studied the models corresponding to $m = 1$ with a method which lends itself to a reasonably simple extension, appropriate to the case we are investigating. In order to make our presentation self-contained, we must briefly review the essentials of the method.

One first considers the second Legendre transform with respect to the field and the two-point function [23]. We indicate by $D_\phi^{\alpha a,\beta b}(p)$, $D_S(p)$, and $D_T^{\alpha\beta,\gamma\delta}(p)$ the dressed propagators of the field $\phi^{\alpha a}$ and of the auxiliary fields S and $T^{\alpha\beta}$. Here α, β, γ, and δ go from 1 to m, while a and b go from 1 to n. It is useful to factorize the group dependence and to introduce scalar propagators

$$D_\phi^{\alpha a,\beta b}(p) = \delta^{\alpha\beta}\delta^{ab}\widehat{D}_\phi(p), \tag{4.6}$$

$$D_T^{\alpha\beta,\gamma\delta}(p) = \frac{1}{2}\left(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma} - \frac{2}{m}\delta^{\alpha\beta}\delta^{\gamma\delta}\right)\widehat{D}_T(p). \tag{4.7}$$

[6] This result can be obtained by using Eq. (4.2) and by taking the limit considered before Eq. (2.3). Notice that, in order to keep M^2 finite in the limit, σ must converge to $m\eta_1$ as $v_0 \to \infty$.

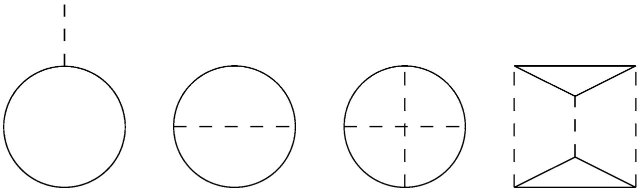

Fig. 2. The graphs appearing in the second Legendre transform. The continuous line represents the dressed propagator of the ϕ field, while the dashed line indicates the dressed propagator of an auxiliary field.

Also we can reabsorb the coupling v_0 and w_0 in the fields. Using the same notations of Ref. [21], we have for the second Legendre transform

$$\begin{aligned}\Gamma = {} & \frac{1}{2}\operatorname{Tr}\log D_\phi + \frac{1}{2}\operatorname{Tr}\log D_S + \frac{1}{2}\operatorname{Tr}\log D_T + \frac{n}{2}\big(\gamma_1(S) + \gamma_1(T)\big) \\ & + \frac{1}{4}nm\gamma_2(S) + \frac{1}{8}n(m-1)(m+2)\gamma_2(T) + \frac{1}{8}nm\gamma_3(SS) \\ & + \frac{1}{8}n(m-1)(m+2)\gamma_3(TS) + \frac{n}{32m}(m-1)(m^2-4)\gamma_3(TT) \\ & + \frac{1}{12}n^2m^2\gamma_4(SSS) + \frac{1}{8}n^2(m-1)(m+2)\big(\gamma_4(SST) + \gamma_4(STT)\big) \\ & + \frac{n^2}{96m}(m-1)(m^2-4)(m+4)\,\gamma_4(TTT) + \cdots . \end{aligned} \tag{4.8}$$

In this equation $\gamma_1, \dots, \gamma_4$, are the graphs reported in Fig. 2 and the letters in parentheses indicate which auxiliary fields are propagating in the graph. In these graphs one should use the scalar dressed propagators $\widehat{D}_\phi$ and $\widehat{D}_T$ while each vertex is trivially one. In the equation we have of course reported only those graphs that are relevant for the computation of the critical indices at order $1/n^2$. The group-theoretical factors in Eq. (4.8) have been obtained by using Eqs. (4.6) and (4.7) and keeping into account that the vertex $\phi\phi T$ has the form:

$$\phi_{\alpha a}\phi_{\beta b}T^{\gamma\delta} \to \frac{1}{2}\delta^{ab}\left(\delta^{\alpha\gamma}\delta^{\beta\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma} - \frac{2}{m}\delta^{\alpha\beta}\delta^{\gamma\delta}\right). \tag{4.9}$$

Eq. (4.8) is completely general and can be used in the computation of the critical indices for all fixed points: while we should keep into account all terms for the chiral fixed point, we should set $T = 0$ and $S = 0$ for the Heisenberg and the antichiral fixed points, respectively.

From Eq. (4.8) we can derive the skeleton Dyson equations for the dressed propagators. It is enough to compute the variation of Γ with respect to the dressed propagators. We obtain for the field ϕ the equation

$$\begin{aligned}& \widehat{D}_\phi^{-1} - \Delta + u + g_1(S) + \frac{1}{2m}(m-1)(m+2)g_1(T) + g_2(SS) \\ & \quad + \frac{1}{m}(m-1)(m+2)\,g_2(ST) + \frac{1}{4m^2}(m-1)\big(m^2-4\big)\,g_2(TT) + mng_3(SS)\end{aligned}$$

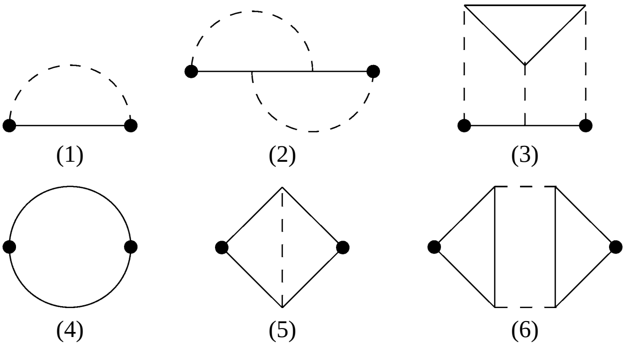

Fig. 3. The graphs appearing in the Dyson equations. The continuous line represents the dressed propagator of the ϕ field, while the dashed line indicates the dressed propagator of an auxiliary field.

$$+\frac{n}{2m}(m-1)(m+2)\big(g_3(STS)+2g_3(TSS)+g_3(TST)+2g_3(TTS)\big)$$
$$+\frac{n}{8m^2}(m-1)\big(m^2-4\big)(m+4)\,g_3(TTT)+\cdots=0, \tag{4.10}$$

while for the auxiliary fields we have

$$D_S^{-1}+c_S+\frac{nm}{2}g_4+nmg_5(S)+\frac{n}{2}(m+2)(m-1)g_5(T)$$
$$+\frac{n^2m^2}{2}g_6(SS)+\frac{n^2}{2}(m+2)(m-1)g_6(TT)+\cdots=0, \tag{4.11}$$
$$\widehat{D}_T^{-1}+c_T+\frac{n}{2}g_4+\frac{n}{2}g_5(S)+\frac{n}{2m}(m-2)g_5(T)$$
$$+n^2g_6(ST)+\frac{n^2}{8m}(m+2)(m-4)g_6(TT)+\cdots=0. \tag{4.12}$$

Here c_S and c_T are two constants, u a momentum-independent contribution due to the tadpoles, $\Delta=p^2+M^2$, and g_i are the graphs reported in Fig. 3. As before, in parentheses we report which auxiliary fields are propagating. Each line is associated to a scalar dressed propagator, while the vertices are one.

The critical exponents are determined following closely the method of Ref. [21]. As in Ref. [21] we introduce two auxiliary functions:

$$a(x)\equiv\frac{\Gamma\big(\frac{d}{2}-x\big)}{\Gamma(x)},\qquad p(x)\equiv\frac{a\big(x-\frac{d}{2}\big)}{\pi^d a(x)}, \tag{4.13}$$

$$q(x,y)\equiv\frac{a(x-y)\,a\big(x+y-\frac{d}{2}\big)}{a(x)a\big(x-\frac{d}{2}\big)}. \tag{4.14}$$

Note the trivial symmetry of the function $q(x,y)$ which will play a role below:

$$q(x,y)=q\big(x,\tfrac{d}{2}-y\big). \tag{4.15}$$

The calculation of the $1/n$ correction starts from assuming for $x\to 0$ the following behavior of the dressed propagators:

$$D_X(x)=\frac{A_X}{x^{2\alpha_X}}\big(1+B_Xx^{2\lambda}+\cdots\big), \tag{4.16}$$

where X is any of the fields. Here α_X is related to the dimension of the field. For $X = \phi$ we have $\alpha_\phi = d/2 - 1 + \eta/2$. The correction term we report is the analytic one in the temperature and therefore $\nu = 1/(2\lambda)$. From Eq. (4.16) we obtain for the inverse functions:

$$D_X^{-1}(x) = \frac{p(\alpha_X)}{A_X x^{2d-2\alpha_X}}\left[1 - B_X q(\alpha_X, \lambda) x^{2\lambda} + \cdots\right]. \tag{4.17}$$

Plugging these expressions in the skeleton equations and equating the corresponding terms we obtain six equations for the amplitudes. Such equations have nontrivial solutions only if

$$\alpha_S = \alpha_T = 2 - \eta \equiv \beta, \tag{4.18}$$

and the following consistency equations are satisfied:

$$p(\alpha_\phi) = 2\frac{M}{n} p(\beta), \tag{4.19}$$

$$\left[q(\alpha_\phi, \lambda) + 1\right] q(\beta, \lambda) = 2, \tag{4.20}$$

where M is a group-theoretical factor that depends on the fixed point:

$$M = \begin{cases} M^+ \equiv \frac{1}{2}(m+1) & \text{(chiral f.p.)}, \\ M^- \equiv \frac{1}{2m}(m-1)(m+2) & \text{(antichiral f.p.)}, \\ M^H \equiv \frac{1}{m} & \text{(Heisenberg f.p.)}. \end{cases} \tag{4.21}$$

Note that $M^- = 0$ for $m = 1, -2$, a result which follows from the fact that only a symmetric traceless tensor propagates. Eq. (4.19) allows the determination of the first large-n coefficient appearing in the expansion of the exponent η,

$$\eta = \frac{\eta_1}{n} + \frac{\eta_2}{n^2} + \frac{\eta_3}{n^3} + \cdots. \tag{4.22}$$

For η_1 we obtain

$$\eta_1 = M\eta_{11}, \tag{4.23}$$

where

$$\eta_{11} = -\frac{4\Gamma(d-2)}{\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}-2\right)\Gamma\left(\frac{d}{2}+1\right)}, \tag{4.24}$$

and the dependence on the fixed point is encoded in the factor M. The quantity η_{11} is the well-known result for the $m = 1$ model; among its most important properties we wish to mention that $\eta_{11} \to 0$ both in the $d \to 4$ and in the $d \to 2$ limit.

Eq. (4.20) should allow the determination of the exponent ν. However, there is a subtle point that has been overlooked in the previous analyses. It is convenient for our discussion to introduce the auxiliary function

$$r(\eta, \lambda) \equiv \left[q\left(\frac{d}{2} + \frac{\eta}{2} - 1, \lambda\right) + 1\right] q(2-\eta, \lambda), \tag{4.25}$$

which corresponds to the left-hand side of Eq. (4.20). Because of Eq. (4.15), the function $r(\eta,\lambda)$ has the symmetry

$$r(\eta,\lambda) \equiv r\left(\eta, \tfrac{d}{2}-\lambda\right). \tag{4.26}$$

As a consequence, once η is fixed by solving Eq. (4.19), we still have the possibility of finding *two* different solutions for $\lambda \equiv 1/2\nu$. It is convenient to parametrize the two solutions by

$$\lambda \equiv \frac{d}{4} \pm \left(\frac{d}{4} - 1 + \rho\right). \tag{4.27}$$

Using Eq. (4.20) and the fact that η is of order $1/n$, one can show that also ρ is of order $1/n$ and therefore has an expansion of the form

$$\rho = \frac{\rho_1}{n} + \frac{\rho_2}{n^2} + \cdots. \tag{4.28}$$

The coefficient ρ_1 is computed from Eq. (4.20) using the fact that for n large

$$\begin{aligned} q\left(\tfrac{d}{2}-1+\tfrac{1}{2}\eta, \tfrac{d}{2}-1+\rho\right) &= q\left(\tfrac{d}{2}-1+\tfrac{1}{2}\eta, 1-\rho\right) \\ &= \frac{4-d}{d}\left(1-\frac{2\rho_1}{\eta_1}\right) + O\left(n^{-1}\right). \end{aligned} \tag{4.29}$$

We then obtain

$$\rho_1 = M\rho_{11}, \tag{4.30}$$

where

$$\rho_{11} = \frac{(d-1)(d-2)}{4-d}\eta_{11}. \tag{4.31}$$

As we observed the consistency equations are satisfied for two independent choices of λ. In order to associate to correct one to each fixed point we use the large-n estimates of the exponent ν given in Eqs. (4.4) and (4.5). Then we have

$$\lambda = \begin{cases} \dfrac{d}{2} - 1 + \dfrac{1}{n}M^{+}\rho_{11} & \text{chiral f.p.,} \\ 1 - \dfrac{1}{n}M^{-}\rho_{11} & \text{antichiral f.p.,} \\ \dfrac{d}{2} - 1 + \dfrac{1}{n}M^{H}\rho_{11} & \text{Heisenberg f.p.} \end{cases} \tag{4.32}$$

The corrections of order $1/n^2$ can be obtained by generalizing the arguments of Refs. [21, 22]. One must only pay attention to insert proper group-theoretical factors in front of the corresponding $m = 1$ contributions: they can be obtained from Eqs. (4.10), (4.11), and (4.12).

We introduce the following definitions:

$$\eta_{21}^{HF} = (\eta_{11})^2\left[\Psi + \frac{d^2+2d-4}{2d(d-2)}\right], \tag{4.33}$$

$$\rho_{21}^{HF} = (\eta_{11})^2\frac{d-1}{(d-4)^2}\left[(d-2)\left(4+2d-d^2\right)\Psi + \frac{32+8d-30d^2+7d^3}{d(d-4)}\right], \tag{4.34}$$

where

$$\Psi \equiv \psi(d-2) + \psi\left(2-\frac{d}{2}\right) - \psi\left(\frac{d}{2}-2\right) - \psi(2). \tag{4.35}$$

Also:

$$\begin{aligned}
\eta_{21}^{(a)} &= (\eta_{11})^2\left(\frac{d}{4-d}\Psi + \frac{d(6-d)}{2(4-d)^2}\right), \\
\eta_{21}^{(b)} &= (\eta_{11})^2\left(\frac{d(d-3)}{4-d}\Psi + \frac{d(d-2)}{4-d}\right), \\
\rho_{21}^{(a)} &= \frac{d(d-2)}{2(4-d)^2}(\eta_{11})^2\left[2(d-1)\Psi + \frac{3}{2}d(d-3)R_1 + 2d - 8\right. \\
&\qquad \left. + \frac{6}{4-d} - \frac{4}{d-2} + \frac{12}{(d-2)^2}\right], \\
\rho_{21}^{(b)} &= \frac{d(d-2)}{2(4-d)^2}(\eta_{11})^2 \\
&\quad \times \left[\frac{4(d-3)}{(d-2)}\Psi + \frac{3}{2}d(3d-8)R_1 + \frac{2d(d-3)^2}{(4-d)}\left(6R_1 - R_2 - R_3^2\right) + d^2\right. \\
&\qquad \left. + d + 20 - \frac{12(4-d)}{(d-2)^2} - \frac{16}{4-d}\right],
\end{aligned} \tag{4.36, 4.37}$$

where

$$\begin{aligned}
R_1 &\equiv \psi'\left(\frac{d}{2}-1\right) - \psi'(1), \\
R_2 &\equiv \psi'(d-3) - \psi'\left(2-\frac{d}{2}\right) - \psi'\left(\frac{d}{2}-1\right) + \psi'(1), \\
R_3 &\equiv \psi(d-3) + \psi\left(2-\frac{d}{2}\right) - \psi\left(\frac{d}{2}-1\right) - \psi(1).
\end{aligned} \tag{4.38}$$

Recalling the above considerations about the choice of λ in conjunction with the choice of the critical point, we can now write down our final results for the $1/n$ expansion of the critical exponents in the $O(n) \times O(m)$ models, both for the chiral (stable) critical point and for the antichiral (unstable) one:

$$\begin{aligned}
\eta^+ &= \frac{m+1}{2n}\eta_{11} + \frac{1}{n^2}\left[\frac{(m+1)^2}{4}\eta_{21}^{HF} + \frac{m+3}{4}\eta_{21}^{(a)} + \frac{m^2+3m+4}{8}\eta_{21}^{(b)}\right] \\
&\quad + \mathrm{O}\left(\frac{1}{n^3}\right),
\end{aligned} \tag{4.39}$$

$$\begin{aligned}
\eta^- &= \frac{(m-1)(m+2)}{2mn}\eta_{11} \\
&\quad + \frac{1}{n^2}\left[\frac{(m-1)^2(m+2)^2}{4m^2}\eta_{21}^{HF} + \frac{(m-1)(m^2-4)}{4m^2}\eta_{21}^{(a)}\right. \\
&\qquad \left. + \frac{(m-1)(m^2-4)(m+4)}{8m^2}\eta_{21}^{(b)}\right] + \mathrm{O}\left(\frac{1}{n^3}\right),
\end{aligned} \tag{4.40}$$

$$\nu^+ = \frac{1}{d-2} - \frac{2}{(d-2)^2}\frac{m+1}{2n}\rho_{11} - \frac{2}{(d-2)^2}\frac{1}{n^2}\left[\frac{(m+1)^2}{4}\rho_{21}^{HF} + \frac{m+3}{4}\rho_{21}^{(a)} + \frac{m^2+3m+4}{8}\rho_{21}^{(b)} - \frac{2}{d-2}\frac{(m+1)^2}{4}\rho_{11}^2\right] + \mathrm{O}\left(\frac{1}{n^3}\right), \tag{4.41}$$

$$\nu^- = \frac{1}{2} + \frac{1}{2}\frac{(m-1)(m+2)}{2mn}\rho_{11} + \frac{1}{2}\frac{1}{n^2}\left[\frac{(m-1)^2(m+2)^2}{4m^2}\rho_{21}^{HF} + \frac{(m-1)(m^2-4)}{4m^2}\rho_{21}^{(a)} + \frac{(m-1)(m^2-4)(m+4)}{8m^2}\rho_{21}^{(b)} + \frac{(m-1)^2(m+2)^2}{4m^2}\rho_{11}^2\right] + \mathrm{O}\left(\frac{1}{n^3}\right). \tag{4.42}$$

The expressions for the stable fixed point at order $1/n$ coincide with those of Ref. [14]. Note that $\eta^- = 0$, $\nu^- = 1/2$ for $m = -2, 1$, in agreement with our ϵ-expansion results.

It is possible to expand the above large-n results in powers of $\epsilon = 4 - d$. The resulting expressions can be compared with the the ϵ-expansion results for the LGW Hamiltonian presented in the previous section. We find full agreement both for the stable and the unstable fixed point for all m, thus confirming our identification of the large-n fixed points with the perturbative ones.

For $d = 3$ the large-n expansions simplify to:

$$\eta^+ = \frac{4(m+1)}{3n\pi^2} + \frac{16(m^2-7m-26)}{27n^2\pi^4} + \mathrm{O}\left(\frac{1}{n^3}\right), \tag{4.43}$$

$$\eta^- = \frac{4(m-1)(m+2)}{3mn\pi^2} + \frac{16(m-1)(m+2)(m^2-8m-2)}{27m^2n^2\pi^4} + \mathrm{O}\left(\frac{1}{n^3}\right), \tag{4.44}$$

$$\nu^+ = 1 - \frac{16(m+1)}{3n\pi^2} - \frac{1}{n^2}\left(\frac{4(m^2+3m+4)}{\pi^2} - \frac{64(5m^2+19m+32)}{27\pi^4}\right) + \mathrm{O}\left(\frac{1}{n^3}\right), \tag{4.45}$$

$$\nu^- = \frac{1}{2} + \frac{4(m+2)(m-1)}{3mn\pi^2} + \frac{(m-1)(m+2)}{27m^2n^2\pi^4}\left[16\left(13m^2+4m+28\right) + 27(m+4)(m-2)\pi^2\right] + \mathrm{O}\left(\frac{1}{n^3}\right). \tag{4.46}$$

5. The $1/n$-expansion in the presence of a vector field

It is quite instructive to extend the discussion of the previous paragraph to the more general case in which gauge-invariant vector degrees of freedom are allowed. This corresponds to studying the general Lagrangian (2.1) with $t_0 = 0$.

In the large-n limit one starts from Eqs. (2.8) and (2.9) with $t_0 = 0$. As discussed by Hikami [18], the gauge kinetic term is irrelevant for $d < 4$, as well as the T^2 and S^2 terms in Eq. (2.8). Thus, the large-n limit can be studied by keeping only into account $H_{\rm eff}$. The discussion of the fixed points is identical to that presented in the previous Section. If $g_0 \neq 0$ a new set of fixed points appear: for generic $v_0 > 0$, $w_0 > 0$ we have the chiral-gauge fixed point in which all excitations (S, $T^{\alpha\beta}$, $A^{\alpha\beta}_\mu$) propagate; for $w_0 = 0$ we have the antichiral-gauge fixed point ($T^{\alpha\beta}$, $A^{\alpha\beta}_\mu$), for $v_0 = 0$ the Heisenberg-gauge fixed point (S, $A^{\alpha\beta}_\mu$), and for $u_0 = v_0 = w_0 = 0$ the pure gauge fixed point ($A^{\alpha\beta}_\mu$).

Here, we want to compute the critical behavior for $n \to \infty$, keeping only the leading correction. Since the model is gauge-invariant, the large-n propagator of the field $A^{\alpha\beta}_\mu$ is not uniquely defined. Indeed, by integration over the field $\boldsymbol{\phi}$ we obtain a coupling $\frac{1}{2} A^{\alpha\beta}_\mu A^{\gamma\delta}_\nu M^{\alpha\beta,\gamma\delta}_{\mu\nu}$, where in momentum space

$$M^{\alpha\beta,\gamma\delta}_{\mu\nu}(p) = \frac{1}{2}\left(\delta^{\alpha\gamma}\delta^{\beta\delta} - \delta^{\alpha\delta}\delta^{\beta\gamma}\right)\left(p_\mu p_\nu - p^2\delta_{\mu\nu}\right)\widehat{M}\left(p^2\right), \tag{5.1}$$

which is not invertible. A propagator for the field $A^{\alpha\beta}_\mu$ is obtained by adding a gauge-fixing term, that introduces a longitudinal term, makes the matrix invertible, but does not contribute to physical quantities [24].

The calculation is completely analogous to that performed in the previous section. For the second Legendre transform, we obtain to order $O(1/n)$

$$\Gamma = \Gamma(A = 0) + \frac{1}{2}\operatorname{Tr}\log D_A + \frac{n}{2}\gamma_1(A) + \frac{nm(m-1)}{8}\gamma_2(A) + \cdots, \tag{5.2}$$

where γ_1 and γ_2 correspond to the graphs reported in Fig. 2, all group-theoretical factors have been explicitly singled out, and $\Gamma(A = 0)$ is the expression reported in Eq. (4.8).

Generalizing the results of Refs. [18,24] we obtain then

$$\rho_1 = \rho_{11}\left[M + \frac{d^2-1}{2}(m-1)\right], \tag{5.3}$$

where M is a group-theoretical factor defined in Eq. (4.21) (for the pure-gauge fixed point $M = 0$) and ρ_1 is the $1/n$ contribution to the exponent ρ defined in Eq. (4.27). Note that the result (5.3) does not depend on the gauge fixing used to define the propagator of the field A. In principle, one could also compute a gauge-fixing dependent exponent η but its significance is not so clear, since, because of the gauge invariance, the field $\boldsymbol{\phi}$ does not have a well-defined anomalous dimension.

As a check we can compare our results with those obtained in perturbation theory for the gauge Hamiltonian (2.1) with $t_0 = 0$. Hikami [18] determined the following one-loop β functions in the $\overline{\rm MS}$ scheme:

$$\beta_u = -\epsilon u + \left[\frac{mn+8}{6}u^2 + \frac{(m-1)(n-1)}{6}\left(\frac{1}{2}v^2 - uv\right)\right.$$
$$\left. - \frac{3}{2}(m-1)u\alpha + \frac{9}{8}(m-1)\alpha^2\right],$$
$$\beta_v = -\epsilon v + \left[\frac{m+n-8}{6}v^2 + 2uv - \frac{3}{2}(m-1)v\alpha + \frac{9}{4}(m-2)\alpha^2\right],$$
$$\beta_\alpha = -\epsilon\alpha + \left[\frac{n}{12} - \frac{11}{3}(m-2)\right]\alpha^2, \tag{5.4}$$

where $\alpha = N_d g^2$. These expressions generalize the results presented in Section 3. Choosing the $\alpha^* = 0$ solution of the fixed-point equations we obtain the four critical points already discussed. However, if we choose the solution

$$\alpha^* = \epsilon\left[\frac{n}{12} - \frac{11}{3}(m-2)\right]^{-1} + O(\epsilon^2), \tag{5.5}$$

we find another set of four critical points, corresponding to the distinct roots of a quartic algebraic equation. This equation cannot be solved in closed form, but it is easy to find its roots in the form of a series in the powers of $1/n$. The relevant terms in the expansion of the roots are:

- Chiral-gauge fixed point:

$$\frac{u^*}{6\epsilon} = \frac{1}{n} + \frac{2-10m}{n^2} + O(\epsilon), \qquad \frac{v^*}{6\epsilon} = \frac{1}{n} + \frac{32-10m}{n^2} + O(\epsilon). \tag{5.6}$$

- Antichiral-gauge fixed point:

$$\frac{u^*}{6\epsilon} = \frac{m-1}{mn} + \frac{(m-1)(16+88m-10m^2)}{m^2n^2} + O(\epsilon),$$
$$\frac{v^*}{6\epsilon} = \frac{1}{n} + \frac{12+32m-10m^2}{mn^2} + O(\epsilon). \tag{5.7}$$

- Heisenberg-gauge fixed point:

$$\frac{u^*}{6\epsilon} = \frac{1}{mn} + \frac{27m^3-117m^2+90m-8}{m^2n^2} + O(\epsilon),$$
$$\frac{v^*}{6\epsilon} = \frac{27(m-2)}{n^2} + O(\epsilon). \tag{5.8}$$

- Pure-gauge fixed point:

$$\frac{u^*}{6\epsilon} = \frac{27(m-1)}{n^2} + O(\epsilon), \qquad \frac{v^*}{6\epsilon} = \frac{27(m-2)}{n^2} + O(\epsilon). \tag{5.9}$$

Thus, the gauge model has in general 8 fixed points and, at least for large n, the chiral-gauge fixed point is the stable one.[7] Substituting these expressions into the relationship [18]

$$\nu^{-1} = 2 - \frac{mn+2}{6}u^* + \frac{(m-1)(n-1)}{6}v^* + \frac{3(m-1)}{4}\alpha^* + O(\epsilon^2), \tag{5.10}$$

[7] This is not true for generic n and m. In order to obtain the general fixed-point structure, one should generalize the analysis performed in Section 3.

we find a $1/n$ expanded form of the $\mathrm{O}(\epsilon)$ contribution to the critical exponent ν for each of the four solutions:

$$\frac{1}{\nu} = \begin{cases} 2-\epsilon+(48m-42)\dfrac{\epsilon}{n}+\mathrm{O}(\epsilon^2,n^{-2}) & \text{chiral-gauge f.p.,} \\ 2-\dfrac{6(m-1)(8m+1)}{m}\dfrac{\epsilon}{n}+\mathrm{O}(\epsilon^2,n^{-2}) & \text{antichiral-gauge f.p.,} \\ 2-\epsilon+\dfrac{45m^2-45m+6}{m}\dfrac{\epsilon}{n}+\mathrm{O}(\epsilon^2,n^{-2}) & \text{Heisenberg-gauge f.p.,} \\ 2-45(m-1)\dfrac{\epsilon}{n}+\mathrm{O}(\epsilon^2,n^{-2}) & \text{pure-gauge f.p.} \end{cases} \tag{5.11}$$

It is then a matter of trivial algebra to verify that these expressions are in full agreement with the ϵ-expansion of the four solutions discussed above in the context of the $1/n$ expansion, which explains the names we have given to each fixed point. Again, we think it is important to notice that the conformal bootstrap approach can naturally accommodate for the expansion of *all* solutions, not only the stable ones.

6. The $\tilde{\epsilon}$-expansion of $O(n) \times O(m)$ nonlinear σ models

The LGW Hamiltonian is the natural tool for the study of the critical behavior of systems near the upper critical dimension $d = 4$. If one is interested in the critical behavior near the lower critical dimension, one can still use perturbation theory, applied however to the nonlinear σ models (NLσ). The degrees of freedom of the NLσ models should correspond to the interacting Goldstone modes of the system, while the effect of the massive modes is only taken into account in the form of constraints for the massless fields. In this context, it is possible to perform an expansion in powers of $\tilde{\epsilon} \equiv d-2$. In the present paper we extend the results of Refs. [14,15,32,33] to a general $O(n) \times O(m)$ symmetry group and to $\mathrm{O}(\tilde{\epsilon}^2)$. Comparing with our previous $1/n$ expansion results we will be able to identify the nature of the fixed points of the NLσ. In particular, we will show explicitly that the stable fixed point of the generic model can be identified with the stable fixed point of the LGW theory.

We consider the Hamiltonian (2.6). This Hamiltonian is geometric in nature, and its variables are best understood as generalized coordinates spanning a manifold. The cases we shall be interested in correspond to manifolds that are coset spaces. More specifically, we must study the coset space (remember that $n \geqslant m$)

$$\frac{O(n) \times O(m)}{O(n-m) \times O(m)}, \tag{6.1}$$

which is topologically equivalent to

$$\frac{O(n)}{O(n-m)}. \tag{6.2}$$

Associating fields π^I with the Goldstone modes that correspond to the broken generators

$$\big\{\mathrm{Lie}\big(O(n)\big) - \mathrm{Lie}\big(O(n-m)\big)\big\}, \tag{6.3}$$

the Hamiltonian may be formulated in purely geometric terms, i.e.,

$$\widetilde{H} = \frac{1}{2} g_{IJ}(\pi) \nabla \pi^I \nabla \pi^J. \tag{6.4}$$

The couplings $T_i \equiv 1/\eta_i$ are related to the independent entries of the tangent-space metric η_{IJ}.

A number of important RG properties of the NLσ models have been derived in the general case by Friedan [25] and specialized to the models of interest in Refs. [15,32,33]. If R_{IJKL} and R_{IJ} are, respectively, the Riemann and Ricci tensor for the metric g_{IJ}, the RG β functions of the model can be written to two-loop order as

$$\beta_{IJ} \equiv s \frac{\partial g_{IJ}}{\partial s} = -\tilde{\epsilon} \eta_{IJ} + R_{IJ} + \frac{1}{2} R_{IPQR} R_J^{\ PQR} + \cdots. \tag{6.5}$$

The number of algebraically independent β-functions β_i coincides with the number of independent couplings T_i. Therefore, we should consider two β-functions associated with T_1 and T_2. The fixed points are determined from the equations

$$\beta_i\left(T_1^*, T_2^*\right) = 0, \tag{6.6}$$

that can be perturbatively solved in powers of $\tilde{\epsilon}$.

The evaluation of the two-loop β functions for arbitrary m and n requires no special skills, but it takes some time and effort in view of the many computational steps involved. Without belaboring on the intermediate steps, we report here our final results:

$$\begin{aligned}
\beta_1 &\equiv -s \frac{\partial T_1}{\partial s} \\
&= -\tilde{\epsilon} T_1 + \left[n - 2 - \frac{m-1}{2} X \right] T_1^2 + \left[A + BX + CX^2 \right] T_1^3 + \mathrm{O}\left(T_i^4\right), \\
\beta_2 &\equiv -s \frac{\partial T_2}{\partial s} \\
&= -\tilde{\epsilon} T_2 + \left[\frac{m-2}{2} + \frac{n-m}{2} X^2 \right] T_2^2 + \left[DX^2 + EX^3 + FX^4 + G \right] T_2^3 \\
&\quad + \mathrm{O}\left(T_i^4\right),
\end{aligned} \tag{6.7}$$

where X is shorthand for the ratio T_1/T_2 and we have defined the coefficients:

$$\begin{aligned}
A(n,m) &\equiv 2m(n-m) - n + \frac{3}{8}(m-1)(m-2), \\
B(n,m) &\equiv -(m-1)\left[\frac{3}{2}(n-m) + \frac{3}{8}(m-2) \right], \\
C(n,m) &\equiv (m-1)\left[\frac{3}{8}(n-m) + \frac{m}{8} \right], \\
D(n,m) &\equiv \frac{3}{4}(n-m)(m-2), \\
E(n,m) &\equiv -\frac{3}{4}(n-m)(m-2),
\end{aligned}$$

$$F(n,m) \equiv \frac{1}{8}(n-m)(2m-3),$$
$$G(n,m) \equiv \frac{1}{8}(m-2)^2. \tag{6.8}$$

Our results were submitted to a number of basic consistency checks:

(i) when $m = 1$ there is no η_2 coupling, and $\beta_1(T_1)$ reduces to the well-known β function for the vector σ-model defined on the coset space $O(n)/O(n-1)$;

(ii) when $m = 2$ our expressions reduce to those of Ref. [15];

(iii) when $m = n$ there is no η_1 coupling,[8] and the model reduces to a standard $O(n) \times O(n)$ principal chiral model. One may verify that $\beta_2(T_2)$ is directly related to the known β function of these models [26].

One may also consider the "gauge" limit $\eta_2 = 0$, which was studied by Hikami [18]: the identification with Hikami's coupling is $\eta_1 \to 1/t$. One must however recognize that the limit is singular, and as a consequence the function $\beta_1^{\text{gauge}}(t)$ is *not* obtainable from our expressions by setting $X = 0$ (with the notable exception of $m = 2$ models). If we assume $\eta_2 = 0$ from the very beginning of our calculation, the result is

$$\beta_1^{\text{gauge}}(t) = -\tilde{\epsilon}t + (n-2)t^2 + [2m(n-m) - n]t^3 + \mathrm{O}(t^4), \tag{6.9}$$

consistent with that reported in Ref. [18].

The β-functions (6.7) are the starting point for the perturbative evaluation of the critical points and exponents to $\mathrm{O}(\tilde{\epsilon}^2)$. A consistent ansatz for the simultaneous solutions of the equations $\beta_i(T_1^*, T_2^*) = 0$ is the following:

$$T_1^* = t_1\tilde{\epsilon} + t_2\tilde{\epsilon}^2 + \mathrm{O}(\tilde{\epsilon}^3), \tag{6.10}$$
$$X^* = X_0 + X_1\tilde{\epsilon} + \mathrm{O}(\tilde{\epsilon}^2). \tag{6.11}$$

It is straightforward to obtain the following algebraic equations for t_1 and X_0:

$$-\frac{1}{t_1} + (n-2) - \frac{m-1}{2}X_0 = 0, \tag{6.12}$$
$$-\frac{1}{t_1}X_0 + \frac{m-2}{2} + \frac{n-m}{2}X_0^2 = 0. \tag{6.13}$$

They are trivially solved by

$$X_0^{\pm} = \frac{n-2 \pm \sqrt{(n-2)^2 - (n-1)(m-2)}}{n-1},$$
$$\frac{1}{t_1^{\pm}} = n - 2 - \frac{m-1}{2}X_0^{\pm}. \tag{6.14}$$

Iterating the procedure we may also obtain

$$X_1^{\pm} = \frac{(D-A)(X_0^{\pm})^2 + (E-B)(X_0^{\pm})^3 + (F-C)(X_0^{\pm})^4 + G}{X_0^{\pm}\big[(n-2) - (n-1)X_0^{\pm}\big]\big(n-2-\frac{m-1}{2}X_0^{\pm}\big)}, \tag{6.15}$$

[8] If $n = m$ the vectors $\mathbf{e}^\alpha$ are an orthogonal basis in R^m and therefore satisfy the completeness relation $\sum_\alpha(\mathbf{v}\cdot\mathbf{e}^\alpha)(\mathbf{w}\cdot\mathbf{e}^\alpha) = \mathbf{v}\cdot\mathbf{w}$ for all vectors $\mathbf{v}$, $\mathbf{w}$. Then, it is a simple matter to show that the second term in Eq. (2.6) is one half of the first one.

$$t_2^{\pm} = \left[\frac{m-1}{2} X_1^{\pm} - \frac{A + B X_0^{\pm} + C (X_0^{\pm})^2}{n - 2 - \frac{m-1}{2} X_0^{\pm}}\right] \frac{1}{\left(n - 2 - \frac{m-1}{2} X_0^{\pm}\right)^2}. \tag{6.16}$$

This analysis shows the existence of a couple of nontrivial fixed points of the RG equations. However, it is evident from Eq. (6.14) that such a pair of solutions does not exist for all m and n: for some values $X_0^{\pm}$ is indeed complex. Repeating the analysis we performed in Section 3 for the ϵ-expansion, we see that these two fixed points exist only for $n > \widetilde{n}^+$ and $n < \tilde{n}^-$, where

$$\tilde{n}^{\pm} = \frac{m + 2 \pm \sqrt{m^2 - 4}}{2} \pm \frac{1}{2}\sqrt{\frac{m+2}{m-2}} \frac{m^2 + 4 \pm m\sqrt{m^2 - 4}}{(1 \pm \sqrt{m^2 - 4})(m \pm \sqrt{m^2 - 4})} \tilde{\epsilon} + \mathrm{O}(\tilde{\epsilon}^2). \tag{6.17}$$

Note that for $\tilde{\epsilon}$ small, we have $m \leqslant \tilde{n}^+ < m + 1$ and $\tilde{n}^- < m$. Thus, since $n \geqslant m$, all models with integer $n \geqslant m + 1$, have a a pair of nontrivial fixed points, at least for $\tilde{\epsilon}$ small. Beside these two fixed points, there is also a fixed point for $T_1 = 0$ and $T_2 = T_2^*$ that belongs to the universality class of the $O(m) \times O(n)$ principal chiral model. Such a fixed point always exists perturbatively for $m > 2$ and in particular is the only present for $n = m$.

As one may easily notice, the expansion (6.17) is singular when $m = 2$. This is related to the following peculiar feature of $m = 2$ models: for any value of n the (unstable) fixed point corresponds to the solution $X^* = 0$, and as a consequence we observe its coalescence with the "gauge" fixed point obtained by setting $\eta_2 = 0$. This phenomenon does not happen for $m > 2$. In this case, the gauge fixed point and the antichiral fixed point are distinct.

The exponent η is easily computed. To two-loop order we have

$$\eta = -\tilde{\epsilon} + (n - m) T_1^* + (m - 1) T_2^* + \mathrm{O}(\tilde{\epsilon}^3). \tag{6.18}$$

Substituting the expression of the fixed point, we obtain

$$\eta = (n - m)\left(t_1 \tilde{\epsilon} + t_2 \tilde{\epsilon}^2\right) + (m - 1)\left[\frac{t_1}{X_0}\tilde{\epsilon} + \left(\frac{t_2}{X_0} - \frac{t_1 X_1}{X_0^2}\right)\tilde{\epsilon}^2\right] - \tilde{\epsilon} + \mathrm{O}(\tilde{\epsilon}^3). \tag{6.19}$$

We can expand η at the stable critical point in powers of $1/n$, obtaining

$$\eta = \left[\frac{m+1}{2n} + \frac{3m^2 + 7m + 6}{8n^2} + \mathrm{O}\left(\frac{1}{n^3}\right)\right]\tilde{\epsilon} - \left[\frac{m+1}{2n} + \frac{3(m+1)^2}{4n^2} + \mathrm{O}\left(\frac{1}{n^3}\right)\right]\tilde{\epsilon}^2 + \mathrm{O}(\tilde{\epsilon}^3). \tag{6.20}$$

If we compare such expression with the $\tilde{\epsilon}$-expansion of η^+ as obtained from the large-n expansion of Section 4, Eq. (4.40), we find complete agreement, confirming the identification of the two fixed points.

In NLσ models the evaluation of stability goes together with the evaluation of the critical exponent ν, since both are related to the eigenvalues of the derivative matrix

$$\frac{\partial \beta_i}{\partial T_j}\left(T_1^*, T_2^*\right). \tag{6.21}$$

More precisely, stability requires that the above matrix possesses only one positive eigenvalue $\lambda_+ = \nu^{-1}$. The presence of two positive eigenvalues signals the instability of the fixed point. It is possible to evaluate the above-mentioned eigenvalues in the context of the $\tilde{\epsilon}$ expansion, obtaining:

$$\lambda_+ = \tilde{\epsilon} - \nu_2(n,m)\tilde{\epsilon}^2 + \mathrm{O}(\tilde{\epsilon}^3), \tag{6.22}$$

$$\lambda_- = \lambda_1(n,m)\tilde{\epsilon} + \lambda_2(n,m)\tilde{\epsilon}^2 + \mathrm{O}(\tilde{\epsilon}^3), \tag{6.23}$$

where

$$\begin{aligned}
&\nu_2(n,m) = \frac{a_{112}a_{221} + a_{111}a_{222} - a_{122}a_{211} - a_{121}a_{212}}{a_{111} + a_{221}},\\
&\lambda_1(n,m) = 1 - a_{111} - a_{221}, \qquad \lambda_2(n,m) = \nu_2(n,m) - a_{112} - a_{222}.
\end{aligned} \tag{6.24}$$

Here we defined

$$\begin{aligned}
a_{111} &\equiv \frac{m-1}{2} t_1 X_0 = -a_{121} X_0,\\
a_{211} &\equiv -(n-m) t_1 X_0^2 = -a_{221} X_0,\\
a_{112} &\equiv \frac{m-1}{2}(t_1 X_1 + t_2 X_0) - t_1^2\big(A + 2B X_0 + 3C X_0^2\big),\\
a_{122} &\equiv -\frac{m-1}{2} t_2 + t_1^2 (B + 2C X_0),\\
a_{212} &\equiv -(n-m)\big(2t_1 X_0 X_1 + t_2 X_0^2\big) - t_1^2\big(2D X_0 + 3E X_0^2 + 4F X_0^3\big),\\
a_{222} &\equiv (n-m)(t_1 X_1 + t_2 X_0) + t_1^2\big(D + 2E X_0 + 3F X_0^2 - G/X_0^2\big).
\end{aligned} \tag{6.25}$$

The $1/n$ expansion of ν evaluated at the stable fixed point coincides with the $\tilde{\epsilon}$ expansion of ν^+ obtained in Section 4. The result of the expansion is:

$$\nu_2 = \frac{m+1}{2}\frac{1}{n} + \frac{(m+1)^2}{2}\frac{1}{n^2} + \mathrm{O}\left(\frac{1}{n^3}\right). \tag{6.26}$$

Notice that the coalescence value $\tilde{n}^\pm$ can be easily determined within the $\tilde{\epsilon}$ expansion by imposing the condition

$$\lambda_-\big[\tilde{n}^\pm, m\big] = 0. \tag{6.27}$$

While the stable fixed point is identified with the chiral fixed point of the $O(n) \times O(m)$ LGW model, the unstable one is unrelated to those of the LGW model. In order to understand its nature, it is again useful to consider the large-n limit. From Eq. (6.14) one observes that $X_0^- \sim 1/n$ as $n \to \infty$. Therefore, the fixed point survives in the large-n limit if we scale the coupling constants as $T_1 = \mathrm{O}(1/n)$, $T_2 = \mathrm{O}(1)$. Then, for large n the β functions decouple. Moreover, while $\beta_1(T_1)$ gets no contributions beyond one loop (as usual in vector models), $\beta_2(T_2)$ turns into the β function of an $O(m) \times O(m)$ principal chiral model [26]. Therefore, in this case the pattern of spontaneous symmetry breaking is highly nontrivial, even in the strict $n \to \infty$ limit. This can be understood from Eqs. (2.8) and (2.9). In the large-n limit the relevant Hamiltonian is

$$H = H_{\mathrm{eff}} + \frac{t_0}{2} A^2. \tag{6.28}$$

Now, the field $\boldsymbol{\phi}$ couples only to the gauge-invariant degrees of freedom, and thus at the saddle point the field $A_\mu^{\alpha\beta}$ is a pure gauge transformation, i.e.,

$$A_\mu^{\alpha\beta} = \left(O^{-1}\partial_\mu O\right)^{\alpha\beta}, \tag{6.29}$$

where O is an $O(m)$ matrix. Thus, for $n \to \infty$, the Hamiltonian can be rewritten as the sum of two terms:

$$H = \frac{1}{2}\sum_\alpha \boldsymbol{\phi}_\alpha \cdot \left(\partial_\mu\partial_\mu + M^2\right)\boldsymbol{\phi}_\alpha + \frac{t_0}{2}\,\mathrm{Tr}\,\partial_\mu O^{-1}\partial_\mu O, \tag{6.30}$$

where, as in Section 4, $M^2 = r_0 + w_0\sigma$. Thus, the unstable fixed point is directly related to the nontrivial fixed point of the principal $O(m) \times O(m)$ chiral model.

Finally, we consider the gauge limit $\eta_1 = 1/t$, $\eta_2 = 0$. We find a nontrivial fixed point:

$$t^* = \frac{1}{n-2}\tilde{\epsilon} + \frac{n - 2m(n-m)}{(n-2)^3}\tilde{\epsilon}^2 + \mathrm{O}\left(\tilde{\epsilon}^3\right). \tag{6.31}$$

Correspondingly we obtain:

$$\nu^{-1} \equiv \beta'(t^*) = \tilde{\epsilon} + \frac{2m(n-m) - n}{(n-2)^2}\tilde{\epsilon}^2 + \mathrm{O}\left(\tilde{\epsilon}^3\right). \tag{6.32}$$

The $1/n$ expansion of the above result gives

$$\nu^{-1} = \tilde{\epsilon} + \frac{2m-1}{n}\tilde{\epsilon}^2 + \mathrm{O}\left(\tilde{\epsilon}^3\right), \tag{6.33}$$

and one may easily check that it agrees with the $\tilde{\epsilon}$ expansion of the results found in Section 5 for the stable fixed point of the gauge model, see Eq. (5.3), with $M = M^+$.

The behavior of NLσ models in the case $m = n-1$ is worth a special discussion [33]. In this case one may naturally define a new n-component field $\mathbf{e}_n$ such that $\mathbf{e}_n \cdot \mathbf{e}_n = 1$ and $\mathbf{e}_\alpha \cdot \mathbf{e}_n = 0$, and one can show that

$$\widetilde{H} = \frac{1}{4}\eta_2 \sum_{\alpha=1}^{n-1} \partial_\mu \mathbf{e}_\alpha \cdot \partial_\mu \mathbf{e}_\alpha + \left(\frac{1}{2}\eta_1 - \frac{1}{4}\eta_2\right)\partial_\mu \mathbf{e}_n \cdot \partial_\mu \mathbf{e}_n. \tag{6.34}$$

When $\eta_1 = \eta_2$, this is the Hamiltonian of an $O(n) \times O(n)$ principal chiral model. The stable fixed point is characterized by the property that $X^* = 1$ and one finds:

$$t_1 = \frac{2}{n-2}, \qquad t_2 = -\frac{1}{n-2}. \tag{6.35}$$

Direct substitution shows that

$$\nu^{-1} = \tilde{\epsilon} + \frac{1}{2}\tilde{\epsilon}^2 + \mathrm{O}\left(\tilde{\epsilon}^3\right), \tag{6.36}$$

$$\eta = \frac{n}{n-2}\tilde{\epsilon} - \frac{n-1}{n-2}\tilde{\epsilon}^2 + \mathrm{O}\left(\tilde{\epsilon}^3\right). \tag{6.37}$$

As one may easily check, these exponents coincide with those obtained in the case $m = n$. Therefore, the symmetry of the $O(n) \times O(n-1)$ model is dynamically promoted to $O(n) \times O(n)$ at the stable fixed point, thus generalizing the results of Ref. [15] concerning the case $O(3) \times O(2)$. This property is certainly true for sufficiently small $d > 2$, but at this level of analysis it is impossible to establish the maximum dimension $\tilde{d}$ for which a stable critical point possessing the enlarged symmetry can be found.

7. Conclusions

At this stage of our analysis, we can draw quite general conclusions on the general fixed-point structure of the models with $O(n) \times O(m)$ for all dimensions $2 \leqslant d \leqslant 4$. By comparing the $\tilde{\epsilon}$-expansion results near two dimensions, the ϵ-expansion results near four dimensions and the large-n results, we have been able to identify the nature of all (stable and unstable) fixed points of these models. In particular, the LGW stable fixed point coincides with the stable one of the NLσ model. We thus quantitatively confirm one of the conclusions of Refs. [15,32]: above $\bar{n}(2, d)$ in the (m, d) plane a second-order phase transition occurs, and with varying d, for n large enough, the critical exponents smoothly interpolate between NLσ and GLW model values.

The unstable fixed points, which give rise to different types of tricritical behavior and crossover phenomena, are instead unrelated and correspond to systems with completely different types of excitations.

The correspondence we have found holds only for sufficiently large values of n, i.e., for $n \geqslant \bar{n}(m, d)$, which is the region of analyticity of the large-n expansion. Since the $1/n$ expansion commutes with the $\epsilon = 4 - d$ expansion of the LGW Hamiltonian, one may expect $\bar{n}(m, d)$ to coincide with $n^+(m, d)$ in a neighborhood of $d = 4$; $n^+(m, d)$ might in turn be evaluated within the $1/n$ expansion by solving the coalescence equation:

$$\nu^+(m, n^+, d) = \nu^-(m, n^+, d). \tag{7.1}$$

The estimate obtained from the lowest order approximation for $\nu^\pm$ is:

$$n^+(m, d) \approx 2\left(m + 1 - \frac{1}{m}\right)\frac{\rho_{11}(d)}{4 - d}. \tag{7.2}$$

This expression has been obtained by solving exactly the equation $1/\nu^+(m, n^+, d) = 1/\nu^-(m, n^+, d)$, where $1/\nu^+(m, n^+, d)$ and $1/\nu^-(m, n^+, d)$ are expanded to order $1/n$. This expression shows the correct qualitative behavior for all $2 \leqslant d \leqslant 4$ and a rough quantitative agreement. It is possible to improve the approximation by including the $1/n^2$ correction in Eq. (7.1). For $m = 2$ and $d = 3$, it predicts $n^+(2, 3) \approx 5.3$, in substantial agreement with the results obtained by using the ERG approach [8,9], the perturbative expansion in fixed dimension [10], and, as we shall show below, the ϵ-expansion.

We want now to understand the behavior of $\bar{n}(m, d)$ near two dimensions. Near two dimensions, using the NLσ model results we know that the (LGW and NLσ) stable fixed point exists only for $n > \widetilde{n}^+(m, d)$, so that in this case $\bar{n}(m, d) = \widetilde{n}^+(m, d)$. Thus, for generic values of d we conjecture

$$\bar{n}(m, d) = \begin{cases} n^+(m, d) & \text{for } d_c(m) \leqslant d \leqslant 4, \\ \widetilde{n}^+(m, d) & \text{for } 2 \leqslant d \leqslant d_c(m), \end{cases}$$

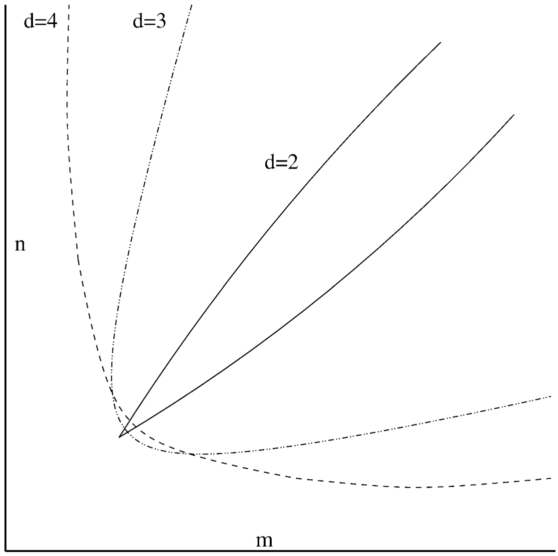

Fig. 4. Sketch of the coalescence line as a function of the dimension d.

where $d_c(m)$ is a critical dimension that we cannot determine with our means. Of course, this expression is valid for $n \geqslant m$. The symmetry under exchange of n and m, implies the existence of a similar boundary curve in the region $n \leqslant m$, obtained by interchanging n and m. A sketch of $\bar{n}(m, d)$ is reported in Fig. 4.

Now, let us discuss the behavior of the LGW fixed points for $d \to 2$. Since the LGW stable fixed point is equivalent to that of the NLσ model, and, for all $n \geqslant 2$, $m \geqslant 2$ except $n = 2$, $m = 2$, the NLσ model is asymptotically free, we expect $\nu^+ = \infty$, a conclusion that is confirmed by the large-n expression (4.41). On the other hand, for $d = 2$, Eq. (4.42) predicts $\nu^- = 1/2$ without $1/n$ and $1/n^2$ corrections. It is thus natural to conjecture that $\nu^- = 1/2$ for all $n \geqslant 2$ and $m \geqslant 2$, i.e., that the LGW antichiral fixed point is a Gaussian fixed point. The case $m = 2$, $n = 2$ needs a special discussion. Using the fact that the $O(2) \times O(2)$ LGW model is equivalent to the so-called mn model[9] with $m = n = 2$ [16,27] one can show that in the $v < 0$ region a stable fixed point exists for all values of d [1,2]. Finally, for $d \to 2$, using the σ-model results of the previous section, one finds that it becomes Gaussian.

We want now to use the knowledge of $\bar{n}(m, 2)$ in order to obtain some informations on $\bar{n}(m, 3)$. For this purpose we will make two hypotheses: first we will assume $d_c(m) < 3$, so that $\bar{n}(m, 3) = n^+(m, 3)$; second, we will assume $\bar{n}(m, d)$ to be sufficiently smooth in d at m fixed, so that we can use the interpolation method of Ref. [29]. Such a method has provided very precise estimates of critical quantities (see, e.g., Refs. [29,30]).

[9] The mn model with $m = 2$ describes n XY models coupled by an $O(n)$-symmetric interaction. Using essentially nonperturbative arguments (see, e.g., Ref. [27]) related to the specific-heat exponent of the XY universality class, one can argue that for $d < d_c$ with $d_c > 3$ there is a stable fixed point belonging to the XY universality class, while for $d > d_c$ there is a stable fixed point with the tetragonal symmetry. This fact has been recently confirmed by high-order field-theoretical calculations in three dimensions [2,28].

Let us first consider the case $m = 2$. We start from [17]

$$\begin{aligned}\bar{n}(2, 4-\epsilon) &= 12 + 4\sqrt{6} - \left(12 + \frac{14}{3}\sqrt{6}\right)\epsilon \\ &\quad + \left[\frac{137}{150} + \frac{91}{300}\sqrt{6} + \left(\frac{13}{5} + \frac{47}{60}\sqrt{6}\right)\zeta(3)\right]\epsilon^2 + \mathrm{O}(\epsilon^3) \\ &= 21.80 - 23.43\epsilon + 7.09\epsilon^2 + \mathrm{O}(\epsilon^3).\end{aligned} \tag{7.3}$$

Following Ref. [29], we rewrite this equation in the following form

$$\bar{n}(2, 4-\epsilon) = 2 + (2-\epsilon)\left(9.90 - 6.67\epsilon + 0.16\epsilon^2\right) + \mathrm{O}(\epsilon^3). \tag{7.4}$$

Note that the new perturbative series is much better behaved than the original one, the coefficients of the series decreasing rapidly. Setting $\epsilon = 1$, we obtain an estimate for $\bar{n}(2, 3)$:

$$\bar{n}(2, 3) \approx 5.3(2), \tag{7.5}$$

where the "error" indicates how the estimate varies from two loops to three loops. It should not be taken seriously; it should only provide an order of magnitude for the precision of the results. The estimate (7.5) is in good agreement with the determinations of Ref. [9,10]: $\bar{n}(2, 3) \approx 5$ (Ref. [9]), ≈ 6 (Ref. [10]). We can try to estimate the exponents for $n = \bar{n}(2, 3)$, by using Eq. (3.30). The coefficients decrease steadily with ϵ and thus we can simply set $\epsilon = 1$, obtaining

$$\eta \approx 0.038, \qquad \nu \approx 0.63. \tag{7.6}$$

It is also interesting to compute the exponents for $n = 6$, $m = 2$, and $d = 3$, in order to make a numerical comparison with the results of Ref. [31] who found $\nu = 0.700(11)$, $\gamma = 1.383(36)$, and the ERG results of Ref. [9] who found $\nu \approx 0.707$, $\gamma \approx 1.377$. If we use our $\mathrm{O}(n^{-2})$ expansions for the critical exponents, we obtain $\gamma \approx 1.22$ and $\nu \approx 0.63$. We can also use the ϵ-expansion, by using the method explained at the end of Section 3. In this case, we must fix $\Delta n = 6 - n^{+}(2, 3)$. Conservatively, we have $0 \lesssim \Delta n \lesssim 1$. Then, from the perturbative series we estimate $\nu \approx 0.63$–0.64, $\gamma \approx 1.24$–1.26, which is rather close to the large-n result, and somewhat lower than the numerical results of Refs. [9,31]. It is also worth mentioning that for $n = 6$ the fixed-dimension field-theoretical approach does not find fixed points that are sufficiently stable with respect to the order of the expansion up to six loops [10]. We believe that these apparent discrepancies among the various approaches deserve further investigation.

Our expressions may also be employed in order to establish an upper bound on the critical dimensionality $d_c(n, m)$ for the existence of a stable fixed point analytically connected with the critical point found in the $1/n$ expansion. This bound can be obtained by forcing the condition $\bar{n}(m, d_c) = n$. In particular, one may determine the dimension d_c such that, for $d < d_c$ the $O(3) \times O(2)$ has a nontrivial fixed point with symmetry $O(4)$ [15]. This corresponds to solving the equation $\bar{n}(2, d_c) = 3$. If we use Eq. (7.4), we find $d_c \approx 2.71$. We may compare our result to those obtained in the ERG approach: $d_c = 2.83$ (Ref. [7]), 2.87 (Ref. [8]). They are in substantial agreement with our result, when allowing for the systematic errors of both approaches. It should also be noticed that our interpolation (7.5) is also in very good agreement with the ERG results of Refs. [7,8] for all ϵ [34].

These analyses can be repeated for larger values of m. Since only $n^+(m,3)$ seems to be rather precisely determined, we only report the results for this quantity. For $m = 3$ and 4 we have the constrained estimates

$$\bar{n}(3,3) \approx 9.1(9), \tag{7.7}$$

$$\bar{n}(4,3) \approx 12(1). \tag{7.8}$$

For large m we have

$$\begin{aligned}\bar{n}(m, 4-\epsilon) &= m\big(9.90 - 10.10\epsilon + 2.66\epsilon^2 + \mathrm{O}\big(\epsilon^3, m^{-1}\big)\big)\\ &= m + (2-\epsilon)m\big[4.45 - 2.83\epsilon - 0.084\epsilon^2 + \mathrm{O}\big(\epsilon^3, m^{-1}\big)\big],\end{aligned} \tag{7.9}$$

where, as already observed, the coefficients of the constrained series are smaller than the original ones. Setting $\epsilon = 1$, we obtain $\bar{n}(m,3) \approx 2.5m$. Note that the large-$m$ approximation is already good at $m = 4$.

It is very important to notice that, since all extrapolation techniques are adiabatic in their parameters, it is not possible to catch the (essentially nonperturbative) features of the models in the region below $\bar{n}$. As a consequence there is no inconsistency between the present statements and our results [10] concerning $O(n) \times O(2)$ models for $n = 2, 3$ in fixed dimension $d = 3$. The fixed points we found for $n = 2, 3$ are certainly not analytically connected with the large-n and small-$\epsilon \equiv 4 - d$ criticalities discussed in this paper.

Finally, we recall that an enlarged parameter space for the $O(n) \times O(m)$ symmetric models with critical dimension $d = 4$ leads to the appearance of several new, generally unstable, fixed points, that physically correspond to tricritical transitions and give rise to crossover phenomena. It is important to recognize that the conformal bootstrap approach to the $1/n$ expansion allows a consistent treatment of all these criticalities. Systems with $O(n) \times O(m)$ symmetry may also possess a "gauge" criticality, which can be described by the appropriate $1/n$ expansion as well as within the ϵ expansion of the Hamiltonian for scalar chromodynamics and within the $\tilde{\epsilon}$ expansion of a class of gauge-invariant NLσ models.

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