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Critical behavior of the correlation function of three-dimensional O(N) models in the symmetric phase

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We present new strong-coupling series for O(N) spin models in three dimensions, on the cubic and diamond lattices. We analyze these series to investigate the two-point Green's function $G(x)$ in the critical region of the symmetric phase. This analysis shows that the low-momentum behavior of $G(x)$ is essentially Gaussian for all N from zero to infinity. This result is also supported by a large-N analysis.

1. INTRODUCTION

Three-dimensional O(N)-symmetric spin models describe many important critical phenomena in nature: the case $N = 3$ describes ferromagnetic materials, where the order parameter is the magnetization; the case $N = 2$ describes the helium superfluid transition, where the order parameter is the quantum amplitude; the case $N = 1$ (Ising model) describes liquid-vapor transitions, where the order parameter is the density.

The critical behavior of the two-point correlation function $G(x)$ is related to critical scattering, which is observed in many experiments, e.g., neutron scattering in ferromagnetic materials, light and X-rays scattering in liquid-gas systems.

In the following we will focus on the low-momentum behavior of the Fourier-transformed correlation function $\widetilde{G}(k)$ in the critical region of the symmetric phase, i.e., for

$$|k| \lesssim 1/\xi, \qquad 0 < T/T_c - 1 \ll 1.$$

2. LATTICE MODELS

Let us consider an O(N)-symmetric lattice spin models described by the nearest-neighbor action

$$S = -N\beta \sum_{\rm links} \vec{s}_{x_l} \cdot \vec{s}_{x_r}\,, \tag{1}$$

where $\beta = 1/T$, $\vec{s}$ is an N-component real vector, and x_l, x_r are the endpoints of the link. The two-point correlation function is defined by

$$G(x) = \langle \vec{s}_x \cdot \vec{s}_0 \rangle. \tag{2}$$

In order to simplify the study the critical behavior of $G(x)$, we introduce the dimensionless RG-invariant function

$$L(k;\beta) \equiv \frac{\widetilde{G}(0;\beta)}{\widetilde{G}(k;\beta)}\,. \tag{3}$$

In the critical region of the symmetric phase, $L(k,\beta)$ is a function only of the ratio $y \equiv k^2/M_G^2$, where $M_G \equiv 1/\xi_G$; the second-moment correlation length ξ_G is defined by

$$\xi_G^2 \equiv \frac{1}{6}\frac{\sum_x x^2 G(x)}{\sum_x G(x)}. \tag{4}$$

M_G is the mass-scale which can be directly observed in scattering experiments. $L(y)$ can be expanded in powers of y around $y = 0$:

$$L(y) = 1 + y + l(y), \qquad l(y) = \sum_{i=2}^{\infty} c_i y^i. \tag{5}$$

$l(y)$ parameterizes the difference from a generalized Gaussian propagator. The coefficients c_i can be expressed as the critical limit of appropriate dimensionless RG-invariant ratios of the spherical moments

$$m_{2j} = \sum_x x^{2j} G(x). \tag{6}$$

Another interesting quantity related to the low-momentum behavior of G is the ratio $s = M^2/M_G^2$, where M is the mass-gap of the theory. Its critical value is $s^* = -y_0$, where y_0 is the zero of $L(y)$ closest to the origin.

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In the large-N limit, $l(y)$ is depressed by a factor of $1/N$. The coefficients c_i can be obtained from a $1/N$ expansion in the continuum [1]:

$$c_2 \simeq -\frac{0.0044486}{N}, \quad c_3 \simeq \frac{0.0001344}{N},$$
$$c_4 \simeq -\frac{0.00000658}{N}, \quad c_5 \simeq \frac{0.00000040}{N} ... \qquad (7)$$

We are presently computing the order $1/N^2$ of the expansion. We expect that the pattern established by the $1/N$ expansion

$$c_i \ll c_2 \ll 1, \quad i \geq 3 \qquad (8)$$

will be followed by all models with sufficiently large N. This implies $s^* - 1 \simeq c_2$: indeed, in the large-N limit,

$$s^* - 1 \simeq -\frac{0.0045900}{N}. \qquad (9)$$

The coefficients c_i can also be computed from an ε-expansion of the corresponding ϕ^4 theory around $d = 4$ [2]:

$$c_i \simeq \varepsilon^2 \frac{N+2}{(N+8)^2} e_i, \qquad (10)$$

where $\varepsilon = 4 - d$ and

$$e_2 \simeq -0.007520, \qquad e_3 \simeq 0.0001919. \qquad (11)$$

3. STRONG-COUPLING EXPANSION

We computed the strong-coupling expansion of $G(x)$ up to 15th order on the cubic lattice, and up to 21st order on the diamond lattice. Our technique for the strong-coupling expansion of O(N) spin models was presented in Ref. [3].

We took special care in the choice of estimators for the "physical" quantities c_i and s^*. This step is very important from a practical point of view: better estimators can greatly improve the stability of the extrapolation to the critical point. Our search for optimal estimators was guided by the requirement of a regular strong-coupling expansion (e.g., no $\ln \beta$ terms) and by the knowledge of the large-N limit (we chose estimators which are "perfect" for $N = \infty$).

The strong-coupling series of the estimators were analyzed by Padé approximants, Dlog-Padé approximants and first-order integral approximants (see Ref. [4] for a review of the resummation techniques; see also Ref. [5]). For diamond lattice models with $N \neq 0$, β_c was not known, and we estimated it from the strong coupling series of the magnetic susceptibility.

Our strong-coupling results on cubic and diamond lattices are compared with the results of the $1/N$ expansion and of the ε-expansion in Table 1. One may notice that universality between cubic and diamond lattice is always confirmed; furthermore, the agreement with the ε-expansion and with the $1/N$ expansion is satisfactory.

The predicted pattern $c_3 \ll c_2 \ll 1$ is verified for all N. We can conclude that the two-point Green's function is essentially Gaussian for all momenta with $|k^2| \lesssim M_G^2$, and that the small corrections are dominated by the $(k^2)^2$ term.

4. APPROACH TO CRITICALITY

We investigated the approach to criticality, with special attention devoted to anisotropy (violation of rotational invariance). Let us introduce the anisotropy estimators

$$l_4 = \sum_{x,y,z} [f_4(x,y) + f_4(y,z) + f_4(z,x)]\, G(x,y,z),$$
$$f_4(x,y) = (x^2+y^2)^2 - 8x^2y^2; \qquad (12)$$
$$l_{6,1} = \sum_{x,y,z} [f_6(x,y) + f_6(y,z) + f_6(z,x)] \times G(x,y,z),$$
$$f_6(x,y) = (x^2+y^2)^3 - 8(x^4y^2 + x^2y^4); \qquad (13)$$
$$l_{6,2} = \sum_{x,y,z} [x^6 + y^6 + z^6 - 45x^2y^2z^2]\, G(x,y,z). \qquad (14)$$

In the critical limit, l_{2j} are depressed with respect to the spherical moments m_{2j}. In the large-N limit one can show that

$$A_{2j,i} \equiv \frac{l_{2j,i}}{m_{2j}} \sim \xi_G^{-2}. \qquad (15)$$

We analyzed the strong-coupling series of

$$B_{2j,i} \equiv \frac{l_{2j,i}}{m_{2j-2}}; \qquad (16)$$

Table 1
Comparison of strong-coupling expansion on cubic and diamond lattices with $1/N$ and ε-expansion

N	lattice	$10^4 c_2$	$10^5 c_3$	$10^4(s^* - 1)$
0	cubic	$\lvert 10^4 c_2\rvert \lesssim 2$	1.2(1)	1.2(3)
	diamond	$\lvert 10^4 c_2\rvert \lesssim 1$	1.0(1)	1.0(5)
	ε-expansion	−2.35	0.60	
1	cubic	−2.9(2)	1.1(1)	−2.3(5)
	diamond	−3.1(2)	1.0(2)	−2.2(3)
	ε-expansion	−2.78	0.71	
2	cubic	−3.8(3)	1.1(1)	−3.5(5)
	diamond	−4.2(3)	1.1(3)	−3.5(2)
	ε-expansion	−3.01	0.77	
3	cubic	−4.0(2)	1.1(2)	−4.0(4)
	diamond	−4.2(3)	1.1(3)	−3.5(2)
	ε-expansion	−3.11	0.79	
4	cubic	−4.1(2)	1.2(1)	−4.0(4)
	diamond	−4.7(2)	1.0(2)	−4.0(2)
	ε-expansion	−3.13	0.80	
	$1/N$	−11.12	3.36	−11.48
8	cubic	−3.5(2)	1.0(2)	−3.7(3)
	diamond	−4.0(1)	0.7(5)	−4.0(4)
	ε-expansion	−2.94	0.75	
	$1/N$	−5.56	1.18	−5.74
16	cubic	−2.4(2)	0.70(5)	−2.7(2)
	diamond	−2.65(5)	0.5(5)	−2.9(2)
	ε-expansion	−2.35	0.60	
	$1/N$	−2.78	0.84	−2.87

for all values of N, we found that $B_{2j,i}$ have a finite (but non-universal) $T \to T_c$ limit. This supports the validity of Eq. (15) for all N.

Ratios of $A_{2j,i}$ are universal quantities; we found that at criticality $A_{6,1}/A_4 \simeq 0.95$ and $A_{6,2}/A_{6,1} \simeq 0.75$ (within one per mill) for all N.

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