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Master Wilson loop operators in large-N lattice QCD_2

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Abstract

An explicit solution is found for the most general independent correlation functions in lattice QCD_2 with Wilson action. The large-N limit of these correlations may be used to reconstruct the eigenvalue distributions of Wilson loop operators for arbitrary loops. Properties of these spectral densities are discussed in the region $\beta < \beta_c = \frac{1}{2}$.

In view of the renewed interest in the master field approach to large-N matrix field theories [1,2], it may be convenient to improve our knowledge of those solvable models that will eventually become suitable laboratories for testing new methods and techniques. Lattice QCD_2 with free boundary conditions is such a solvable system. In principle all correlation functions can be computed by a proper gauge choice leading to a complete factorization of the functional integral and by knowledge of the properties of invariant group integration. The underlying single matrix problem has been solved a long time ago [3], and at first glance not much can be added to what is already known.

However, as extensively discussed in Ref. [1], one of the problems that must be faced in even simpler models (like a model of n independent Hermitian matrices) is that of finding a master field description of correlations involving different, and in general non-commuting, matrix field variables. The purpose of the present letter is that of establishing a benchmark for this problem, in the form of an explicit large-N expression for such correlations at arbitrary distance, and studying some properties of the eigenvalue distributions of the (master) Wilson loop operators associated with these correlations.

Our starting point is the axial gauge formulation of lattice QCD_2 with Wilson's action presented in Refs. [4,3,5]. In this formulation the expectation value of a Wilson loop is reduced to that of an invariant product of field variables in a one-dimensional principal chiral model with free boundary conditions. Let's therefore focus on these models, defined by the lattice action

$$S = 2\beta N \sum_i \mathrm{Re}\,\mathrm{Tr}\,[U_i U_{i+1}^\dagger]\,, \tag{1}$$

where i are the lattice sites. By straightforward manipulations one may show that the most general nontrivial correlation one really needs to compute is

$$W_{l,k} = \frac{1}{N}\langle \mathrm{Tr}\,(U_0 U_l^\dagger)^k\rangle\,, \tag{2}$$

where l plays the role of the space distance and k is a sort of "winding number" for the loop. In turn by the variable change $V_l = U_{l-1}U_l^\dagger$, the evaluation of $W_{l,k}$ is reduced to computing the group integral appearing in the relationship

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$$W_{l,k} = \frac{\int dV_1 ... dV_l \ \frac{1}{N}\mathrm{Tr}\,(V_1...V_l)^k \ \exp(2N\beta\sum_{i=1}^{l} \mathrm{Re}\,\mathrm{Tr}V_i)}{\left[\int dV \exp(2N\beta \mathrm{Re}\,\mathrm{Tr}V)\right]^l} . \quad (3)$$

This problem can be solved for arbitrary N by a character expansion (see e.g. [6]):

$$\exp\left[2N\beta \mathrm{Re}\,\mathrm{Tr}\,V\right] \propto \sum_{(r)} d_{(r)}\tilde{z}_{(r)}(\beta)\chi_{(r)}(V), \quad (4)$$

where $\chi_{(r)}$ are the characters and $d_{(r)}$ the dimensions of the irreducible representations of $U(N)$. Let's remind that the character coefficients $\tilde{z}_{(r)}(\beta)$ are explicitly known for all groups $U(N)$. As a consequence of Eqs. (3), (4) we obtain

$$\begin{aligned} W_{l,k} &= \int \prod_{i=1}^{l} \left[dV_i \sum_{(r)} d_{(r)}\tilde{z}_{(r)}(\beta)\chi_{(r)}(V_i)\right] \\ &\times \frac{1}{N}\mathrm{Tr}\,(V_1...V_l)^k \\ &= \frac{1}{N}\sum_{(r)} C_{k,(r)} d_{(r)} \tilde{z}^l_{(r)} , \end{aligned} \quad (5)$$

where we have introduced the coefficients $C_{k,(r)}$ of the decomposition

$$\mathrm{Tr}\,X^k = \sum_{(r)} C_{k,(r)} \ \chi_{(r)}(X) . \quad (6)$$

It is now an exercise in representation theory to show that, for all $k < N$, the only representations occurring in the r.h.s. of Eq. (6) are those in the form

$$\chi_{(j,k)} \equiv \chi_{(k-j+1,1^{j-1};0)} , \quad (7)$$

and one may show that

$$\mathrm{Tr}\,X^k = \sum_{j=1}^{k} (-1)^{j+1}\chi_{(j,k)}(X) . \quad (8)$$

As a consequence we immediately obtain the final result holding for $k < N$ in all $U(N)$ groups:

$$W_{l,k} = \frac{1}{N}\sum_{j=1}^{k}(-1)^{j+1} d_{(j,k)} \tilde{z}^l_{(j,k)} , \quad (9)$$

where we also know that

$$d_{(j,k)} = \frac{(N+k-j)!}{(N-j)!} \frac{(k-1)!}{k!(j-1)!(k-j)!} . \quad (10)$$

Further closed-form results may be obtained by restricting attention to large N and focusing on the strong coupling domain. When $\beta < 1/2$ and N is large [9]

$$\tilde{z}_{(j,k)} = \frac{N^k(N-j)!}{(N+k-j)!}\beta^k \left[1 + O\left(\beta^{2N}\right)\right] . \quad (11)$$

Under the same assumptions therefore

$$\begin{aligned} W_{l,k} &= \frac{N^{k-1}}{k}\sum_{j=1}^{k}\frac{(-1)^{j+1}}{(j-1)!(k-j)!} \\ &\times \left[\prod_{i=0}^{k-1}\left(1 + \frac{k-j-i}{N}\right)\right]^{1-l}\beta^{kl} , \end{aligned} \quad (12)$$

and one can finally show that

$$\lim_{N\to\infty} W_{l,k} = \frac{(-1)^{k-1}}{k}\binom{lk-2}{k-1}\beta^{kl} . \quad (13)$$

We have explicitly checked that Eq. (13) satisfies the Makeenko-Migdal equations [7] for the large-N limit of the corresponding Wilson loops in QCD_2.

Eq. (13) is our main result, in that it is the quantity that a master field approach to QCD_2 on the lattice should be able to reproduce for arbitrary l and k by substituting a single field configuration in the correlation functions.

It is interesting to explore the properties of the Wilson loop operator at a distance l, and in particular its eigenvalue distribution, by constructing a generating function of the expectations $W_{l,k}$:

$$A_l(x) = \sum_{k=0}^{\infty} W_{l,k}x^k . \quad (14)$$

From the Makeenko-Migdal equations it is possible to derive the Schwinger-Dyson equation satisfied by $A_l(x)$ in the strong coupling regime $\beta < 1/2$ and in the large-N limit

$$(A_l(x) - 1)\,A_l(x)^{l-1} = \beta^l x . \quad (15)$$

We checked explicitly the consistency of Eq. (15) with our solutions (13). When $l = 1$ one recognizes Gross-Witten's solution for the strong coupling phase of the single plaquette models. When $l = 2$ the solution

$$A_2(x) = \tfrac{1}{2}\left(\sqrt{1+4\beta^2 x} + 1\right) \quad (16)$$

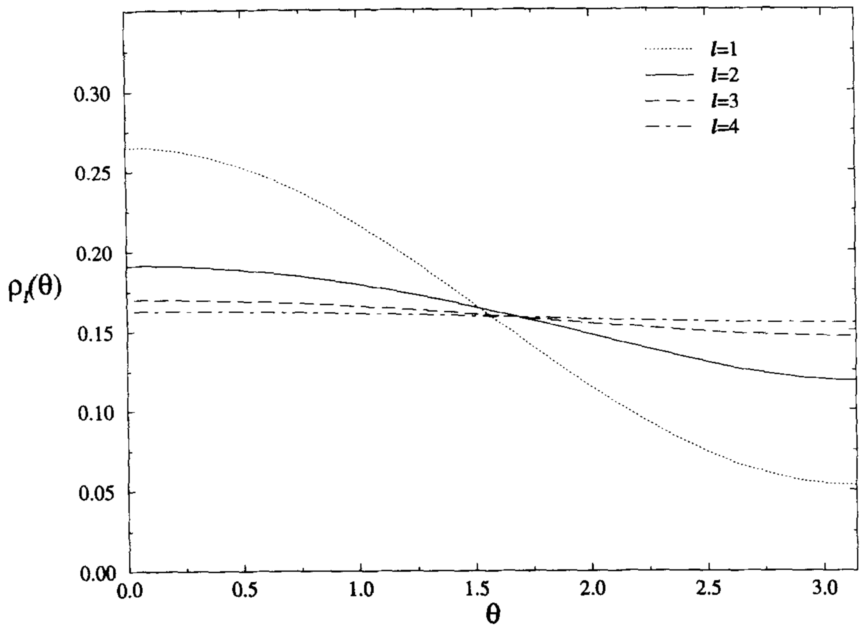

Fig. 1. $\rho_l(\theta)$ at $\beta = \frac{1}{3}$ and for various value of l. $\rho_l(-\theta) = \rho_l(\theta)$ by symmetry.

is related to the generating function for the moments of the energy density

$$\frac{1}{N}\langle \mathrm{Tr}\frac{1}{1-\beta x(V_n + V_{n+1}^\dagger)}\rangle$$
$$= 1 + 2x\beta^2 + 2\sum_{k=1}^{\infty}(\beta x)^{2k} W_{2,k}$$
$$= 2\beta^2 x + \sqrt{1+4\beta^4 x^2} \,. \tag{17}$$

Incidentally, Eq. (17) corrects a mistake in Ref. [8].

More generally, for arbitrary l, we may reexpress our results in terms of the eigenvalue distribution $\rho_l(\theta)$ of the Wilson loop operator $\prod_{i=1}^{l} V_i$, which is related to the generating function $A_l(x)$ by

$$\rho_l(\theta) = \frac{1}{\pi}\left[\mathrm{Re}\, A_l(e^{i\theta}) - \tfrac{1}{2}\right] , \tag{18}$$

and satisfying

$$\int_{-\pi}^{\pi} d\theta\, \rho_l(\theta) = 2W_{l,0} - 1 = 1 \,. \tag{19}$$

The study of the area dependence in the spectral density of continuum QCD_2 was pioneered a long time ago [5,10] and recently reconsidered by many authors. The lattice counterpart of this problem is the study of the dependence on l of the functions $\rho_l(\theta)$, which can be performed numerically, for arbitrary l, by summing up sufficiently many terms in the Fourier expansion of Eq. (18). In Figs. 1 and 2 we plotted the spectral densities as functions of θ for different values of l at fixed values of β, and in particular when $\beta = \beta_c$.

As one may notice, the property

$$\rho_l(\theta) \underset{l\to\infty}{\longrightarrow} \frac{1}{2\pi} , \tag{20}$$

which can be independently derived from the confinement properties of the model [10], is exhibited by our solutions. It is however quite interesting to notice that at $\beta = \beta_c$ and $\theta = \pi$ the equation satisfied by the eigenvalue density is

$$\left[1 - 2\pi\rho_l(\pi)\right]\left[1 + 2\pi\rho_l(\pi)\right]^{l-1} = 1 \,. \tag{21}$$

Besides the strong coupling solution, which corresponds to the values plotted in Fig. 2, Eq. (21) admits a second solution $\rho_l(\pi) = 0$ (coinciding with the first one when $l = 1,2$). This phenomenon leaves

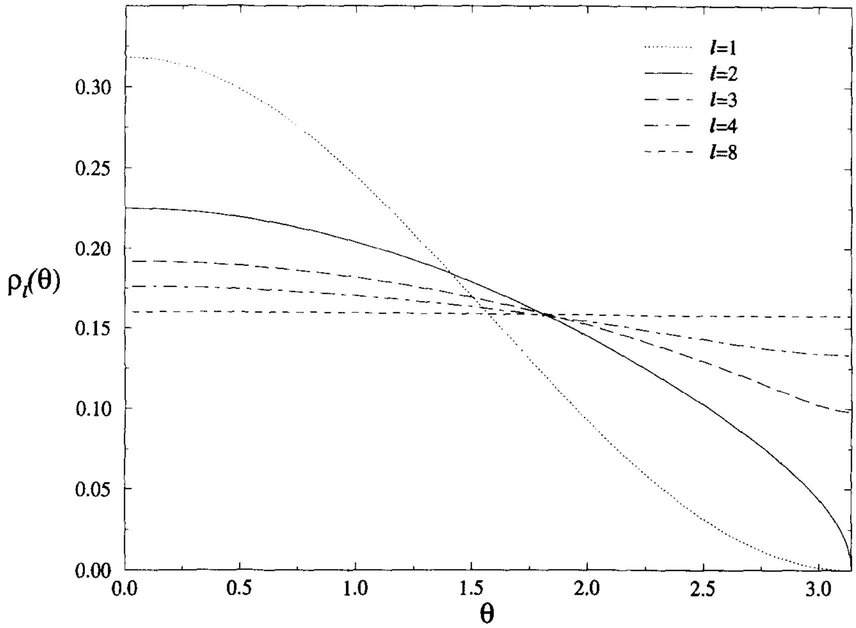

Fig. 2. $\rho_l(\theta)$ at $\beta = \beta_c = \frac{1}{2}$ and for various value of l.

open the possibility of a compactified solution in the weak coupling phase for arbitrary values of l.

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