Running coupling constant and correlation length from Wilson loops

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Abstract

We consider a definition of the QCD running coupling constant $\alpha(\mu)$ related to Wilson loops of size $r \times t$ with arbitrary fixed t/r. The schemes defined by these couplings are very close to the $\overline{\rm MS}$ scheme (i.e. the one-loop perturbative correction to the coupling is small) for all values of t/r; in the $t/r \to \infty$ limit, the "$q\bar{q}$ force" scheme is recovered, where the coupling constant is related to the quark-antiquark force. We discuss the possibility of applying finite-size scaling techniques to the Monte Carlo evaluation of $\alpha(\mu)$ up to very large momentum scales. We propose a definition of correlation length, also related to Wilson loops, which should make such a computation feasible.

One of the most important goals of a non-perturbative approach to QCD such as the lattice is the determination of $\Lambda_{\overline{\rm MS}}$ in units of a physical mass scale; this determination requires a very precise measurement of the running coupling constant in the $\overline{\rm MS}$ scheme $\alpha_{\overline{\rm MS}}(\mu)$ in the high-momentum region, where perturbation theory is reliable and accurate. A determination of $\alpha_{\overline{\rm MS}}(\mu \sim m_{Z_0})$ would be most welcome from a phenomenological point of view. Although considerable progress has been achieved in the last few years, especially in quenched QCD [1–6], this task turns out to be quite hard, due to the difficulty of performing lattice calculations at large momentum scales.

In the present letter we will address the two problems of: choice of an optimal $\alpha(\mu)$ for the lattice computation; determination of $\alpha(\mu)$ up to the highest possible momentum scale.

The requirements for an optimal choice of running coupling constant to be used in a lattice Monte Carlo simulation are:

1) fast and accurate lattice measurement;

2) proximity to $\alpha_{\overline{\rm MS}}$, to reduce systematic errors due to neglected orders in the perturbative conversion to the $\overline{\rm MS}$ scheme;

3) well-defined finite-size scaling properties.

We shall now present a new definition of $\alpha(\mu)$ addressing the above points.

A natural definition of running coupling is derived from the static quark-antiquark force $F(r)$ by the relationship

$$F(r) = -c_F \frac{\alpha_{q\bar{q}}(1/r)}{r^2} \, , \tag{1}$$

where $c_F = (N^2 - 1)/(2N) = \frac{4}{3}$. (We prefer to write $\alpha_{q\bar{q}}$ with a momentum scale dependence, in analogy with $\alpha_{\overline{MS}}(\mu)$, rather then with a length scale dependence, as often seen in the literature.) In perturbation theory the $q\bar{q}$ and the $\overline{MS}$ couplings are related by a very small first-order coefficient [7], making the determination of $\alpha_{\overline{MS}}$ from $\alpha_{q\bar{q}}$ by a perturbative redefinition of the coupling at large momentum very precise.

The force $F(r)$ between two static quarks separated by a distance r can be evaluated from rectangular Wilson loops $W(r,t)$:

$$F(r) = -\frac{\partial V(r)}{\partial r} \, , \quad V(r) = -\lim_{t \to \infty} \frac{1}{t} \ln W(r,t). \tag{2}$$

Recent developments in the calculation of $\alpha_{q\bar{q}}$ were reported in Refs. [4,5].

On the lattice the implementation of this idea to extract $\alpha_{q\bar{q}}(r)$ presents two major problems:

1) it requires a $t \to \infty$ limit procedure in order to evaluate $V(r)$;

2) it requires control of the force from short to long distance in one simulation, with $a \ll r \ll L$ holding for the whole range of physical r involved. This strongly limits the accessible values of r since the lattice size L/a available on today's supercomputers is not very large.

In this letter we extract $\alpha(\mu)$ from rectangular Wilson loops $W(r,t)$ of size $r \times t$ with $x \equiv t/r$ fixed. This solves the problem (1) above, and opens the road to the use of finite size techniques in order to reach very small distances without the necessity of very large lattices. Moreover, the close connection with the $\overline{MS}$ scheme is retained, as we shall see from a perturbative calculation.

In weak coupling,

$$\chi(r,t) \equiv \frac{\partial^2 \ln W(r,t)}{\partial r \partial t} = -c_F A(x) \frac{\alpha}{r^2} \left(1 + O(\alpha)\right) \, , \tag{3}$$

where $x = t/r$ and

$$A(x) = \frac{2}{\pi} \left[\arctan(x) + \frac{1}{x} + \frac{\arctan(1/x)}{x^2} \right] . \tag{4}$$

The renormalization properties of the Wilson loop operator [8,9] allow the definition of a running coupling constant $\alpha_x(1/r)$ parametrized by $x = t/r$:

$$\chi(r,t) = -c_F A(x) \frac{\alpha_x(1/r)}{r^2} \, . \tag{5}$$

Without loss of generality, we will choose r to be the shorter side of the loop, and therefore $x \geq 1$. Since

$$\lim_{t \to \infty} \chi(r,t) = F(r), \tag{6}$$

we have

$$\lim_{x \to \infty} \alpha_x(1/r) = \alpha_{q\bar{q}}(1/r). \tag{7}$$

The perturbative expansion of $\chi(r,t)$ in dimensional regularization can be obtained from the corresponding expansion of the Wilson loop $W(r,t)$ [10]. In the $\overline{MS}$ renormalization scheme

$$\chi(r,t) = -c_F A(x) \frac{\alpha_{\overline{MS}}(\mu)}{r^2} \left[1 + b_0 \left(\ln(r\mu) + R(x)\right) \alpha_{\overline{MS}}(\mu) + O\left(\alpha^2\right)\right], \tag{8}$$

where b_0 is the first coefficient of the β-function:

$$\mu \frac{\partial}{\partial \mu} \alpha_{\overline{\text{MS}}} = -b_0 \alpha_{\overline{\text{MS}}}^2 + O(\alpha^3), \quad b_0 = \frac{1}{4\pi} \left(\frac{22}{3} N - \frac{4}{3} N_f \right),$$ (9)

and $R(x)$ is a finite function of x:

$$R(x) = \gamma_E - 1 + \frac{1}{4\pi b_0} \left(\frac{31}{9} N - \frac{10}{9} N_f \right)$$

$$+ \frac{2}{\pi A(x)} \left[I_1(x) + I_2(x) + \frac{N}{4\pi b_0} (I_3(x) + I_4(x) + I_5(x)) \right].$$ (10)

We present for the interested reader explicit formulae for the functions I:

$$I_1(x) = \frac{\pi}{2} A(x) - r \frac{\partial}{\partial t} [f(x) + f(1/x)] - r^2 \frac{\partial^2}{\partial r \partial t} [f(1/x) \ln x],$$

$$I_2(x) = -\frac{\pi \ln 2}{2} A(x) - \frac{1}{2} r^2 \frac{\partial^2}{\partial r \partial t} [g(x) + g(1/x)],$$

$$I_3(x) = -r^2 \frac{\partial^2}{\partial r \partial t} [h(x) + h(1/x)],$$

$$I_4(x) = -r^2 \frac{\partial^2}{\partial r \partial t} k(x),$$

$$I_5(x) = l(x) + \frac{l(1/x)}{x^2},$$ (11)

where $x = t/r$ and

$$f(x) = x \arctan(x) - \frac{1}{2} \ln(1 + x^2),$$

$$g(x) = x \int_0^x dy \frac{\ln(1 + y^2)}{1 + y^2} - \frac{1}{4} \ln^2(1 + x^2),$$

$$h(x) = f(x)(6 - 4 \ln x) - (1 + x^2) \arctan^2(x) + 2 \int_0^x dy \left[2 \ln y (x - y) + f(x - y) \right] \arctan(y),$$

$$k(x) = -4f(x)f(1/x),$$

$$l(x) = \frac{4}{\pi} \int_0^x dy \int_{-\infty}^{\infty} dz_1 dz_2 \int_0^{\infty} d\rho$$

$$\times \frac{\rho z_2 (y^2 - x^2 + 2xz_1 - 2yz_1)}{[z_1^2 + (z_2 - 1)^2 + \rho^2][(z_1 - x)^2 + z_2^2 + \rho^2]^2[(z_1 - y)^2 + z_2^2 + \rho^2]^2}.$$ (12)

All the functions I vanish in the large-x limit, where the perturbative expansion of the static quark-antiquark force [7] is recovered.

$R(x)$ determines the first order connection between the couplings $\alpha_x(1/r)$ and $\alpha_{\overline{\text{MS}}}(\mu)$:

$$\alpha_{\overline{\text{MS}}}(1/r) = \alpha_x(1/r) - b_0 R(x) \alpha_x^2(1/r) + O\left(\alpha_x^3(1/r)\right), \quad \ln \left(\frac{\Lambda_x}{\Lambda_{\overline{\text{MS}}}} \right) = R(x),$$ (13)

where Λ_x is the x-dependent Λ-parameter associated with the x-schemes (5).

Table 1
We report $R(x) = \ln \Lambda_x/\Lambda_{\overline{MS}}$ for $N_f = 0, 2, 4$ and for some values of x

x	$N_f = 0$	$N_f = 2$	$N_f = 4$
1	−0.20758	−0.27875	−0.37270
5/4	−0.11177	−0.18145	−0.27343
4/3	−0.08975	−0.15852	−0.24930
3/2	−0.05567	−0.12242	−0.21053
5/3	−0.03131	−0.09603	−0.18146
7/4	−0.02178	−0.08554	−0.16970
2	−0.00069	−0.06189	−0.14266
3	0.03187	−0.02341	−0.09637
4	0.03979	−0.01317	−0.08308
10	0.04557	−0.00500	−0.07175
∞	0.04691	−0.00324	−0.06945

In Table 1 we report $R(x)$ for $N_f = 0$, 2, and 4, for various values of x. In quenched QCD ($N_f = 0$), $R(x)$ turns out to be independent of N; the values for $N_f \neq 0$ in the Table are computed for SU(3). As anticipated above, the first order perturbative coefficient relating α_x to $\alpha_{\overline{MS}}$ remains small for all $x \geq 1$.

On the lattice we can define the quantity

$$\phi(r, x) = \chi_C \left(\frac{r}{a}, \frac{xr}{a} \right), \tag{14}$$

where χ_C is the usual Creutz ratio

$$\chi_C(R, T) = -\ln \frac{W(R, T) \, W(R+1, T+1)}{W(R, T+1) \, W(R+1, T)}. \tag{15}$$

$\phi(r, x)$ is a natural estimator of $\chi(r, xr)$. By measuring $\phi(r, x)$ at different scales r keeping x fixed, one may easily extract the corresponding running coupling $\alpha_x(1/r)$.

We would like to conclude with some considerations concerning the possibility of employing finite size techniques to reach very small distances, without the need of very large lattices. The power of this kind of techniques was recently illustrated by the exploration of the extremely small distance regime of two-dimensional spin models [11].

In the scaling region the following finite size scaling relations holds

$$\xi_L(\beta) \simeq f_\xi(L/\xi_L) \, \xi_\infty(\beta), \tag{16}$$

where ξ is a correlation length of the theory, and

$$\phi(\beta, r, x, L) \simeq f_\phi(x, L/r, L/\xi_L) \, \phi(\beta, r, x, \infty), \tag{17}$$

Notice that the existence of an additional scale r in $\phi(r, x)$ causes an extra dependence on the ratio L/r in the finite size scaling function f_ϕ. Finite size scaling functions like f_ξ and f_ϕ may be reconstructed by performing simulations at relatively small lattices, by employing the scheme outlined in Ref. [11]. In this context finite size methods should allow us to keep $a \ll r$ even when r is very small in physical unit.

A crucial point in the realization of this program is the definition of a correlation length ξ suitable for an accurate finite size scaling study. Once found such a scale, the study of the finite size scaling of ϕ should be relatively easy. Finite size studies of two-dimensional spin models [11,12] suggest that a suitable correlation length is one defined from the second moment of a correlation function. Such a correlation function should have appropriate properties: it must renormalize multiplicatively with a renormalization factor independent of the distance, fall down exponentially at large distances, and of course be easily measurable in Monte Carlo

simulations. We identified a possible candidate for pure gauge theory, which is easily constructed from Wilson loops. The correlation function

$$Y(r,t) = \frac{W(r,t)}{W\left(\frac{1}{2}r, \frac{1}{2}t\right)^2}$$ (18)

is renormalized by a constant factor, since the divergence associated with the perimeter term of the Wilson loop operator [9] cancels in the ratio; it is exponentially suppressed at large distances, due to the area law of pure gauge theory; it is constructed from Wilson loops, which are a standard "easy" observable of Monte Carlo simulations.

From $Y(r,t)$ we can define a second moment type correlation length:

$$\xi^2 = \frac{1}{2} \frac{\int_0^\infty dr \int_r^{\kappa r} dt\, Y(r,t)\, rt}{\int_0^\infty dr \int_r^{\kappa r} dt\, Y(r,t)} ,$$ (19)

where κ is a free parameter ($\kappa > 1$), which can be chosen to optimize the measurement. Assuming an exact area law for the Wilson loop, Eq. (19) would give $\xi^2 = 1/\sigma$, where σ is the string tension.

The measurement of $\alpha_x(\mu)$ in the above-mentioned scheme up to a large μ leads to a direct determination of the β-function of the SU(3) lattice gauge theory and of the adimensional quantity $\xi\Lambda_{\overline{MS}}$. This quantity still needs to be converted to a more phenomenological scale, such as r_0 [13]; but this is a rather minor problem, since it involves only a measurement of ξ/r_0, which can be performed at the values of β of our choice.

The program presented here is not specific to pure gauge theories, and both the definition of coupling constant and the finite-size scaling relationships hold unchanged for full QCD, with the inclusion of dynamical fermions. The definition of correlation length has to be changed; however a "physical" definition such as the inverse nucleon mass is perfectly viable in this case.

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