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THE LARGE-*N* EXPANSION OF UNITARY-MATRIX MODELS

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Abstract

The general features of the $1/N$ expansion in statistical mechanics and quantum field theory are briefly reviewed both from the theoretical and from the phenomenological point of view as an introduction to a more detailed analysis of the large-N properties of spin and gauge models possessing the symmetry group $\mathrm{SU}(N)\times\mathrm{SU}(N)$.

An extensive discussion of the known properties of the single-link integral (equivalent to YM_2 and one-dimensional chiral models) includes finite-N results, the external field solution, properties of the determinant, and the double scaling limit.

Two major classes of solvable generalizations are introduced: one-dimensional closed chiral chains and models defined on a $d-1$ dimensional simplex. In both cases, large-N solutions are presented with emphasis on their double scaling properties.

The available techniques and results concerning unitary-matrix models that correspond to asymptotically free quantum field theories (two-dimensional chiral models and four-dimensional QCD) are discussed, including strong-coupling methods, reduced formulations, and the Monte Carlo approach.

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It will be good enough that these (results) be judged useful by those willing to find the truth of the facts. They were put together as a lasting possession, rather than in competition for temporary audience.

[Thucydides, *The Peloponnesian War*, I, 22]

1. Introduction

1.1. General motivation for the 1/N expansion

The approach to quantum field theory and statistical mechanics based on the identification of the large-N limit and the perturbative expansion in powers of $1/N$, where N is a quantity related to the number of field components, is by now almost thirty years old. It goes back to the original work by Stanley [1] on the large-N limit of spin systems with O(N) symmetry, soon followed by Wilson's suggestion that the $1/N$ expansion may be a valuable alternative in the context of renormalization-group evaluation of critical exponents, and by 't Hooft's extension [2] to gauge theories and, more generally, to fields belonging to the adjoint representation of SU(N) groups. More recently, the large-N limit of random-matrix models was put into a deep correspondence with the theory of random surfaces, and therefore it became relevant to the domain of quantum gravity.

In order to understand why the $1/N$ expansion should be viewed as a fundamental tool in the study of quantum and statistical field theory, it is worth emphasizing a number of relevant features:

1. N is an intrinsically dimensionless parameter, representing a dependence whose origin is basically group-theoretical, and leading to well-defined field representations for all integer values, hence it is not subject to any kind of renormalization;
2. N does not depend on any physical scale of the theory, hence we may expect that physical quantities should not show any critical dependence on N (with the possible exception of finite-N scaling effects in the double-scaling limit);
3. the large-N limit is a thermodynamical limit, in which we observe the suppression of fluctuations in the space of internal degrees of freedom; hence we may expect notable simplifications in the algebraic and analytical properties of the model, and even explicit integrability in many instances.

Since integrability does not necessarily imply triviality, the large-N solution to a model may be a starting point for finite-N computations, because it shares with interesting finite values of N many physical properties. (This is typically not the case for the standard free-field solution which forms the starting point for the usual perturbative expansions.) Moreover, for reasons which are clearly, if not obviously, related to the three points above, the physical variables which are naturally employed to parameterize large-N results and $1/N$ expansions are usually more directly related to the observables of the models than the fields appearing in the original local Lagrangian formulation.

More reasons for a deep interest in the study of the large-N expansion will emerge from the detailed discussion we shall present in the rest of this introductory section. We must however anticipate that many interesting review papers have been devoted to specific issues in the context of the large-N limit, starting from Coleman's lectures [3], going through Yaffe's review on the reinterpretation of the large-N limit as classical mechanics [4], Migdal's review on loop equations [5], and Das' review on reduced models [6], down to Polyakov's notes [7] and to the recent large commented collection of original papers by Brezin and Wadia [8], not to mention Sakita's booklet [9] and Ma's contributions [10,11]. Moreover, the $1/N$ expansion of two-dimensional spin models has been reviewed by two of the present authors a few years ago [12]. As a consequence, we decided to devote only a bird's eye overview to the general issues, without pretension of offering a self-contained presentation of all the many conceptual and technical developments that have appeared in an enormous and ever-growing literature; we even dismissed the purpose of offering a complete reference list grouped by arguments, because the task appeared to be beyond our forces.

We preferred to focus on a subset of all large-N topics, which has never been completely and systematically reviewed: the issue of unitary-matrix models. Our self-imposed limitation should not appear too restrictive, when considering that it still involves such topics as $\mathrm{U}(N)\times\mathrm{U}(N)$ principal chiral models, virtually all that concerns large-N lattice gauge theories, and an important subset of random-matrix models with their double-scaling limit properties, related to two-dimensional conformal field theory.

The present paper is organized on a logical basis, which will neither necessarily respect the sequence of chronological developments, nor it will keep the same emphasis that was devoted by the authors of the original papers to the discussion of the different issues.

Section 2 is devoted to a presentation of the general and common properties of unitary-matrix models, and to an analysis of the different approaches to their large-N solution that have been discussed in the literature.

Section 3 is a long and quite detailed discussion of the most elementary of all unitary-matrix systems. Since all essential features of unitary-matrix models seem to emerge already in the simplest example, we thought it worthwhile to make this discussion as complete and as illuminating as possible.

Section 4 is an application of results obtained by studying the single-link problem, which exploits the equivalence of this model with lattice YM_2 and principal chiral models in one dimension.

Section 5 is devoted to a class of reasonably simple systems, whose physical interpretation is that of closed chiral chains as well as of gauge theories on polyhedra.

Section 6 presents another class of integrable systems, corresponding to chiral models defined on a d-dimensional simplex, whose properties are relevant both in the discussion of the strong-coupling phase of more general unitary-matrix models and in the context of random-matrix models.

Section 7 deals with the physically more interesting applications of unitary-matrix models: two-dimensional principal chiral models and four-dimensional lattice gauge theories, sharing the properties of asymptotic freedom and "confinement" of the Lagrangian degrees of freedom. Special issues, like numerical results and reduced models, are considered.

1.2. Large N as a thermodynamical limit: factorization

As we already mentioned briefly in the introduction, one of the peculiar features of the large-N limit is the occurrence of notable simplifications, that become apparent at the level of the quantum equations of motion, and tend to increase the degree of integrability of the systems. These simplifications are usually related to a significant reduction of the number of algebraically independent correlation functions, which in turn is originated by the property of factorization.

This property is usually stated as follows: connected Green's functions of quantities that are invariant under the full symmetry group of the system are suppressed with respect to the corresponding disconnected parts by powers of $1/N$. Hence when $N \to \infty$ one may replace expectation values of products of invariant quantities with products of expectation values.

One must however be careful, since factorization is not a property shared by all invariant operators without further qualifications. In particular, experience shows that operators associated with very high rank representations of the symmetry group, when the rank is O(N), do not possess the factorization property. A very precise characterization has been given by Yaffe [4], who showed that factorization is a property of "classical" operators, i.e., those operators whose coherent state matrix elements have a finite $N \to \infty$ limit.

It is quite interesting to investigate the physical origin of factorization. The property

$$\lim_{N \to \infty} \langle AB \rangle = \langle A \rangle \langle B \rangle \tag{1.1}$$

implies in particular that

$$\lim_{N \to \infty} \langle A^2 \rangle = \langle A \rangle^2 , \tag{1.2}$$

i.e., the vacuum state of the model, seen as a statistical ensemble, seems to possess no fluctuations. To be more precise, all the field configurations that correspond to a nonvanishing vacuum wavefunction can be related to each other by a symmetry transformation. This residual infinite degeneracy of the vacuum configurations makes the difference between the large-N limit and a strictly classical limit $\hbar \to 0$, and allows the possibility of violations of factorization when infinite products of operators are considered; this is in a sense the case with representations whose rank is O(N).

More properly, we may view large N as a thermodynamical limit [13], since the number of degrees of freedom goes to infinity faster than any other physical parameter, and as a consequence the "macroscopic" properties of the system, i.e., the invariant expectation values, are fixed in spite of the great number of different "microscopic" realizations. This realization does not rule out the possibility of searching for the so-called "master field", that is a representative of the equivalence class of the field configurations corresponding to the large-N vacuum, such that all invariant expectation values of the factorized operators can be obtained by direct substitution of the master field value into the definition of the operators themselves [3].

There has been an upsurge of interest on master fields in recent years [14,15], triggered by new results in non-commutative probability theory applied to the stochastic master field introduced in Ref. [16].

1.3. 1/N expansion of vector models in statistical mechanics and quantum field theory

The first and most successful application of the approach based on the large-N limit and the $1/N$ expansion to field theories is the analysis of vector models enjoying O(N) or SU(N) symmetry.

Actually, "vector models" is a nickname for a wide class of different field theories, characterized by bosonic or fermionic Lagrangian degrees of freedom lying in the fundamental representation of the symmetry group (cf. Ref. [12] and references therein).

A quite general feature of these models is the possibility of expressing all self-interactions of the fundamental degrees of freedom by the introduction of a Lagrange multiplier field, a boson and a singlet of the symmetry group, properly coupled to the Lagrangian fields, such that the resulting effective Lagrangian is quadratic in the N-component fields. One may therefore formally perform the Gaussian integration over these fields, obtaining a form of the effective action which is nonlocal, but depends only on the singlet multiplier, acting as a collective field; in this action N appears only as a parameter.

The considerations developed in Section 1.2 make it apparent that all fluctuations of the singlet field must be suppressed in the large-N limit (no residual degeneracy is left in the trivial representation). As a consequence, solving the models in this limit simply amounts to finding the singlet field configuration minimizing the effective action. The problem of nonlocality is easily bypassed by the consideration that translation invariance of the physical expectation values requires the action-minimizing field configuration to be invariant in space-time; hence the saddle-point equations of motion become coordinate-independent and all nonlocality disappears.

As one may easily argue from the above considerations, the large-N solution of vector models describes some kind of Gaussian field theory. Nevertheless, this result is not as trivial as one might imagine, since the free theory realization one is faced with usually enjoys quite interesting properties, in comparison with the naïve Lagrangian free fields. Typical phenomena appearing in the large-N limit are an extension of the symmetry and spontaneous mass generation. Moreover, when the fundamental fields possess some kind of gauge symmetry, one may also observe dynamical generation of propagating gauge degrees of freedom; this is the case with two-dimensional CP^{N-1} models and their generalizations [17,18].

The existence of an explicit form of the effective action offers the possibility of a systematic expansion in powers of $1/N$. The effective vertices of the theory turn out to be Feynman integrals over a single loop of the free massive propagator of the fundamental field. In two dimensions, where the physical properties of many vector models are especially interesting (e.g., asymptotic freedom), these one-loop integrals can all be computed analytically in the continuum version, and even on the lattice many analytical results have been obtained.

The $1/N$ expansion is the starting point for a systematic computation of critical exponents, which are nontrivial in the range $2 < d < 4$, for the study of renormalizability of superficially nonrenormalizable theories in the same dimensionality range, and for the computation of physical amplitudes. Notable is the case of the computation of amplitude ratios, which are independent of the coupling in the scaling region, and therefore are functions of $1/N$ alone; hopefully, their $1/N$ expansion possesses a nonvanishing convergence radius. The $1/N$ expansion was also useful to explore the double-scaling limit properties of vector models [19–21].

The properties of the large-N limit and of the $1/N$ expansion of continuum and lattice vector models were already reviewed by many authors. We therefore shall not discuss this topic further. We only want to stress that this kind of studies can be very instructive, given the physical interest of vector models as realistic prototypes of critical phenomena in two and three dimensions and as models for dynamical Higgs mechanism in four dimensions. Moreover, some of the dynamical properties emerging mainly from the large-N studies of asymptotically free models (in two dimensions) may be used to mimic some of the features of gauge theories in four dimensions; however, at least one of the essential aspects of gauge theories, the presence of matrix degrees of freedom (fields in the adjoint representation), cannot be captured by any vector model.

1.4. $1/N$ expansion of matrix models: planar diagrams

The first major result concerning the large-N limit of matrix-valued field theories was due to G. 't Hooft, who made the crucial observation that, in the $1/N$ expansion of continuum gauge theories, the set of Feynman diagrams contributing to any given order admits a simple topological interpretation. More precisely, by drawing the U(N) fundamental fields ("quarks") as single lines and the U(N) adjoint fields ("gluons") as double lines, each line carrying one color index, a graph corresponding to a nth-order contribution can be drawn on a genus n surface (i.e., a surface possessing n "holes"). In particular, the zeroth-order contribution, i.e., the large-N limit, corresponds to the sum of all planar diagrams. The extension of this topological expansion to gauge models enjoying O(N) and Sp($2N$) symmetry has been described by Cicuta [22]. Large-N universality among O(N), U(N), and Sp($2N$) lattice gauge theories has been discussed by Lovelace [23].

This property has far-reaching consequences: it allows for reinterpretations of gauge theories as effective string theories, and it offers the possibility of establishing a connection between matrix models and the theory of random surfaces, which will be exploited in the study of the double-scaling limit.

As a byproduct of this analysis, 't Hooft performed a summation of all planar diagrams in two-dimensional continuum Yang–Mills theories, and solved QCD_2 to leading nontrivial order in $1/N$, finding the meson spectrum [24,25].

Momentum-space planarity has a coordinate-space counterpart in lattice gauge theories. It is actually possible to show that, within the strong-coupling expansion approach, the planar diagrams surviving in the large-N limit can be identified with planar surfaces built up of plaquettes by gluing them along half-bonds [26–28]. This construction however leads quite far away from the simplest model of planar random surfaces on the lattice originally proposed by Weingarten [29,30], and hints at some underlying structure that makes a trivial free-string interpretation impossible.

1.5. The physical interpretation: QCD phenomenology

The sum of the planar diagrams has not till now been performed in the physically most interesting case of four-dimensional SU(N) gauge theories. It is therefore strictly speaking impossible to make statements about the relevance of the large-N limit for the description of the physically relevant case $N = 3$. However, it is possible to extract from the large-N analysis a number of qualitative and semi-quantitative considerations leading to a very appealing picture of the phenomenology predicted by the $1/N$ expansion of gauge theories. These predictions can be improved further by adopting Veneziano's form of the large-N limit [31], in which not only the number of colors N but also the number of flavors N_f is set to infinity, while their ratio N/N_f is kept finite. We shall not enter a detailed discussion of large-N QCD phenomenology, but it is certainly useful to quote the relevant results.

1.5.1. The large-N property of mesons

Mesons are stable and noninteracting; their decay amplitudes are $O(N^{-1/2})$, and their scattering amplitudes are $O(N^{-1})$.

Meson masses are finite.

The number of mesons is infinite.

Exotics are absent and Zweig's rule holds.

1.5.2. The large-N property of glueballs

Glueballs are stable and noninteracting, and they do not mix with mesons; a vertex involving k glueballs and n mesons is $O(N^{1-k-n/2})$.

The number of glueballs is infinite.

1.5.3. The large-N property of baryons

A large-N baryon is made out of N quarks, and therefore it possesses peculiar properties, similar to those of solitons [32].

Baryon masses are $O(N)$.

The splitting of excited states is $O(1)$.

Baryons interact strongly with each other; typical vertices are $O(N)$.

Baryons interact with mesons with $O(1)$ couplings.

1.5.4. The η' mass formula

The spontaneous breaking of the $SU(N_f)$ axial symmetry in QCD gives rise to the appearance of a multiplet of light pseudoscalar mesons. This symmetry-breaking pattern was explicitly demonstrated in the context of large-N QCD by Coleman and Witten [33]. However, the singlet pseudoscalar is not light, due to the anomaly of the U(1) axial current. Since the anomaly equation

$$\partial_\mu J^5_\mu = \frac{g^2 N_f}{16\pi^2} \operatorname{Tr} \tilde{F}_{\mu\nu} F^{\mu\nu} \tag{1.3}$$

has a vanishing right-hand side in the limit $N_c \to \infty$ with N_f and $g^2 N_c$ fixed (the standard large-N limit of non-Abelian gauge theories), the leading-order contribution to the mass of the η' should be $O(1/N_c)$. The proportionality constant should be related to the symmetry-breaking term, which in turn is related to the so-called topological susceptibility, i.e., the vacuum expectation value of the square of the topological charge. The resulting relationship shows a rather satisfactory quantitative agreement with experimental and numerical results [34–38].

1.6. The physical interpretation: two-dimensional quantum gravity

In the last ten years, a new interpretation of the $1/N$ expansion of matrix models has been put forward. Starting from the relationship between the order of the expansion and the topology of two-dimensional surfaces on which the corresponding diagrams can be drawn, several authors [39–43] proposed that large-N matrix models could provide a representation of random lattice two-dimensional surfaces, and in turn this should correspond to a realization of two-dimensional quantum gravity. These results were found consistent with independent approaches, and proper modifications of the matrix self-couplings could account for the incorporation of matter.

The functional integrals over two-dimensional closed Riemann manifolds can be replaced by the discrete sum over all (piecewise flat) manifolds associated with triangulations. It is then possible to identify the resulting partition function with the vacuum energy

$$E_0 = -\log Z_N , \tag{1.4}$$

obtained from a properly defined $N \times N$ matrix model, and the topological expansion of two-dimensional quantum gravity is nothing but the $1/N$ expansion of the matrix model.

The partition function of two-dimensional quantum gravity is expected to possess well-defined scaling properties [44]. These may be recovered in the matrix model by performing the so-called "double-scaling limit" [45–47]. This limit is characterized by the simultaneous conditions

$$N \to \infty , \qquad g \to g_c , \tag{1.5}$$

where g is a typical self-coupling and g_c is the location of some large-N phase transition. The limits are however not independent. In order to get nontrivial results, one is bound to tune the two conditions (1.5) in such a way that the combination

$$x = (g - g_c) N^{2/\gamma_1} \tag{1.6}$$

is kept finite and fixed. γ_1 is a computable critical exponent, usually called "string susceptibility". According to Ref. [44], it is related to the central charge c of the model by

$$\gamma_1 = \frac{1}{12}[25 - c + \sqrt{(1-c)(2-c)}] . \tag{1.7}$$

An interesting reinterpretation of the double-scaling limit relates it to some kind of finite-size scaling in a space where N plays the role of the physical dimension L [21,48,49]. Research in this field has exploded in many directions. A wide review reflecting the state of the art as of the year 1993 appeared in the already-mentioned volume by Brezin and Wadia [8]. Here we shall only consider those results that are relevant to our more restricted subject.

2. Unitary matrices

2.1. General features of unitary-matrix models

Under the header of unitary-matrix models we class all the systems characterized by dynamical degrees of freedom that may be expressed in terms of the matrix representations of the unitary groups U(N) or special unitary groups SU(N) and by interactions enjoying a global or local $U(N)_L \times U(N)_R$ symmetry. Typically we shall consider lattice models, with no restriction on the lattice structure and on the number of lattice points, ranging from 1 (single-matrix problems) to infinity (infinite-volume limit) in an arbitrary number of dimensions.

In the field-theoretical interpretation, i.e., when considering models in infinite volume and in proximity of a fixed point of some (properly defined) renormalization group transformation, such models will have a continuum counterpart, which in turn shall involve unitary-matrix valued fields in the case of spin models, while for gauge models the natural continuum representation will be in terms of hermitian matrix (gauge) fields.

A common feature of all unitary-matrix models will be the group-theoretical properties of the functional integration measure: for each dynamical variable the natural integration procedure is based on the left- and right-invariant Haar measure

$$\mathrm{d}\mu(U) = \mathrm{d}\mu(UV) = \mathrm{d}\mu(VU), \qquad \int \mathrm{d}\mu(U) = 1 . \tag{2.1}$$

An explicit use of the invariance properties of the measure and of the interactions (gauge fixing) can sometimes lead to formulations of the models where some of the symmetries are not apparent. Global U(N) invariance is however always assumed, and the interactions, as well as all physically interesting observables, may be expressed in terms of invariant functions.

It is convenient to introduce some definitions and notations. An arbitrary matrix representation of the unitary group U(N) is denoted by $\mathscr{D}^{(r)}_{ab}(U)$. The characters and dimensions of irreducible representations are $\chi_{(r)}(U) = \mathscr{D}^{(r)}_{aa}(U)$ and $d_{(r)}$ respectively. (r) is characterized by two sets of decreasing positive integers $\{l\} = l_1, \dots, l_s$ and $\{m\} = m_1, \dots, m_t$. We may define the ordered set of integers $\{\lambda\} = \lambda_1, \dots, \lambda_N$ by the relationships

$$\begin{aligned} &\lambda_k = l_k, \quad (k = 1, \dots, s), \qquad \lambda_k = 0, \quad (k = s+1, \dots, N-t) , \\ &\lambda_k = -m_{N-k+1}, \quad (k = N-t+1, \dots, N) . \end{aligned} \tag{2.2}$$

It is then possible to write down explicit expressions for all characters and dimensions, once the eigenvalues $\exp \mathrm{i}\phi_i$ of the matrix U are known:

$$\chi_{(\lambda)}(U) = \frac{\det\| \exp\{\mathrm{i}\phi_i(\lambda_j + N - j)\}\|}{\det\| \exp\{\mathrm{i}\phi_i(N-j)\}\|} , \tag{2.3}$$

$$d_{(\lambda)} = \frac{\prod_{i<j} (\lambda_i - \lambda_j + j - i)}{\prod_{i<j} (j-i)} = \chi_{(\lambda)}(1) . \tag{2.4}$$

The general form of the orthogonality relations is

$$\int \mathrm{d}\mu(U)\, \mathscr{D}^{(r)}_{ab}(U)\, \mathscr{D}^{(s)\,*}_{cd}(U) = \frac{1}{d_{(r)}} \delta_{r,s}\, \delta_{a,c}\, \delta_{b,d} . \tag{2.5}$$

Further relations can be found in Ref. [50].

The matrix U_{ab} itself coincides with the fundamental representation (1) of the group, and enjoys the properties

$$\chi_{(1)}(U) = \operatorname{Tr} U, \qquad d_{(1)} = N, \qquad \sum_a U_{ab} U^*_{ac} = \delta_{bc} . \tag{2.6}$$

The measure $\mathrm{d}\mu(U)$ (which we shall also denote simply by $\mathrm{d}U$), when the integrand depends only on invariant combinations, may be expressed in terms of the eigenvalues [51].

2.2. Chiral models and lattice gauge theories

Unitary matrix models defined on a lattice can be divided into two major groups, according to the geometric and algebraic properties of the dynamical variables: when the fields are defined in association with lattice sites, and the symmetry group is global, i.e., a single $U(N)_L \times U(N)_R$ transformation is applied to all fields, we are considering a spin model (principal chiral model); in turn, when the dynamical variables are defined on the links of the lattice and the symmetry is local, i.e., a different transformation for each site of the lattice may be performed, we are dealing with a gauge model (lattice gauge theory). As we shall see, these two classes are not unrelated to each other: an analogy between d-dimensional chiral models and $2d$-dimensional gauge theories can be found according to the following correspondence table [52]:

spin	**gauge**
site, link	link, plaquette
loop	surface
length	area
mass	string tension
two-point correlation	Wilson loop

While this correspondence in arbitrary dimensions is by no means rigorous, there is some evidence supporting the analogy.

In the case $d = 1$, which we shall carefully discuss later, one can prove an identity between the partition function (and appropriate correlation functions) of the two-dimensional lattice gauge theory and the corresponding quantities of the one-dimensional principal chiral model. Both theories are exactly solvable, both on the lattice and in the continuum limit, and the correspondence can be explicitly shown.

Approximate real-space renormalization recursion relations obtained by Migdal [53] are identical for d-dimensional chiral models and $2d$-dimensional gauge models.

The two-dimensional chiral model and the (phenomenologically interesting) four-dimensional non-Abelian gauge theory share the property of asymptotic freedom and dynamical generation of a mass scale. In both models these properties are absent in the Abelian case (XY model and U(1) gauge theory respectively), which shows no coupling-constant renormalization in perturbation theory.

The structure of the high-temperature expansion and of the Schwinger–Dyson equations is quite similar in the two models.

It will be especially interesting for our purposes to investigate the Schwinger–Dyson equations of unitary-matrix models and discuss the peculiar properties of their large-N limit.

2.3. Schwinger–Dyson equations in the large-N limit

In order to make our analysis more concrete, we must at this stage consider specific forms of interactions among unitary matrices, both in the spin and in the gauge models. The most dramatic restriction that we are going to impose on the lattice action is the condition of considering only nearest-neighbor interactions. The origin of this restriction is mainly practical, because non nearest-neighbor interactions lead to less tractable problems. We assume that, for the systems we are interested in, it will always be possible to find a lattice representation in terms of nearest-neighbor interactions within the universality class.

Let us denote by x an arbitrary lattice site, and by x, μ an arbitrary lattice link originating in the site x and ending in the site $x+\mu$: μ is one of the d positive directions in a d-dimensional hypercubic lattice. A plaquette is identified by the label x, μ, ν, where the directions μ and ν ($\mu \neq \nu$) specify the plane where the plaquette lies. The dynamical variables (which we label by U in the general case) are site variables U_x in spin models and link variables $U_{x,\mu}$ in gauge models.

The general expression for the partition function is

$$Z = \int \prod \mathrm{d}\mu(U) \exp[-\beta S(U)] , \tag{2.7}$$

where β is the inverse temperature (inverse coupling) and the integration is extended to all dynamical variables. The action $S(U)$ must be a function enjoying the property of extensivity and of (global and local) group invariance, and respect the symmetry of the lattice. Adding the requisite that the interactions involve only nearest neighbors, we find that a generic contribution to the action of spin models must be proportional to

$$\sum_{x,\mu} \chi_{(r)}(U_x U^\dagger_{x+\mu}) + \text{h.c.} , \tag{2.8}$$

and for gauge models to

$$\sum_{x,\mu,\nu} \chi_{(r)}(U_{x,\mu} U_{x+\mu,\nu} U^\dagger_{x+\nu,\mu} U^\dagger_{x,\nu}) + \text{h.c.} , \tag{2.9}$$

where (r) is in principle arbitrary, and the summation is extended to all oriented links of the lattice in the spin case, to all the oriented plaquettes in the gauge case. In practice we shall mostly focus on the simplest possible choice, corresponding to the fundamental representation. In order to reflect the extensivity of the action, i.e., the proportionality to the number of space and internal degrees of freedom, it will be convenient to adopt the normalizations

$$S(U) = -\sum_{x,\mu} N(\mathrm{Tr}\, U_x U^\dagger_{x+\mu} + \text{h.c.}) \quad \text{(spin)} , \tag{2.10}$$

$$S(U) = -\sum_{x,\mu,\nu} N(\mathrm{Tr}\, U_{x,\mu} U_{x+\mu,\nu} U^\dagger_{x+\nu,\mu} U^\dagger_{x,\nu} + \text{h.c.}) \quad \text{(gauge)} . \tag{2.11}$$

Once the lattice action is fixed, it is easy to obtain sets of Schwinger–Dyson equations relating the correlation functions of the models. These are the quantum field equations and solving them corresponds to finding a complete solution of a model. It is extremely important to notice the simplifications occurring in the Schwinger–Dyson equations when the large-N limit is considered. These simplifications are such to allow, in selected cases, explicit solutions to the equations.

Before proceeding to a derivation of the equations, we must preliminarily identify the sets of correlation functions we are interested in. For obvious reasons, these correlations must involve the dynamical fields at arbitrary space distances, and must be invariant under the symmetry group of the model. Without pretending to achieve full generality, we may restrict our attention to such typical objects as the invariant correlation functions of a spin model

$$G^{(n)}(x_1, y_1, \dots, x_n, y_n) = \frac{1}{N}\left\langle \mathrm{Tr} \prod_{i=1}^{n} U_{x_i} U^{\dagger}_{y_i} \right\rangle \tag{2.12}$$

and to the so-called Wilson loops of a gauge model

$$W(\mathscr{C}) = \frac{1}{N}\left\langle \mathrm{Tr} \prod_{l \in \mathscr{C}} U_l \right\rangle, \tag{2.13}$$

where $\mathscr{C}$ is a closed arbitrary walk on the lattice, and $\prod_{l \in \mathscr{C}}$ is the ordered product over all the links along the walk. It is worth stressing that the action itself is a sum of elementary Green's functions (elementary Wilson loops).

More general invariant correlation functions may involve expectation values of products of invariant operators similar to those appearing in the r.h.s. of Eqs. (2.12) and (2.13). The already mentioned property of factorization allows us to express the large-N limit expectation value of such products as a product of expectation values of the individual operators. As a consequence, the large-N form of the Schwinger–Dyson equations is a (generally infinite) set of equations involving only the above-defined quantities.

For sake of clarity and completeness, we present the explicit large-N form of the Schwinger–Dyson equations for the models described by the standard actions Eqs. (2.10) and (2.11). For principal chiral models [54],

$$\begin{aligned} 0 = {} & G^{(n)}(x_1, y_1, \dots, x_n, y_n) \\ & + \beta \sum_{\mu} \left[G^{(n+1)}(x_1, x_1+\mu, x_1, y_1, \dots, x_n, y_n) - G^{(n)}(x_1+\mu, y_1, \dots, x_n, y_n) \right] \\ & + \sum_{s=2}^{n} \left[\delta_{x_1, x_s} G^{(s-1)}(x_1, y_1, \dots, x_{s-1}, y_{s-1})\, G^{(n-s+1)}(x_s, y_s, \dots, x_n, y_n) \right. \\ & \left. - \delta_{x_1, y_s} G^{(s)}(x_1, y_1, \dots, x_s, y_s)\, G^{(n-s)}(x_{s+1}, y_{s+1}, \dots, x_n, y_n) \right] . \end{aligned} \tag{2.14}$$

For lattice gauge theories [55,56],

$$\beta \left[\sum_{\mu} W(\mathscr{C}_{x,\mu\nu}) - W(\mathscr{C}_{x-\mu,\mu\nu}) \right] = \sum_{y \in \mathscr{C}} \delta_{x,y}\, W(\mathscr{C}_{x,y})\, W(\mathscr{C}_{y,x}) , \tag{2.15}$$

where $W(\mathscr{C}_{x,\mu\nu})$ is obtained by replacing $U_{x,\nu}$ with $U_{x,\mu} U_{x+\mu,\nu} U^{\dagger}_{x+\nu,\mu}$ in the loop $\mathscr{C}$, and $\mathscr{C}_{x,y}$, $\mathscr{C}_{y,x}$ are the sub-loops obtained by splitting $\mathscr{C}$ at the intersection point, including the "trivial" splitting. Eqs. (2.15) are commonly known as the lattice Migdal–Makeenko equations. The derivation of the Schwinger–Dyson equations is obtained by performing infinitesimal variations of the integrand in the functional integral representation of expectation values and exploiting invariance of the measure.

2.4. Survey of different approaches

Schwinger–Dyson equations are the starting point for most techniques aiming at the explicit evaluation of large-N vacuum expectation values for nontrivial unitary-matrix models. The form exhibited in Eqs. (2.14) and (2.15) involves in principle an infinite set of variables, and it is therefore not immediately useful to the purpose of finding explicit solutions.

Successful attempts to solve large-N matrix systems have in general been based on finding reformulations of Schwinger–Dyson equations involving more restricted sets of variables and more compact representations (collective fields). As a matter of fact, in most cases it turned out to be convenient to define generating functions, whose moments are the correlations we are interested in, and whose properties are usually related to those of the eigenvalue distributions for properly chosen covariant combinations of matrix fields.

By "covariant combination" we mean a matrix-valued variable whose eigenvalues are left invariant under a general $SU(N) \times SU(N)$ transformation of the Lagrangian fields. Such objects are typically those appearing in the r.h.s. of Eqs. (2.12) and (2.13) *before* the trace operation is performed. Under the $SU(N) \times SU(N)$ transformation $U \to VUW^\dagger$, these operators transform accordingly to $\mathcal{O} \to V\mathcal{O}V^\dagger$, and therefore their eigenvalue spectrum is left unchanged.

Without belaboring on the details (some of which will however be exhibited in the discussion of the single-link integral presented in Section 3), we only want to mention that the approach based on extracting appropriate Schwinger–Dyson equations for the generating functions is essentially algebraic in nature, involving weighted sums of infinite sets of equations in the form (2.14) or (2.15), identification of the relevant functions, and resolution of the resulting algebraic equations, where usually a number of free parameters appear, whose values are fixed by boundary and/or asymptotic conditions and analyticity constraints. The approach based on direct replacement of the eigenvalue distributions in the functional integral and the minimization of the resulting effective action leads in turn to integral equations which may be solved by more or less straightforward techniques. These two approaches are however intimately related, since the eigenvalue density is usually connected with the discontinuity along some cut in the complex-plane extension of the generating function, and one may easily establish a step-by-step correspondence between the algebraic and functional approach.

Let us finally mention that the procedure based on introducing invariant degrees of freedom and eigenvalue density operators has been formalized by Jevicki and Sakita [57,58] in terms of a "quantum collective field theory", whose equations of motion are the Schwinger–Dyson equations relevant to the problem at hand.

A quite different application of the Schwinger–Dyson equations is based on the strong-coupling properties of the correlation functions. In the strong-coupling domain, expectation values are usually analytic in the coupling β within some positive convergence radius, and their boundary value at $\beta = 0$ can easily be evaluated. As a consequence, it is formally possible to solve Eqs. (2.14) and (2.15) in terms of strong-coupling series by sheer iteration of the equations. This procedure may in practice turn out to be too cumbersome for practical purposes; however, in some circumstances, it may lead to rather good approximations [59,60] and even to a complete strong-coupling solution. Continuation to the weak-coupling domain is however a rather nontrivial task.

As a special application of the strong-coupling approach, we must mention the attempt (pioneered by Kazakov et al. [61]) to construct an effective action for the invariant degrees of freedom by means of a modified strong-coupling expansion, and explore the weak-coupling regime by solving the saddle-point equations of the resulting action. This technique might be successful at least in predicting the location and features of the large-N phase transition which is relevant to many physical problems, as mentioned in Section 1.

A numerical approach to large-N lattice Schwinger–Dyson equations based on the minimization of an effective large-N Fokker–Plank potential and suited for the weak-coupling regime was proposed by Rodrigues [62].

Another relevant application of the Schwinger–Dyson equations is found in the realm of the so-called "reduced" models. These models, whose prototype is the Eguchi–Kawai formulation of strong-coupling large-N lattice gauge theories [63], are based on the physical intuition that, in the absence of fluctuations, due to translation invariance, the space extension of the lattice must be essentially irrelevant in the large-N limit, since all invariant physics must be already contained in the expectation values of (properly chosen) purely local variables. More precisely, one might say that, when $N \to \infty$, the SU(N) group becomes so large that it accommodates the full Poincarè group as a subgroup, and in particular it should be possible to find representations of the translation and rotation operators among the elements of SU(N). As a consequence, one must be able to reformulate the full theory in terms of a finite number of matrix field variables defined at a single space–time site (or on the d links emerging from the site in the case of a lattice gauge theory) and of the above-mentioned representations of the translation group. This reformulation is called "twisted Eguchi–Kawai" reduced version of the theory [64,65].

We shall spend a few more words on the reduced models in Section 7. Moreover, a very good review of their properties has already appeared many years ago [6]. In this context, we must only mention that the actual check of validity of the reduction procedure is based on deriving the Schwinger–Dyson equations of the reduced model and comparing them with the Schwinger–Dyson equations of the original model. Usually the equivalence is apparent already at a superficial level when naïvely applying to correlation functions of the reduced model the symmetry properties of the action itself. This procedure however requires some attention, since the limit of infinitely many degrees of freedom within the group itself allows the possibility of spontaneous breakdown of some of the symmetries which would be preserved for any finite value of N. In this context, we recall once more that large N is a thermodynamical limit: N must go to infinity before any other limit is considered, and sometimes the limiting procedures do not commute. It is trivial to recognize that, when the strong-coupling phase is considered, symmetries are unbroken, and the equivalence between original and reduced model may be established without further ado. Problems may occur in the weak-coupling side of a large-N phase transition.

An unrelated and essentially numeric approach to solving the large-N limit of lattice matrix models is the coherent state variational algorithm introduced by Yaffe and coworkers [66,67]. We refer to the original papers for a presentation of the results that may be obtained by this approach.

3. The single-link integral

3.1. The single-link integral in external field: finite-N solution

All exact and approximate methods of evaluation of the functional integrals related to unitary-matrix models must in principle face the problem of performing the simplest of all relevant integrations: the single-link integral. The utmost importance of such an evaluation makes it proper to devote to it an extended discussion, which will also give us the opportunity of discussing in a prototype example the different techniques that may be applied to the models we are interested in.

A quite general class of single-link integrals may be introduced by defining

$$Z(A^\dagger A) = \int \mathrm{d}U \exp\left[N \operatorname{Tr}(A^\dagger U + U^\dagger A)\right] , \tag{3.1}$$

where as usual U is an element of the group U(N) and A is now an arbitrary $N \times N$ matrix. The U(N) invariance of the Haar measure implies that the one link integral (3.1) must depend only on the eigenvalues of the Hermitian matrix $A^\dagger A$, which we shall denote by $x_1, \dots, x_N$. The function $Z(x_1, \dots, x_N)$ must satisfy a Schwinger–Dyson equation: restricting the variables to the U(N) singlet subspace, the Schwinger–Dyson equation was shown to be equivalent to the partial differential equation [68,69]

$$\frac{1}{N^2} x_k \frac{\partial^2 Z}{\partial x_k^2} + \frac{1}{N}\frac{\partial Z}{\partial x_k} + \frac{1}{N^2}\sum_{s \neq k} \frac{x_s}{x_k - x_s}\left(\frac{\partial Z}{\partial x_k} - \frac{\partial Z}{\partial x_s}\right) = Z , \tag{3.2}$$

with the boundary condition $Z(0, \dots, 0) = 1$ and the request that Z be completely symmetric under exchange of the x_i.

It is convenient to reformulate the equation in terms of the new variables $z_k = 2N\sqrt{x_k}$, and to parameterize the solution in terms of the completely antisymmetric function $\hat{Z}(z_1, \dots, z_N)$ by defining

$$Z(z) = \frac{\hat{Z}(z)}{\prod_{i<j}(z_i^2 - z_j^2)} . \tag{3.3}$$

The equation satisfied by $\hat{Z}$ can be shown to reduce to

$$\left[\sum_k z_k^2 \frac{\partial^2}{\partial z_k^2} + (3 - 2N)\sum_k z_k \frac{\partial}{\partial z_k} - \sum_k z_k^2 + \frac{2}{3}N(N-1)(N-2)\right]\hat{Z} = 0 . \tag{3.4}$$

Eq. (3.4) has the structure of a fermionic many-body Schrödinger equation. With some ingenuity it may be solved in the form of a Slater determinant of fermion wavefunctions. In conclusion, we obtain, after proper renormalization [70] (see also [71]),

$$Z(z_1, \dots, z_N) = 2^{N(N-1)/2}\left(\prod_{k=0}^{N-1} k!\right)\frac{\det\|z_j^{i-1} I_{i-1}(z_j)\|}{\det\|z_j^{2(i-1)}\|} , \tag{3.5}$$

where $I_i(z)$ is the modified Bessel function. Eq. (3.5) is therefore a representation of the single-link integral in external field for arbitrary U(N) groups. By taking proper derivatives with respect to its arguments one may in principle reconstruct all the cumulants for the group integration of an arbitrary string of (uncontracted) matrices [72,73].

Some special limits of the general expression (3.5) may prove useful. Let us first of all consider the case when A is proportional to the identity matrix: $A = a1$ and therefore $z_i = 2Na$ and

$$Z(2Na, \dots, 2Na) = \det\|I_{i-j}(2Na)\| . \tag{3.6}$$

As we shall see, this is exactly Bars' and Green's solution for U(N) lattice gauge theory in two dimensions [74].

When only one eigenvalue of A is different from zero the result is

$$Z(2Na,0,\dots,0)=(N-1)!\,(Na)^{1-N}I_{N-1}(2Na)\,. \tag{3.7}$$

The large-N limit will be discussed in the next subsection.

3.2. The external field problem: large-N limit

For our purposes it is extremely important to extract the limiting form of Eq. (3.5) when $N\to\infty$. In principle, it is a very involved problem, since the dependence on N comes not only through the z_i but also from the dimension of the matrices whose determinant we must evaluate. It is however possible to obtain the limit, either by solving separately the large-N version of Eq. (3.2), or by directly manipulating Eq. (3.5).

In the first approach, we introduce the large-N parameterization

$$Z=\exp NW\,, \tag{3.8}$$

where W is now proportional to N; we then obtain from Eq. (3.2), dropping second-derivative terms that are manifestly depressed in the large-N limit [69],

$$x_k\left(\frac{\partial W}{\partial x_k}\right)^2+\frac{\partial W}{\partial x_k}+\frac{1}{N}\sum_{s\neq k}\frac{x_s}{x_s-x_k}\left(\frac{\partial W}{\partial x_s}-\frac{\partial W}{\partial x_k}\right)=1\,. \tag{3.9}$$

It is possible to show that in the large-N limit Eq. (3.9) admits solutions, which can be parameterized by the expression

$$\frac{\partial W}{\partial x_k}=\frac{1}{\sqrt{x_k+c}}\left[1-\frac{1}{2N}\sum_s\frac{1}{\sqrt{x_k+c}+\sqrt{x_s+c}}\right],\quad c\geq 0\,. \tag{3.10}$$

Substitution of Eq. (3.10) into Eq. (3.9) and some algebraic manipulation lead to the consistency condition

$$c\left[\frac{1}{2N}\sum_s\frac{1}{\sqrt{x_s+c}}-1\right]=0\,, \tag{3.11}$$

which in turn admits two possible solutions:

(a) c is determined by the condition

$$\frac{1}{2N}\sum_s\frac{1}{\sqrt{x_s+c}}=1\,, \tag{3.12}$$

implying $c\leq\frac{1}{4}$; this is a "strong coupling" phase, requiring that the eigenvalues satisfy the bound

$$\frac{1}{2N}\sum_s\frac{1}{\sqrt{x_s}}\geq 1\,, \tag{3.13}$$

i.e., at least some of the x_s are sufficiently small;

(b) when

$$\frac{1}{2N}\sum_s \frac{1}{\sqrt{x_s}} \le 1 , \tag{3.14}$$

then the solution corresponds to the choice $c=0$; this is a "weak coupling" phase, and all eigenvalues are large enough.

Direct integration of Eq. (3.10) with proper boundary conditions leads to the large-N result [69]

$$W(x) = 2\sum_k \sqrt{x_k+c} - \frac{1}{2N}\sum_{k,s} \log(\sqrt{x_k+c}+\sqrt{x_s+c}) - Nc - \frac{3}{4}N , \tag{3.15}$$

which must be supplemented with Eq. (3.12) in the strong-coupling regime (3.13), while $c=0$ reproduces the weak-coupling result by Brower and Nauenberg. Amazingly enough, setting $c=0$ in Eq. (3.15) one obtains the naïve one-loop estimate of the functional integral, which turns out to be exact in this specific instance.

It is possible to check that Eq. (3.15) is reproduced by carefully taking the large-N limit of Eq. (3.5), which requires use of the following asymptotic limits of Bessel functions [70]

$$k!\left(\frac{2}{z}\right)^k I_k(z) \underset{z\to\infty}{\to} \left[\frac{1}{2}\left(1+\sqrt{1+\frac{z^2}{k^2}}\right)\right]^{1-k}\left(1+\frac{z^2}{k^2}\right)^{-1/4} \times \exp(\sqrt{k^2+z^2}-k) \quad \text{(strong coupling)} , \tag{3.16}$$

$$I_k(z) \approx \frac{1}{\sqrt{2\pi z}} \exp z \qquad \text{(weak coupling)} . \tag{3.17}$$

An essential feature of Eq. (3.15) is the appearance of two different phases in the large-N limit of the single-link integral. Such a transition would be mathematically impossible for any finite value of N; however it affects the large-N behavior of all unitary-matrix models and gives rise to a number of interesting phenomena. A straightforward analysis of Eq. (3.15) shows that the transition point corresponds to the condition

$$t \equiv \frac{1}{2N}\sum_s \frac{1}{\sqrt{x_s}} = 1 . \tag{3.18}$$

It is also possible to evaluate the difference between the strong- and weak-coupling phases of W in the neighborhood of $t=1$, finding the relationship [69]

$$W_{\text{strong}} - W_{\text{weak}} \sim (t-1)^3 . \tag{3.19}$$

As a consequence, we may classify this phenomenon as a "third order phase transition".

3.3. The properties of the determinant

The large-N factorization of invariant amplitudes is a well-established property of products of operators defined starting from the fundamental representation of the symmetry group. Operators corresponding to highly nontrivial representations may show a more involved pattern of behavior in the large-N limit. Especially relevant from this point of view are the properties of determinants of covariant combinations of fields [52,75]; we will consider the quantities

$$\Delta(x) = \det[U_0 U_x^\dagger] \tag{3.20}$$

for lattice chiral models and

$$\Delta(\mathscr{C}) = \det \prod_{l \in \mathscr{C}} U_l \tag{3.21}$$

for lattice gauge theories.

The expectation values of these operators may act as an order parameter for the large-N phase transition characterizing the class of models we are taking into consideration. Indeed the determinant picks up the phase characterizing the U(1) subgroup that constitutes the center of U(N). Moreover, since

$$\mathrm{U}(N) \approx \mathrm{U}(1) \times \frac{\mathrm{SU}(N)}{Z_N},$$

SU(N) → U(N) as $N \to \infty$ because $Z_N \to$ U(1); therefore the determinant of the U(N) theory in the large-N limit reflects properties of the center of SU(N).

In lattice models this Abelian U(1) subgroup is not decoupled, as it happens in the continuum theory, and therefore $\langle \Delta \rangle$ does not in general have on the lattice the free-theory behavior it has in the continuum.

The basic properties of the determinant may be explored by focusing once more on the external field problem we discussed above. Let us introduce a class of determinant operators, and define their expectation values as [76]

$$\Delta^{(l)} = \langle \det U^l \rangle = \frac{\int \mathrm{d}U \det U^l \exp[N \operatorname{Tr}(U^\dagger A + A^\dagger U)]}{\int \mathrm{d}U \exp[N \operatorname{Tr}(U^\dagger A + A^\dagger U)]}. \tag{3.22}$$

In order to parameterize the SU(N) external-source integral, besides the eigenvalues x_i of $AA^\dagger$, a new external parameter must be introduced, that couples to the determinant:

$$\theta = \frac{\mathrm{i}}{2N}(\log\det A^\dagger - \log\det A). \tag{3.23}$$

Because of the symmetry properties, $\Delta^{(l)}$ may only depend on the eigenvalues z and on θ. It was found that, when U enjoys U(N) symmetry (with finite N),

$$\Delta^{(l)} = \exp(\mathrm{i}Nl\theta)\frac{\hat{Z}_l}{\hat{Z}_0}, \tag{3.24}$$

where $\hat{Z}_l$ is the solution of the following Schwinger–Dyson equation, generalizing Eq. (3.4):

$$\frac{1}{N}\left[\sum_k z_k^2 \frac{\partial^2}{\partial z_k^2} + (3-2N)\sum_k z_k \frac{\partial}{\partial z_k} - \sum_k z_k^2 + \frac{2}{3}N(N-1)(N-2)\right]\hat{Z}_l = l^2 \hat{Z}_l; \tag{3.25}$$

$\hat{Z}_l$ satisfy the property

$$\hat{Z}_l = \left(\prod_k z_k\right)^{|l|} \left(\prod_k \frac{1}{z_k}\frac{\partial}{\partial z_k}\right)^{|l|} \hat{Z}_0 = \det\| z_i^{j-1} I_{j-1-l}(z_i) \| \,. \tag{3.26}$$

When the weak-coupling condition $t \equiv \sum_k 1/z_k \le 1$ is satisfied, the leading contribution to the large-N limit of all $\hat{Z}_l$ is the same:

$$\hat{Z}_l \to \hat{Z}^{(\infty)} = \exp\left[\sum_k z_k - \frac{1}{2}\sum_k \log 2\pi z_k + \sum_{i<k} \log(z_i - z_k)\right]. \tag{3.27}$$

In order to determine the large-N limit of $\Delta^{(l)}$, one therefore needs to compute the O(1) factor in front of the exponentially growing term (3.27). It is convenient to define

$$X_l = \frac{\hat{Z}_l}{\hat{Z}^{(\infty)}}\,, \tag{3.28}$$

whose Schwinger–Dyson equation may be extracted from Eq. (3.25) and takes the form

$$\frac{1}{N}\left[\sum_k z_k^2 \frac{\partial^2 X_l}{\partial z_k^2} + 2\sum_k z_k^2 \frac{\partial X_l}{\partial z_k} + \sum_{k \ne i} \frac{z_k z_i}{z_k - z_i}\left(\frac{\partial X_l}{\partial z_k} - \frac{\partial X_l}{\partial z_i}\right)\right] = \left(l^2 - \frac{1}{4}\right) X_l\,. \tag{3.29}$$

Let us introduce the large-N Ansatz

$$X_l = X_l(t)\,, \tag{3.30}$$

reducing Eq. (3.29) to

$$\frac{1}{N}\sum_k \frac{1}{z_k^2}\,\frac{\mathrm{d}^2 X_l}{\mathrm{d}t^2} + 2(t-1)\,\frac{\mathrm{d}X_l}{\mathrm{d}t} = \left(l^2 - \frac{1}{4}\right) X_l\,. \tag{3.31}$$

Removing terms that are depressed by two powers of $1/N$, we are left with a consistent equation whose solution is

$$X_l = (1-t)^{\frac{1}{2}(l^2 - \frac{1}{4})}\,. \tag{3.32}$$

Finally we can compute the weak-coupling large-N limit of $\Delta^{(l)}$:

$$\Delta^{(l)} \underset{N\to\infty}{\to} \exp(\mathrm{i}Nl\theta)(1-t)^{l^2/2}, \quad t \le 1\,. \tag{3.33}$$

From the standard strong-coupling expansion we may show that

$$\Delta^{(l)} \underset{N\to\infty}{\to} 0 \quad \text{when } t \ge 1\,. \tag{3.34}$$

An explicit evaluation, starting from the exact expression (3.26), expanded in powers of $1/z_k$ for arbitrary N, allows us to show that the quantities $\hat{Z}_l$ may be obtained from Eqs. (3.27) and (3.28) by expanding Eq. (3.32) up to 2nd order in t with no $O(1/N^2)$ corrections. $\Delta^{(l)}$ according to this result violate factorization; in turn, they take the value which would be predicted by an effective Gaussian theory governing the U(1) phase of the field U.

3.4. Applications to mean field and strong coupling

The single-link external-field integral has a natural domain of application in two important methods of investigation of lattice field theories: mean-field and strong-coupling expansion. Extended papers and review articles have been devoted in the past to these topics (cf. Ref. [77] and references therein), and we shall therefore focus only on those results that are specific to the large-N limit and to the $1/N$ expansion.

Let us first address the issue of the mean-field analysis, considering for sake of definiteness the case of d-dimensional chiral models, but keeping in mind that most results can be generalized in an essentially straightforward manner to lattice gauge theories. The starting point of the mean-field technique is the application of the random field transform to the functional integral:

$$\begin{aligned} Z_N &= \int \mathrm{d}U_n \exp\Big\{N\beta\sum_{n,\mu} \mathrm{Tr}(U_n U^\dagger_{n+\mu} + U_{n+\mu}U^\dagger_n)\Big\} \\ &= \int \mathrm{d}V_n\,\mathrm{d}A_n \exp\Big\{N\beta\sum_{n,\mu}\mathrm{Tr}(V_nV^\dagger_{n+\mu} + V_{n+\mu}V^\dagger_n) - N\sum_n \mathrm{Tr}(A_nV^\dagger_n + V_nA^\dagger_n)\Big\} \\ &\quad\times \int \mathrm{d}U_n \exp\Big\{N\sum_{n,\mu}\mathrm{Tr}(A_nU^\dagger_n + U_nA^\dagger_n)\Big\}\,, \end{aligned} \tag{3.35}$$

where V_n and A_n are arbitrary complex $N\times N$ matrices. Therefore the integration over U_n is just the single-link integral we discussed above. As a consequence, the original chiral model is formally equivalent to a theory of complex matrices with effective action

$$-\frac{1}{N}S_{\mathrm{eff}}(A,V) = \beta\sum_{n,\mu}\mathrm{Tr}(V_nV^\dagger_{n+\mu} + V_{n+\mu}V^\dagger_n) - \sum_n \mathrm{Tr}(A_nV^\dagger_n + V_nA^\dagger_n) + \sum_n W(A_nA^\dagger_n)\,. \tag{3.36}$$

The leading order in the mean-field approximation is obtained by applying saddle-point techniques to the effective action, assuming saddle-point values of the fields A_n and V_n that are translation-invariant and proportional to the identity.

We mention that, in the case at hand, the large-N saddle-point equations in the weak-coupling phase are

$$A_n = a = 2\beta\, dv, \qquad V_n = v = 1 - \frac{1}{4a}, \tag{3.37}$$

and they are solved by the saddle-point values

$$\bar a = \beta d\left(1 + \sqrt{1 - \frac{1}{2\beta d}}\right), \qquad \bar v = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{2\beta d}}, \tag{3.38}$$

leading to a value of the free and internal energy

$$\frac{Fd}{N^2L} = \bar a - \frac{1}{2}\log 2\bar a - \frac{1}{2}, \qquad \frac{1}{2d}\frac{\partial}{\partial\beta}\frac{Fd}{N^2L} = \bar v^2\,. \tag{3.39}$$

The strong-coupling solution is trivial: $v = a = 0$, and there is a first-order transition point at

$$\beta_c d = \tfrac{1}{2}, \qquad \bar{v}_c = \tfrac{1}{2}, \qquad \bar{a}_c = \tfrac{1}{2}\,. \tag{3.40}$$

One may also compute the quadratic fluctuations around the mean-field saddle point by performing a Gaussian integral, whose quadratic form is related to the matrix of the second derivatives of W with respect to the fields, and generate a systematic loop expansion in the effective action (3.36), which in turn appears to be ordered in powers of $1/d$. Therefore mean-field methods are especially appropriate for the discussion of models in large space dimensions, and not very powerful in the analysis of $d = 2$ models. The very nature of the transition cannot be taken for granted, especially at large N. However, when $d \geq 3$ there is independent evidence of a first-order phase transition for $N \geq 3$. We mention that a detailed mean-field study of SU(N) chiral models in d dimensions appeared in Refs. [78,79].

When willing to extend the mean-field approach, it is in general necessary to find a systematic expansion of the functional $W(AA^\dagger)$ in the powers of the fluctuations around the saddle-point configurations. Moreover, one may choose to consider not only the large-N value of the functional, but also its expansion in powers if $1/N^2$, in order to make predictions for large but finite values of N. The expansion of W_0 up to fourth order in the fluctuations was performed in Ref. [80], where explicit analytic results can be found. A technique for the weak-coupling $1/N^2$ expansion of W can be found in Ref. [81]. We quote the complete $\mathrm{O}(1/N^4)$ result:

$$\frac{W}{N} = \frac{1}{N^2}\left[\sum_a z_a - \frac{1}{2}\sum_{a,b} \log\frac{z_a + z_b}{2N} - \frac{3}{4}N^2 + \log(1-t)^{-1/8} + \frac{3}{2^7}(1-t)^{-3}\sum_a \frac{1}{z_a^3}\right] + \mathrm{O}\left(\frac{1}{N^6}\right), \tag{3.41}$$

where $t = \sum_a 1/z_a$. Eq. (3.41) can also be expanded in the fluctuations around a saddle-point configuration. Extension to SU(N) with large N was also considered. A discussion of large-N mean field for lattice gauge theories can be found in Refs. [79,82–85].

Let us now turn to a discussion of the main features of the large-N strong-coupling expansion. A preliminary consideration concerns the fact that it is most convenient to reformulate the strong-coupling expansion (i.e., the expansion in powers of β) into a character expansion, which is ordered in the number of lattice steps involved in the effective path that can be associated with each nontrivial contribution to the functional integral. The large-N character expansion will be discussed in greater detail in Section 4.8. Here we only want to discuss those features that are common to any attempt aimed at evaluating strong-coupling series for expectation values of invariant operators in the context of U(N) and SU(N) matrix models, with special focus on the large-N behavior of such series.

The basic ingredient of strong-coupling computations is the knowledge of the cumulants, i.e., the connected contributions obtained performing the invariant group integration of a string of uncontracted U and $U^\dagger$ matrices. U(N) group invariance insures us that these group integrals can be non-zero only if the same number of U and $U^\dagger$ matrices appear in the integrand. SU(N) is slightly different in this respect, and its peculiarities will be discussed later and are not relevant to the present analysis.

It was observed a long time ago that the cumulants, whose group structure is that of invariant tensors with the proper number of indices, involve N-dependent numerical coefficients. The asymptotic behavior of these coefficients in the large-N limit was studied first by Weingarten [86]. However, for finite N, the coefficients written as function of N are formally plagued by the so-called DeWit–'t Hooft poles [87], that are singularities occurring for integer values of N. The highest singular value of N grows with the number n of U matrices involved in the integration, and therefore for sufficiently high orders of the series it will reach any given finite value. A complete description of the pole structure was presented in Ref. [72]; not only single poles, but also arbitrary high-order poles appear for large enough n, and analyticity is restricted to $N \geq n$. Obviously, since group integrals are well defined for all n and N, this is only a pathology of the $1/N$ expansion. Finite-N results are finite, but they cannot be obtained as a continuation of a large-N strong-coupling expansion. However, it is possible to show that the strict $N \to \infty$ limit of the series exists, and moreover, for sufficiently small β and sufficiently large N, the limiting series is a reasonable approximation to the true result, all nonanalytic effects being $O(\beta^{2N})$ in U(N) models and $O(\beta^N)$ in SU(N) models. As a consequence, computing the large-N limit of the strong-coupling series is meaningful and useful in order to achieve a picture of the large-N strong-coupling behavior of matrix models, but the evaluation of $O(1/N^2)$ or higher-order corrections in the strong-coupling phase is essentially pointless.

The large-N limit of the external-field single-link integral has been considered in detail from the point of view of the strong-coupling expansion. In particular, one may obtain expressions for the coefficients of the expansion of W in powers of the moments of $AA^\dagger$: setting

$$\rho_n = \frac{1}{N}\mathrm{Tr}(AA^\dagger)^n, \quad W = \sum_{n=1}^{\infty} \sum_{\substack{\alpha_1,\ldots,\alpha_n \\ \sum_k k\alpha_k = n}} W_{\alpha_1,\ldots,\alpha_n} \rho_1^{\alpha_1} \cdots \rho_n^{\alpha_n}, \tag{3.42}$$

one gets

$$W_{\alpha_1,\ldots,\alpha_n} = (-1)^n \frac{(2n + \sum_k \alpha_k - 3)!}{(2n)!} \prod_k \left[-\frac{(2k)!}{(k!)^2} \right]^{\alpha_k} \frac{1}{\alpha_k!}. \tag{3.43}$$

Further properties of this expansion can be found in the original reference [88].

A character-expansion representation of the single-link integral was also produced for arbitrary U(N) integrals in Ref. [73]. Strong-coupling expansions for large-N lattice gauge theories have been analyzed in detail by Kazakov [26,89], O'Brien and Zuber [27], and Kostov [28], who proposed reinterpretations in terms of special string theories.

3.5. The single-link integral in the adjoint representation

The integral introduced at the beginning of Section 3 is by no means the most general single-link integral one can meet in unitary-matrix models. As mentioned in Section 2, any invariant function of the U's is in principle a candidate for a lattice action. In practice, the only case that has been considered till now that cannot be reduced to Eq. (3.1) is the integral introduced by Itzykson and Zuber [50]

$$I(M_1, M_2) = \int \mathrm{d}U \exp \mathrm{Tr}\,(M_1 U M_2 U^\dagger), \tag{3.44}$$

where M_1 and M_2 are arbitrary Hermitian matrices. This is a special instance of the single-link integral for the coupling of the adjoint representation of U to an external field.

The result, because of U(N) invariance, can only depend on the eigenvalues m_{1i} and m_{2i} of the Hermitian matrices. Several authors [50,90,91] have independently shown that

$$I(M_1,M_2)=\left(\prod_{p=1}^{N-1}p!\right)\frac{\det\|\exp(m_{1i}m_{2j})\|}{\Delta(m_{11},\dots,m_{1N})\Delta(m_{21},\dots,m_{2N})}\,, \tag{3.45}$$

where $\Delta(m_1,\dots,m_N)=\prod_{i>j}(m_i-m_j)$ is the Vandemonde determinant. A series expansion for $I(M_1,M_2)$ in terms of the characters of the unitary group takes the form

$$I(M_1,M_2)=\sum_{(r)}\frac{1}{|n|!}\frac{\sigma_{(r)}}{d_{(r)}}\chi_{(r)}(M_1)\chi_{(r)}(M_2)\,, \tag{3.46}$$

where $\sigma_{(r)}$ is the dimension of the representation (r) of the permutation group; we will present an explicit evaluation of $\sigma_{(r)}$ in Eq. (4.105). Eq. (3.45) plays a fundamental rôle in the decoupling of the "angular" degrees of freedom when models involving complex Hermitian matrices are considered.

An interesting development based on the use of Eq. (3.45) is the so-called "induced QCD" program, aimed at recovering continuum large-N QCD by taking proper limits in the parameter space of the lattice Kazakov–Migdal model [92]

$$S=N\sum_x \mathrm{Tr}\,V(\Phi_x)-N\sum_{x,\mu}\mathrm{Tr}(\Phi_x U_{x,\mu}\Phi_{x+\mu}U^\dagger_{x,\mu})\,, \tag{3.47}$$

where $U_{x,\mu}$ is the non-Abelian gauge field and Φ_x is a Hermitian $N\times N$ (matrix-valued) Lorentz–scalar field. The Itzykson-Zuber integration (3.44) allows the elimination of the gauge degrees of freedom and reduces the problem to studying the interactions of Hermitian matrix fields (with self-interactions governed by the potential V). Discussion of the various related developments is beyond the scope of the present report. It will be enough to say that, while one may come to the conclusion that this model does *not* induce QCD, it is certainly related to some very interesting (and sometimes solvable) matrix models (cf. Ref. [93] for a review).

4. Two-dimensional lattice Yang–Mills theory

4.1. Two-dimensional Yang–Mills theory as a single-link integral

The results presented in the previous section allow us to analyze the simplest physical system described by a unitary-matrix model. As we shall see, one of the avatars of this system is a Yang–Mills theory in two dimensions (YM_2), in the lattice Wilson formulation. Notwithstanding the enormous simplifications occurring in this model with respect to full QCD, still some nontrivial features are retained, and even in the large-N limit some interesting physical properties emerge. It is therefore worth presenting a detailed discussion of this system, which also offers the possibility of comparing the different technical approaches to the large-N solution in a completely controlled situation.

The lattice formulation of the two-dimensional U(N) gauge theory is based on dynamical variables $U_{x,\mu}$ which are defined on links; however, because of gauge invariance, in two dimensions there are no transverse gauge degrees of freedom, and a one-to-one correspondence can be established between link variables and plaquettes. A convenient way of exploiting this fact consists in fixing the gauge [94]

$$U_{x,0} = 1 \tag{4.1}$$

(the lattice version of the temporal gauge $A_0 = 0$). An extremely important consequence of the gauge choice (4.1) emerges from considering the gauge-fixed form of the single-plaquette contribution to the lattice action:

$$\mathrm{Tr}(U_{x,0}U_{x+0,1}U^\dagger_{x+1,0}U^\dagger_{x,1}) \to \mathrm{Tr}\, U_{x+0,1}U^\dagger_{x,1} . \tag{4.2}$$

This is nothing but the single-link contribution to the one-dimensional lattice action of a principal chiral model whose links lie along the 0 direction. When considering invariant expectation values (Wilson loops), we then recognize that they can be reduced to contracted products of tensor correlations of variables defined on decoupled one-dimensional models. As a consequence, YM_2 factorizes completely into a product of independent chiral models labeled by their 1 coordinate. Not only the partition function, but also all invariant correlations can be systematically mapped into those of the corresponding chiral models. The area law for non self-interacting Wilson loops in YM_2 and the exponential decay of the two-point correlations in one-dimensional chiral models are trivial corollaries of these results [94].

The above considerations allow us to focus on the prototype model defined by the action

$$S = -N\sum_i \mathrm{Tr}(U_iU^\dagger_{i+1} + U^\dagger_i U_{i+1}) , \tag{4.3}$$

where i is the site label of the one-dimensional lattice. By straightforward manipulations we may show that the most general nontrivial correlation one really needs to compute involves product of invariant operators of the form

$$\mathrm{Tr}(U_0U^\dagger_l)^k , \tag{4.4}$$

where l plays the rôle of the space distance, and k is a sort of "winding number".

An almost trivial corollary of the above analysis is the observation that YM_2 and principal chiral models in one dimension enjoy a property of "geometrization", i.e., the only variables that can turn out to be relevant for the complete determination of expectation values are the single-plaquette (single-link) averages of products of powers of moments [95]

$$\prod_k\, [\mathrm{Tr}\,(U_0U^\dagger_1)^k]^{m_k} \tag{4.5}$$

and the geometrical features of the correlations (in YM_2, areas of Wilson loops and subloops; in chiral models, distances of correlated points), such that all coupling dependence is incorporated in the expectation values of the quantities (Eq. (4.5)). This result is sufficiently general to apply not only to the Wilson action formulation, but also to all "local" actions such that the interaction depends only on invariant functions of the single-plaquette (single-link) variable, i.e., any linear combination of the expressions appearing in Eqs. (2.8) and (2.9) [95–97].

In order to proceed to the actual computation, it is convenient to perform a change of variables, allowed by the invariance of the Haar measure, parameterizing the fields by

$$V_l = U_{l-1} U_l^\dagger\,; \tag{4.6}$$

the action (4.3) explicitly factorizes into

$$S = -N \sum_l \mathrm{Tr}(V_l + V_l^\dagger)\,. \tag{4.7}$$

It is now easy to get convinced that in the most general case a Wilson loop expectation value (correlation function) can be represented as a finite product of invariant tensors, each of which is originated by a single-link integration of the form

$$\frac{\int \mathrm{d}V_l f(V_l) \exp[N\beta\,\mathrm{Tr}(V_l + V_l^\dagger)]}{\int \mathrm{d}V_l \exp[N\beta\,\mathrm{Tr}(V_l + V_l^\dagger)]} \equiv \langle f(V_l)\rangle\,, \tag{4.8}$$

where $f(V_l)$ is any (tensor) product of V_l's and $V_l^\dagger$'s, and the only nontrivial contributions to the full expectation value come from integrations extended to plaquettes belonging to the area enclosed by the loop itself (in chiral models, links comprised between the extremal points of the space correlation).

For the sake of definiteness, we may focus on the correlators [98]

$$W_{l,k} \equiv \frac{1}{N} \langle \mathrm{Tr}(U_0 U_l^\dagger)^k \rangle\,, \tag{4.9}$$

and find that

$$W_{l,k} = \frac{\int \mathrm{d}V_1 \ldots \mathrm{d}V_l (1/N)\,\mathrm{Tr}(V_1 \ldots V_l)^k \exp[N\beta \sum_{i=1}^l \mathrm{Tr}(V_i + V_i^\dagger)]}{\prod_i \int \mathrm{d}V_i \exp[N\beta\,\mathrm{Tr}(V_i + V_i^\dagger)]}\,. \tag{4.10}$$

This problem can be formally solved for arbitrary N by a character expansion, which we shall discuss in Section 4.8. It is however immediate to recognize that we are ultimately led to computing the general class of group integrals whose form is

$$\int \mathrm{d}V \prod_k (\mathrm{Tr}\,V^k)^{m_k} \exp[N\beta\,\mathrm{Tr}(V + V^\dagger)] \tag{4.11}$$

(where the product runs over positive and negative values of k), and in turn it is in principle an exercise based on the exploitation of the result for the external field single-link integral introduced in Eq. (3.1).

By the way, integrals of the form (4.11) can easily be expressed as linear combinations of integrals belonging to the class

$$\int \mathrm{d}V\, \chi_{(\lambda)}(V) \exp[N\beta\,\mathrm{Tr}(V + V^\dagger)]\,, \tag{4.12}$$

where λ labels properly chosen representations of U(N). Eq. (4.12) is in turn related to the definition of the character coefficients in the character expansion of $\exp[N\beta\,\mathrm{Tr}(V + V^\dagger)]$. For arbitrary N, as a matter of principle, $\chi_{(\lambda)}(V)$ has a representation in terms of the eigenvalues $\exp \mathrm{i}\phi_i$ of the matrix V, while $\mathrm{Tr}(V + V^\dagger) = 2\sum_i \cos\phi_i$ and the measure itself can in this case be expressed in terms of the eigenvalues as

$$\mathrm{d}\mu(V) \sim \prod_i \mathrm{d}\phi_i\, \Delta^2(\phi_1, \dots, \phi_N)\,, \tag{4.13}$$

where

$$\Delta(\phi_1, \dots, \phi_N) \equiv \det \exp \| \mathrm{i}(i\phi_j) \|\,,$$

$$\Delta^2(\phi_1, \dots, \phi_N) = \prod_{i<j} 4\sin^2 \frac{\phi_i - \phi_j}{2}\,. \tag{4.14}$$

As a consequence, it is always possible to express all U(N) integrals in the class (Eq. (4.12)) in terms of linear combinations of products of modified Bessel functions $I_k(2N\beta)$, with $k < N$.

Let us now come to the specific issue of evaluating the relevant physical quantities in the large-N limit of U(N) models, and comparing the procedures corresponding to different possible approaches. Basic to most subsequent developments is the observation that the large-N factorization property allows us to focus on a very restricted class of interesting correlations, which we label by

$$w_k \equiv \left\langle \frac{1}{N} \mathrm{Tr}\, V^k \right\rangle \equiv W_{1,k}\,. \tag{4.15}$$

The first explicit solution to the problem of evaluating w_k in the large-N limit was offered by Gross and Witten [94]. To this purpose, they introduced the eigenvalue density

$$\rho(\phi) = \frac{1}{N} \sum_i \delta(\phi - \phi_i)\,, \tag{4.16}$$

and considered the group integral defining the partition function of the single-link model

$$Z(\beta) \sim \int \prod_i \mathrm{d}\phi_i\, \Delta^2(\phi_1, \dots, \phi_N) \exp\left(2N\beta \sum_i \cos\phi_i \right)\,. \tag{4.17}$$

The integral (Eq. (4.17)) can be evaluated in the $N \to \infty$ limit by a saddle-point technique [99] applied to the effective action

$$2\beta \int \rho(\phi) \cos\phi\, \mathrm{d}\phi + ⨍ \rho(\phi)\, \rho(\phi') \log \sin \frac{\phi - \phi'}{2}\, \mathrm{d}\phi\, \mathrm{d}\phi'\,, \tag{4.18}$$

with the constraint $\int \rho(\phi)\, \mathrm{d}\phi = 1$. The support of the function $\rho(\phi)$ is dynamically determined. The saddle-point integral equation is

$$2\beta \sin\phi = \int_{-\phi_c}^{\phi_c} \mathrm{d}\phi'\, \rho(\phi') \cot \frac{\phi - \phi'}{2}\,, \tag{4.19}$$

and it is possible to identify two distinct solutions, corresponding to weak and strong coupling. When β is small, it is easy to find out that

$$\rho(\phi) = \frac{1}{2\pi}(1 + 2\beta\cos\phi), \qquad -\pi \le \phi \le \pi\,; \tag{4.20}$$

$\rho(\phi)$ is positive definite whenever $\beta \le \frac{1}{2}$. When β is large, $\phi_c < \pi$ and

$$\rho(\phi) = \frac{2\beta}{\pi}\cos\frac{\phi}{2}\sqrt{\frac{1}{2\beta} - \sin^2\frac{\phi}{2}}, \qquad \sin^2\frac{\phi_c}{2} = \frac{1}{2\beta}, \tag{4.21}$$

submitted to the condition $\beta \ge \frac{1}{2}$. Therefore it is possible to identify the location of the third-order phase transition [94]:

$$\beta_c = \tfrac{1}{2}. \tag{4.22}$$

By direct substitution, one finds the values of the free and internal energy (per unit link or unit plaquette):

$$\frac{F}{N^2} = \begin{cases} \beta^2, & \beta \le \frac{1}{2}, \\ 2\beta - \frac{1}{2}\log 2\beta - \frac{3}{4}, & \beta \ge \frac{1}{2}, \end{cases} \tag{4.23}$$

$$w_1 = \frac{1}{2}\frac{\partial}{\partial\beta}\frac{F}{N^2} = \begin{cases} \beta, & \beta \le \frac{1}{2}, \\ 1 - \dfrac{1}{4\beta}, & \beta \ge \frac{1}{2}. \end{cases} \tag{4.24}$$

More generally, one may evaluate w_k from $\rho(\phi)$, thanks to the relationship

$$w_k = \int_{-\phi_c}^{\phi_c} \mathrm{d}\phi \cos k\phi\, \rho(\phi) = \begin{cases} 0, & \beta \le \frac{1}{2},\ k \ge 2, \\ \left(1 - \dfrac{1}{2\beta}\right)^2 \dfrac{1}{k-1} P_{k-2}^{(1,2)}\left(1 - \dfrac{1}{\beta}\right), & \beta \ge \frac{1}{2}, \end{cases} \tag{4.25}$$

where $P_k^{(\alpha,\beta)}$ are the Jacobi polynomials. All w_k are differentiable once in $\beta = \beta_c$, but their second derivatives are discontinuous. Let us notice that Eqs. (4.22), (4.23) and (4.24) are an immediate consequence of Eq. (3.12) and Eq. (3.15) for the special choice

$$x_s = \beta^2. \tag{4.26}$$

4.2. The Schwinger–Dyson equations of the two-dimensional Yang–Mills theory

It is interesting to obtain the above results from the algebraic approach to the Schwinger–Dyson equations of the model. We can restrict Eq. (2.15) to the set of Wilson loops $\mathscr{C}_k$ consisting of k turns around a single plaquette, in which case by definition $W(\mathscr{C}_k) = w_k$. Formally, the Schwinger–Dyson equations do not close on this set of expectation values; however, one may check by inspection, using the factorization property of two-dimensional functional integral for the Yang–Mills theory, that contributions from other Wilson loops cancel in the equations for w_k (this is strictly a two-dimensional property). As a consequence, we obtain the large-N relationships [100]

$$\beta(w_{n-1} - w_{n+1}) = \sum_{k=1}^{n} w_k w_{n-k}, \tag{4.27}$$

with a boundary condition $w_0 = 1$. The solution is found by defining a generating function

$$\Phi(t) \equiv \sum_{k=0}^{\infty} w_k t^k \tag{4.28}$$

and noticing that Eq. (4.27) corresponds to

$$\Phi t^2 - (\Phi - 1 - w_1 t) = \frac{t}{\beta}(\Phi^2 - \Phi)\,, \tag{4.29}$$

which is solved by

$$\Phi(t) = \frac{\beta}{2t}\sqrt{\left(1 + \frac{t}{\beta} + t^2\right)^2 - 4t^2\left(1 - \frac{w_1}{\beta}\right)} - \frac{\beta}{2t}\left(1 - \frac{t}{\beta} - t^2\right). \tag{4.30}$$

The condition $|w_k| \le 1$ implies that $\Phi(t)$ is holomorphic within the unitary circle. On the boundary of the analyticity domain, $t = \mathrm{e}^{\mathrm{i}\phi}$ and

$$w_k = \frac{1}{\pi}\int_{-\pi}^{\pi} [\mathrm{Re}\,\Phi(\mathrm{e}^{\mathrm{i}\phi}) - \tfrac{1}{2}]\cos k\phi \,\mathrm{d}\phi\,, \tag{4.31}$$

and as a consequence we may identify

$$\frac{1}{\pi}[\mathrm{Re}\,\Phi(\mathrm{e}^{\mathrm{i}\phi}) - \tfrac{1}{2}] = \rho(\phi)\,. \tag{4.32}$$

The positivity condition on $\rho(\phi)$ leads to a complete determination of the solution, implying either

$$w_1 = \beta, \qquad w_k = 0 \quad (k \ge 2), \qquad \beta \le \tfrac{1}{2} \tag{4.33}$$

or $\rho(\pi) = 0$, which in turn leads to

$$w_1 = 1 - \frac{1}{4\beta}, \qquad -\phi_c \le \phi \le \phi_c, \qquad \beta \le \tfrac{1}{2}\,, \tag{4.34}$$

and ϕ_c is given by Eq. (4.21). It is immediate to check that the resulting eigenvalue densities are the same as Eqs. (4.20) and (4.21).

Let us mention that these methods may in principle be applied to more general formulations of the theory based on “local” actions, and in particular Wilson loop expectation values can be computed for the fixed-point version of the model, corresponding to the continuum action [95]. In the general “local” formulation, the expression of the effective action (Eq. (4.18)) is replaced by

$$2\sum_k \beta_k \int \rho(\phi)\cos k\phi \,\mathrm{d}\phi + ⨍ \rho(\phi)\,\rho(\phi')\log\sin\frac{\phi - \phi'}{2}\,\mathrm{d}\phi\,\mathrm{d}\phi'\,, \tag{4.35}$$

where β_k is an (infinite) set of couplings, to be eventually determined by the fixed-point conditions. The solution of the corresponding saddle-point equation

$$\sum_k k\beta_k \sin k\phi = \mathrm{Im}\,\Phi(\mathrm{e}^{\mathrm{i}\phi}) \tag{4.36}$$

leads to an identification of $\rho(\phi)$ by the use of Eq. (4.32). As a consequence, we may express all w_k in terms of the couplings β_k. As long as a strong-coupling phase exists, the solution is simply

$$w_k = k\beta_k\,. \tag{4.37}$$

At the fixed point, a quite explicit transcendental equation can be found for the function $\Phi(t)$:

$$\frac{\Phi(t)-1}{\Phi(t)}\exp\frac{\Phi(t)-\frac{1}{2}}{2\beta}=t\,, \tag{4.38}$$

which leads to the continuum solution

$$w_k=\frac{1}{k}\exp\left[-\frac{k}{4\beta}\right]L_{k-1}^{(1)}\left(\frac{k}{2\beta}\right), \tag{4.39}$$

where $L_k^{(\alpha)}$ are generalized Laguerre polynomials.

The fixed-point action in YM_2 in turn is nothing but the "heat kernel" action [101], discussed in the large-N context in Ref. [102]. Large-N continuum YM_2 is slightly beyond the purpose of the present review. We must however mention that in recent years a number of interesting results have appeared in a string theory context. It is worth quoting Refs. [103–105] and references therein.

While the problem of evaluating the more general expectation values $W_{l,k}$ is solved in principle, in practice it is not always simple to obtain compact closed-form expressions whose general features can be easily understood. In the strong-coupling regime $\beta<\frac{1}{2}$, it is not too difficult to determine from finite-N results the large-N limit in the form [98]

$$\lim_{N\to\infty} W_{l,k}=\frac{(-1)^{k-1}}{k}\binom{lk-2}{k-1}\beta^{kl}\,, \tag{4.40}$$

and one may show that the corresponding Schwinger–Dyson equations close on the set $W_{l,k}$ for any fixed l and are solved by Eq. (4.40). As a matter of fact, by defining

$$\Phi_l(t)\equiv\sum_{k=0}^{\infty} W_{l,k}t^k, \tag{4.41}$$

one may show that the strong-coupling Schwinger–Dyson equations reduce to

$$[\Phi_l(t)-1][\Phi_l(t)]^{l-1}=\beta^l t\,. \tag{4.42}$$

For the interesting values $l=1$ and $l=2$, Eq. (4.40) reduces to

$$\Phi_1(t)=1+\beta t\,, \tag{4.43}$$

consistent with the strong-coupling solution (Eq. (4.30)), and

$$\Phi_2(t)=\tfrac{1}{2}(\sqrt{1+4\beta^2 t}+1)\,, \tag{4.44}$$

related to the generating function for the moments of the energy density

$$\frac{1}{N}\left\langle \mathrm{Tr}\frac{1}{1-\beta t(V_n+V_{n+1}^\dagger)}\right\rangle = 1+2t\beta^2+2\sum_{k=1}^{\infty}(\beta t)^{2k}W_{2,k}=2\beta^2 t+\sqrt{1+4\beta^4t^2}\,. \tag{4.45}$$

Eq. (4.45) is related to a different approach for solving large-N unitary-matrix models, based on an integration of the matrix angular degrees of freedom to be performed in strong coupling [61,106].

The corresponding weak-coupling problem is definitely more difficult. As far as we can see, the Schwinger–Dyson equations close only on a larger set of correlation functions, defined by the generating function [107]

$$D^{(l)}_{k,n}(t) = \frac{1}{N}\mathrm{Tr}\left[(V_k)^{n+1}V_{k+1}\dots V_l\frac{1}{1-tV_1\dots V_l}\right],\quad 0<k\le l,\ n\ge 0\,, \tag{4.46}$$

such that

$$\Phi_l(t) = 1 + tD^{(l)}_{1,0}(t)\,. \tag{4.47}$$

The explicit form of the equations is

$$\sum_{j=0}^{n-1} w_j D^{(l)}_{k,n-j}(t) + D^{(l)}_{l,n-1}(t)\,D^{(l)}_{k,0}(t) + \beta[D^{(l)}_{k,n+1}(t) - D^{(l)}_{k,n-1}(t)] = 0,\quad 1\le k\le l\,. \tag{4.48}$$

When $l = 1,2$ it is possible to find explicit weak-coupling solutions, but the general case $l > 2$ has not been solved so far.

More about the calculability of Wilson loops with arbitrary contour in two-dimensional U(∞) lattice gauge theory can be found in Ref. [108]. The corresponding continuum calculations are presented for arbitrary U(N) groups in Ref. [109].

4.3. Large-N properties of the determinant

It is quite interesting to apply the results of Section 3.3, concerning the properties of the determinant, to YM_2 and principal chiral models in one dimension. Exploiting the factorization of the functional integration and the possibility of performing the variable change (4.6) in the operators as well as in the action, we can easily obtain the relationship

$$\Delta_l \equiv \det[U_0U_l^\dagger] = \det[V_1\dots V_l] = \det V_1\dots\det V_l\,, \tag{4.49}$$

and, as a consequence,

$$\langle\Delta_l\rangle = \langle\det V_1\rangle^l\,. \tag{4.50}$$

The problem is therefore reduced to that of evaluating $\langle\det V\rangle$ in the single-plaquette model. It is immediate to recognize from Eqs. (3.31) and (3.32) that

$$\langle\det V\rangle \to \sqrt{1-\frac{1}{2\beta}},\quad \beta\ge\tfrac{1}{2}\,, \tag{4.51}$$

$$\langle\det V\rangle \to 0,\qquad \beta\le\tfrac{1}{2}\,. \tag{4.52}$$

Apparently, this expectation value acts as an order parameter for the phase transition between the weak- and strong-coupling phases. More precisely, according to Green and Samuel [110,111], one must identify the order parameter with the quantity

$$\langle\Delta_l\rangle^{1/N} \tag{4.53}$$

and notice that

$$\langle \Delta_l \rangle^{1/N} \to 1 \qquad \text{in weak coupling}\,, \tag{4.54}$$

$$\langle \Delta_l \rangle^{1/N} \to \exp(-\sigma l) \quad \text{in strong coupling}\,, \tag{4.55}$$

where σ acts as a U(1) "string tension". Eqs. (4.54) and (4.55) generalize to higher dimensions, when replacing l with the (large) area of the corresponding Wilson loop. Notice that the weak-coupling result is consistent with the decoupling of the U(1) degrees of freedom from the SU(N) degrees of freedom, and with the interpretation of U(1) as a free massless field.

It is therefore interesting to compute

$$\sigma = -\frac{1}{N}\log\langle \det V\rangle \tag{4.56}$$

in the case of the single-matrix model; this requires taking the large-N limit only after the strong-coupling calculation of $\langle \det V\rangle$ has been performed. Since the technique of evaluation of σ has some relevance for subsequent developments, we shall briefly sketch its essential steps. Standard manipulations of the single-link integrals for finite N allow to evaluate

$$A_{m,N}(\beta) = \int \mathrm{d}V \exp[N\beta\,\mathrm{Tr}(V + V^\dagger)](\det V)^m = \det\| I_{k-l-m}(2N\beta)\|\,. \tag{4.57}$$

These quantities can be shown to satisfy the recurrence relations [112]

$$A_{m,N}^2 - A_{m+1,N}A_{m-1,N} = A_{m,N-1}A_{m,N+1}\,. \tag{4.58}$$

Willing to compute expectation values, we define

$$\Delta_{m,N}(\beta) = \langle (\det V)^m\rangle = \frac{A_{m,N}}{A_{0,N}}\,. \tag{4.59}$$

Eq. (4.58) implies that

$$\Delta_{m,N}^2 - \Delta_{m+1,N}\Delta_{m-1,N} = \Delta_{m,N-1}\Delta_{m,N+1}(1-\Delta_{1,N}^2)\,. \tag{4.60}$$

Since all $\Delta_{m,1}$ are known, it is possible to reconstruct all $\Delta_{m,N}$ from Eq. (4.60) once $\Delta_{1,N}$ is determined. Now $\Delta_{1,N}$ is exactly $\langle \det V\rangle$, and it is possible to show that it obeys the following second-order differential equation [113]

$$\frac{1}{s}\frac{\mathrm{d}}{\mathrm{d}s}s\frac{\mathrm{d}}{\mathrm{d}s}\Delta_{1,N} + \frac{1}{1-\Delta_{1,N}^2}\left[\left(\frac{\mathrm{d}}{\mathrm{d}s}\Delta_{1,N}\right)^2 - \frac{N^2}{s^2}\right]\Delta_{1,N} + (1-\Delta_{1,N}^2)\Delta_{1,N} = 0\,, \tag{4.61}$$

where $s = 2N\beta$. Eq. (4.61) can be analyzed in weak and strong coupling and in the large-N limit. In particular the weak-coupling $1/N$ expansion leads to

$$\Delta_{1,N} \to \sqrt{1-\frac{1}{2\beta}} - \frac{1}{N^2}\frac{1}{128\beta^3}\left(1-\frac{1}{2\beta}\right)^{5/2} + \mathrm{O}\left(\frac{1}{N^4}\right), \tag{4.62}$$

thus confirming Eq. (4.51), while in strong coupling one may show that

$$\Delta_{1,N} = J_N(2N\beta) + O(\beta^{3N+2}) \underset{N\to\infty}{\to} J_N(2N\beta)\,, \tag{4.63}$$

where J_N is the standard Bessel function, whose asymptotic behavior is well known. As an immediate consequence, we find

$$-\sigma = \sqrt{1-4\beta^2} - \log\frac{1+\sqrt{1-4\beta^2}}{2\beta}, \quad \beta < \tfrac{1}{2}\,. \tag{4.64}$$

This result was first guessed by Green and Samuel [111], and then explicitly demonstrated in Ref. [113].

4.4. Local symmetry breaking in the large-N limit

Another interesting application of the external-field single-link integral to the large-N limit of two-dimensional Yang–Mills theories is the study of the possibility of breaking a local symmetry, as a consequence of the thermodynamical nature of the limit. If we introduce an infinitesimal explicit U(N) symmetry breaking term in the action [114]

$$S = -\beta N[\mathrm{Tr}\, V + JNV^{ij} + \mathrm{h.c.}]\,, \tag{4.65}$$

corresponding to replacing

$$A_{lm} \to \beta[\delta_{lm} + NJ\delta_{lj}\delta_{mi}] \tag{4.66}$$

in Eq. (3.1), we find that the eigenvalues of $AA^\dagger$ are

$$\begin{aligned} x_{1,2} &= \beta^2[1 + \tfrac{1}{2}N^2J^2 \pm \tfrac{1}{2}\sqrt{J^4N^4 + 4J^2N^2}]\,, \\ x_l &= \beta^2, \quad l > 2\,. \end{aligned} \tag{4.67}$$

When taking the large-N limit of the free energy, we find

$$\lim_{N\to\infty} \frac{\log Z}{N^2} = F_0(\beta) + 2\beta|J|\,, \tag{4.68}$$

and in the limit $J \to 0^\pm$ we then find

$$\langle \mathrm{Re}\, V^{ij} \rangle = \pm 1\,. \tag{4.69}$$

We therefore expect that, for finite k, the U(k) global symmetries of large-N chiral models and U(k) gauge symmetries are broken in any number of dimensions [114]. This phenomenon cannot occur for any finite value of N in two dimensions.

4.5. Evaluation of higher-order corrections

In the context of large-N two-dimensional Yang–Mills theory, it is worth mentioning that it is possible to compute systematically higher-order corrections to physical quantities in the powers of

$1/N^2$. It is interesting to notice that the weak-coupling corrections to the free energy [115] (see also [82])

$$F = F_0 + \frac{1}{N^2}\left[\frac{1}{12} - A - \frac{1}{12}\log N - \frac{1}{8}\log\left(1 - \frac{1}{2\beta}\right)\right] + \frac{1}{N^4}\left[\frac{3}{1024\beta^3}\left(1 - \frac{1}{2\beta}\right)^{-3} - \frac{1}{240}\right] + \cdots\,, \tag{4.70}$$

$$U = 1 - \frac{1}{4\beta} - \frac{1}{N^2}\frac{1}{32\beta^2}\left(1 - \frac{1}{2\beta}\right)^{-1} - \frac{1}{N^2}\frac{1}{1024\beta^4}\left(1 - \frac{1}{2\beta}\right)^{-4} + O\left(\frac{1}{N^4}\right), \tag{4.71}$$

where $A = 0.24875\ldots$, are well defined, but become singular when $\beta \to \frac{1}{2}$. In turn, when evaluating higher-order corrections in the strong-coupling phase, one finds out that there are no corrections proportional to powers of $1/N$, while there are contributions that fall off exponentially with large N, as expected from the general arguments discussed in Section 3.4 in connection with the appearance of the DeWit–'t Hooft poles.

Let us however mention that Eqs. (4.60) and (4.61) are also the starting point for a systematic $1/N$ expansion of the free energy in the weak-coupling regime, alternative to Goldschmidt's procedure. The basic ingredient is the observation that, defining the free energy at finite N by

$$F_N(\beta) = \log A_{0,N}(\beta)\,, \tag{4.72}$$

one may show that

$$\frac{d}{ds}(\log F_N - \log F_{N-1}) = \frac{\Delta_{1,N}}{1 - \Delta_{1,N}^2}\left(\frac{d}{ds}\Delta_{1,N} + \frac{N}{s}\Delta_{1,N}\right), \tag{4.73}$$

and this allows for a systematic reconstruction of F_N, whose strong-coupling form is [112]

$$F_N(\beta) = N^2\beta^2 - \sum_{k=1}^{\infty} k J_{N+k}^2(2N\beta) + O(\beta^{4N+4})\,. \tag{4.74}$$

4.6. Mixed-action models for lattice YM_2

Another instance of the problem of the single-link integration for matrix fields in the adjoint representation of the full symmetry group occurs in the discussion of the so-called "mixed action" models. Consider the following single-link integral [116], resulting from a different formulation of lattice YM_2,

$$Z(\beta_f, \beta_a) = \int dU \exp\{N\beta_f \mathrm{Tr}(U + U^\dagger) + \beta_a |\mathrm{Tr}\, U|^2\}\,. \tag{4.75}$$

It is possible to show that, in the large-N limit, the corresponding free energy can be obtained by the same saddle-point technique presented in Section 4.1, i.e., by introducing a spectral density $\rho(\theta)$ for the eigenvalues of U. This spectral density turns out to be precisely the same as the one obtained when $\beta_a = 0$, if one simply replaces β_f by an effective coupling

$$\beta_{\rm eff} = \beta_f + \beta_a w_1(\beta_{\rm eff})\,, \tag{4.76}$$

where w_1 can be evaluated in terms of $\rho(\theta)$ as

$$w_1(\beta_{\rm eff}) = \int d\theta \cos\theta\, \rho(\theta)\,. \tag{4.77}$$

Eq. (4.77) is a self-consistency condition for w_1, which allows a determination of $\beta_{\rm eff}(\beta_{\rm f}, \beta_{\rm a})$. Finally, by substitution into the effective action, one finds the relationship

$$F(\beta_{\rm f}, \beta_{\rm a}) = F(\beta_{\rm eff}(\beta_{\rm f}, \beta_{\rm a}), 0) - \beta_{\rm a} w_1^2(\beta_{\rm eff}(\beta_{\rm f}, \beta_{\rm a}))\,, \tag{4.78}$$

where $F(\beta, 0)$ is nothing but the free energy obtained in Section 4.1.

The strong- and weak-coupling solutions are separated by the line $2\beta_{\rm f} + \beta_{\rm a} = 1$. In strong coupling one obtains

$$\beta_{\rm eff} = w_1 = \frac{\beta_{\rm f}}{1-\beta_{\rm a}}, \qquad F = \frac{\beta_{\rm f}^2}{1-\beta_{\rm a}}\,, \tag{4.79}$$

while in weak coupling

$$\beta_{\rm eff} = \frac{1}{2}\left[\beta_{\rm f} + \beta_{\rm a} + \sqrt{(\beta_{\rm f}+\beta_{\rm a})^2 - \beta_{\rm a}}\right], \qquad w_1 = \frac{1}{2\beta_{\rm a}}\left[\beta_{\rm a} - \beta_{\rm f} + \sqrt{(\beta_{\rm f}+\beta_{\rm a})^2 - \beta_{\rm a}}\right],$$

$$F = \beta_{\rm f} + \frac{\beta_{\rm a}}{2} - \frac{\beta_{\rm f}^2}{2\beta_{\rm a}} - \frac{1}{2} - \frac{1}{2}\log\left[\beta_{\rm f} + \beta_{\rm a} + \sqrt{(\beta_{\rm f}+\beta_{\rm a})^2 - \beta_{\rm a}}\right] + \frac{1}{2}\left(1 + \frac{\beta_{\rm f}}{\beta_{\rm a}}\right)\sqrt{(\beta_{\rm f}+\beta_{\rm a})^2 - \beta_{\rm a}}\,. \tag{4.80}$$

It may be interesting to quote explicitly the limiting case $\beta_{\rm f} = 0$, where [79]

$$Z(0, \beta_{\rm a}) \equiv \int dU \exp \beta_{\rm a} |{\rm Tr}\, U|^2 = \begin{cases} 0, & \beta_{\rm a} < 1\,, \\ \frac{1}{2}\beta_{\rm a} + \frac{1}{2}\beta_{\rm a}\sqrt{1 - \frac{1}{\beta_{\rm a}}} - \frac{1}{2}\log \beta_{\rm a}\left(1 + \sqrt{1 - \frac{1}{\beta_{\rm a}}}\right), & \beta_{\rm a} > 1\,. \end{cases} \tag{4.81}$$

One may actually show that, in any number of dimensions, a lattice gauge theory with mixed action [117–119] (a trivial generalization of Eq. (4.75)) is solved in the large-N limit in terms of the solution of the corresponding theory with pure Wilson action; Eqs. (4.76) and (4.78) hold as they stand, and

$$w_1(\beta_{\rm eff}) = \frac{1}{N}\langle {\rm Tr}\, U_p \rangle|_{\beta_{\rm f} = \beta_{\rm eff},\ \beta_{\rm a} = 0}\,. \tag{4.82}$$

More about the large-N behavior of variant actions can be found in Refs. [120–122]. Different kinds of variant actions have been studied in the large-N limit in Refs. [123–125].

4.7. Double-scaling limit of the single-link integral

In the Introduction, we mentioned that one of the most interesting phenomena related to the large-N limit of matrix models is the appearance of the so-called "double-scaling limit"

$$\begin{cases} N \to \infty\ , \\ g \to g_c\ , \end{cases} \qquad N^{2/\gamma_1}(g_c - g) = \text{const.}\ , \tag{4.83}$$

where g is a (weak) coupling related to the inverse of β. We already discussed the general physical interpretation of this limit as an alternative description of two-dimensional quantum gravity and its relationship to the theory of random surfaces. Here we only want to consider the double-scaling limit properties for those simple models of unitary matrices that can be reformulated as a single-link model (cf. Ref. [126]).

This specific subject was pioneered by Periwal and Shevitz [127], who discussed the double-scaling limit in models belonging to the class

$$Z_N = \int \mathrm{d}U \exp\left[N\beta \operatorname{Tr} \mathscr{V}(U + U^\dagger)\right]\ , \tag{4.84}$$

where $\mathscr{V}(U)$ is a polynomial in U. Because of the invariance of the measure, Eq. (4.84) can be reduced to

$$Z_N \sim \int \mathrm{d}\phi_i |\Delta(\mathrm{e}^{\mathrm{i}\phi_1}, \ldots, \mathrm{e}^{\mathrm{i}\phi_N})|^2 \exp\left[N\beta \sum_i \mathscr{V}(2\cos\phi_i)\right], \tag{4.85}$$

and solved by the method of orthogonal polynomials. One starts by defining polynomials

$$P_n(z) = z^n + \sum_{k=0}^{n-1} a_{k,n} z^k\ , \tag{4.86}$$

that satisfy

$$\oint \frac{\mathrm{d}z}{2\pi \mathrm{i} z} P_n(z)\, P_m\!\left(\frac{1}{z}\right) \exp\left[N\beta \mathscr{V}\!\left(z + \frac{1}{z}\right)\right] = h_n\, \delta_{mn}\ , \tag{4.87}$$

where the integration runs over the unit circle, and moreover obey the recursion relation

$$P_{n+1}(z) = zP_n(z) + R_n z^n P_n\!\left(\frac{1}{z}\right), \qquad \frac{h_{n+1}}{h_n} = 1 - R_n^2\ , \tag{4.88}$$

where $R_n \equiv a_{0,n+1}$. As a corollary,

$$Z_N \propto N! \prod_i (1 - R_{i-1}^2)^{N-i}\ , \tag{4.89}$$

and one may show that

$$(n+1)(h_{n+1} - h_n) = \oint \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \exp\left[N\beta \mathscr{V}\!\left(z + \frac{1}{z}\right)\right] N\beta \mathscr{V}'\!\left(z + \frac{1}{z}\right)\left(1 - \frac{1}{z^2}\right) P_{n+1}(z)\, P_n\!\left(\frac{1}{z}\right), \tag{4.90}$$

which in turn leads to a nonlinear functional equation for R_n.

The simplest example, corresponding to YM_2, amounts to choosing $\mathscr{V}'=1$, obtaining

$$(n+1)R_n^2 = N\beta R_n(R_{n+1}+R_{n-1})(1-R_n^2)\,, \tag{4.91}$$

and in the large-N limit, setting $n=N$ and $R_N=R$, we obtain the limiting form

$$R^2 = 2\beta R^2(1-R^2)\,, \tag{4.92}$$

showing that $\beta_c=\frac{1}{2}$ (degeneracy of solution $R_c=0$). One may now look for the scaling solution to Eq. (4.91) in the form

$$R_N - R_c = R_N = N^{-\mu} f[N^\rho(g_c-g)]\,, \qquad g=\frac{1}{\beta}\,, \tag{4.93}$$

where f^2 is related to the second derivative of the free energy. This is a consistent Ansatz when

$$\mu=\tfrac{1}{3}\,, \qquad \rho=\tfrac{2}{3}\,, \tag{4.94}$$

leading to the equation

$$-2xf+2f^3=f''\,, \qquad x=N^\rho(g_c-g)\,. \tag{4.95}$$

In the case $\mathscr{V}'=1+\lambda u$, one finds the equation

$$\frac{1}{\beta} = -2(1-R^2)(-1-\lambda+3\lambda R^2)\,, \tag{4.96}$$

which reduces to $1/\beta=\frac{3}{2}(1-R^4)$ when $\lambda=\frac{1}{4}$. A scaling solution to the corresponding difference equation requires $\mu=\frac{1}{5}$ and $\rho=\frac{4}{5}$. When $\mathscr{V}'=1+\lambda_1 u+\lambda_2 u^2$, multicriticality sets at $\lambda_1=-\frac{3}{7}$ and $\lambda_2=\frac{1}{14}$, and $1/\beta=\frac{10}{7}(1-R^6)$, leading to the exponents $\mu=\frac{1}{7}$ and $\rho=\frac{6}{7}$. Rather general results can be obtained for an arbitrary order k of the polynomial $\mathscr{V}$: $\mu=1/(2k+1)$, $\rho=2k/(2k+1)$, and $c=1-6/(k(k+1))$.

The double-scaling limit can also be studied in the case of the external-field single-link integral [128], and it was found that its critical behavior is simple enough to be identified with that of the $k=1$ unitary-matrix model. In the language of quantum gravity, the only effect of introducing N^2 real parameters A_{ij} is that of renormalizing the cosmological constant, without changing the universality class of the critical point.

A few interesting features of the double-scaling limit for the $k=1$ model are worth a more detailed discussion [21]. In particular let us recall that, according to Eq. (4.94),

$$\rho=\frac{2}{\gamma_1}=\frac{2}{3}\,, \tag{4.97}$$

and therefore $\gamma_1=3$, implying $c=-2$. We may now reinterpret the double-scaling limit of matrix models as a finite-size scaling with respect to the "volume" parameter N in a two-dimensional $N\times N$ space. As a consequence, we obtain relationships with more conventional critical exponents through the identification $\gamma_1=2\nu$, which in turn by hyperscaling leads to a determination of the specific heat exponent $\alpha=2(1-\nu)$. Numerically we obtain $\nu=\frac{3}{2}$ and $\alpha=-1$. The result $\alpha=-1$

can be easily tested on the solution of the model

$$C(\beta) = \frac{1}{2}\beta^2 \frac{d^2F}{d\beta^2} = \begin{cases} \beta^2, & \beta \le \beta_c\,, \\ \frac{1}{4}, & \beta \ge \beta_c\,, \end{cases} \tag{4.98}$$

with $\beta_c = \frac{1}{2}$, consistent with a negative critical exponent $\alpha = -1$.

It is also interesting to find tests for the exponent ν, especially in view of the fact that the most direct checks are not possible in absence of a proper definition for the relevant correlation length. Numerical studies have been performed by considering the partition function zero β_0 closest to the transition point $\beta_c = \frac{1}{2}$, finding that the relationship

$$\mathrm{Im}\,\beta_0 \propto N^{-1/\nu} \tag{4.99}$$

is rather well satisfied even for very low values of N; at $N \ge 5$, it is valid within one per mille. Another test concerns the location of the peak in the specific heat in U(N) models, whose position $\beta_{\mathrm{peak}}(N)$ should approach β_c with increasing N. Finite-size scaling arguments predict

$$\beta_{\mathrm{peak}}(N) \cong \beta_c + aN^{-1/\nu}\,, \tag{4.100}$$

and large-N results are very well fitted by the choice $\nu = \frac{3}{2}$, $a \cong 0.60$ [129].

4.8. The character expansion and its large-Nlimit: SU(N) vs. U(N)

The general features of the character expansion for lattice spin and gauge models have been extensively discussed by different authors. In particular, Ref. [77], besides offering a general presentation of the issues, presents tables of character coefficients for many interesting groups, including $\mathrm{U}(\infty) \cong \mathrm{SU}(\infty)$, for the Wilson action. Let us therefore only briefly recall the fundamental points of this approach, which is relevant especially in the analysis of the strong-coupling phase and of the phase transition.

In Section 2 we classified the representations and characters of U(N) groups. Because of the orthogonality and completeness relations, every invariant function of V can be decomposed in a generalized Fourier series in the characters of V. Let us now consider for sake of definiteness chiral models with action given by Eq. (2.10); extension to lattice gauge theories is essentially straightforward, at least on a formal level. We can replace the Boltzmann factor corresponding to each lattice link by its character expansion:

$$\exp\{\beta N\,\mathrm{Tr}[U_x U^\dagger_{x+\mu} + U_{x+\mu}U^\dagger_x]\} = \exp\left\{N^2 F(\beta) \sum_{(r)} d_{(r)} \tilde{z}_{(r)}(\beta)\, \chi_{(r)}(U_x U^\dagger_{x+\mu})\right\}, \tag{4.101}$$

where the sum runs over all the irreducible representations of U(N), $F(\beta)$ is the free energy of the single-link model

$$F(\beta) = \frac{1}{N^2}\log\int dV \exp[N\beta\,\mathrm{Tr}(V + V^\dagger)] = \frac{1}{N^2}\log\det\|I_{j-i}(2N\beta)\|\,, \tag{4.102}$$

and $\tilde{z}_{(r)}(\beta)$ are the character coefficients, defined by orthogonality and representable in terms of single-link integrals as

$$d_{(r)}\tilde{z}_{(r)}(\beta) = \langle \chi_{(r)}(V) \rangle = \frac{\det\|I_{\lambda_i+j-i}(2N\beta)\|}{\det\|I_{j-i}(2N\beta)\|}, \tag{4.103}$$

with λ defined by Eq. (2.2). We may notice that, for any finite N, $\tilde{z}_{(r)}(\beta)$ are meromorphic functions of β, with no poles on the real axis, which is relevant to the series analysis. However, singularities may develop, as usual, in the large-N limit. Eq. (4.101) and Eq. (4.102) become rapidly useless with growing N. However, an extreme simplification occurs in the large-N limit, owing to the property

$$d_{(l,m)}\tilde{z}_{(l,m)}(\beta) = \frac{1}{n_+!}\frac{1}{n_-!}\sigma_{(l)}\sigma_{(m)}(N\beta)^{n_+ + n_-}[1 + \mathrm{O}(\beta^{2N})], \tag{4.104}$$

where $n_+ = \sum_i l_i$, $n_- = \sum_i m_i$, and $\sigma_{(l)}$ is the dimension of the representation (l) of the permutation group, which in turn can be computed explicitly as

$$\frac{1}{n_+!}\sigma_{(l_1,\dots,l_s)} = \frac{\prod_{1\le j\le k\le s}(l_j - l_k + k - j)!}{\prod_{i=1}^{s}(l_i + s - i)!}; \tag{4.105}$$

$d_{(l,m)}$ can be parameterized by

$$d_{(l,m)} = \frac{1}{n_+!}\frac{1}{n_-!}\sigma_{(l)}\sigma_{(m)}C_{(l,m)}, \tag{4.106}$$

where $C_{(l,m)}$ can be expressed as a finite product:

$$C_{(l,m)} = \prod_{i=1}^{s}\frac{(N-t-i+l_i)!}{(N-t-i)!}\prod_{j=1}^{t}\frac{(N-s-j+m_j)!}{(N-s-j)!}\prod_{i=1}^{s}\prod_{j=1}^{t}\frac{(N+1-i-j+l_i+m_j)!}{(N+1-i-j)!}, \tag{4.107}$$

allowing for a conceptually simple $1/N$ expansion. These results are complemented with the result

$$F(\beta) = \beta^2 + \mathrm{O}(\beta^{2N+2}) \tag{4.108}$$

and with the unavoidable large-N constraint $\beta \le \frac{1}{2}$.

The character expansion now proceeds as follows.

We notice that, thanks to Eq. (4.104), only a finite number of nontrivial representations contributes to any definite order in the strong-coupling series expansion in powers of β, and each lattice integration variable can appear only once for each link where a nontrivial representation in chosen. A systematic treatment leads to a classification of contributions in terms of paths (surfaces in a gauge theory) along whose non self-interacting sections a particular representation is assigned. Self-intersection points are submitted to constraints deriving from the orthogonality of representations and their composition rules.

In the case of chiral models, all relevant assignments can be generated by considering the class of the lattice random paths satisfying a non-backtracking condition [130].

Once all nontrivial configurations are classified and counted, one is left with the task of computing the corresponding group integrals. Only integrations at intersection points are nontrivial, since other integrations follow immediately from the orthogonality relationships. Unfortunately, no special computational simplifications occur in the large-N limit of group integrals.

Apparently, the character expansion is the most efficient way of computing the strong-coupling expansion of lattice models. In particular, very long strong-coupling series have been obtained in the large-N limit for the free energy, the mass gap, and the two-point Green's functions of chiral models in two and three dimensions (for the free energy, 18 orders on the square lattice, 26 orders on the honeycomb lattice, and 16 orders on the cubic lattice; for the Green's functions, 15 orders on the square lattice, 20 orders on the honeycomb lattice, and 14 orders on the cubic lattice). The analysis of these series will be discussed in Section 7.

Before leaving the present subsection, we must make a few comments concerning the relationship between SU(N) and U(N) groups. We already made the observation that when $N\to\infty$ there is essentially no difference between SU(N) and U(N) models, at least when considering operators not involving the determinant. In order to explore this relationship more carefully, we may start as usual from the expression of the single-link integral (3.1).

Representations of $Z(A^\dagger A)$ in the SU(N) case can be obtained [131] in terms of the eigenvalues x_i of $A^\dagger A$ and of θ, defined in Eq. (3.23). Introducing the Vandemonde determinant

$$\Delta(\lambda_1,\dots,\lambda_N)=\prod_{j>i}(\lambda_j-\lambda_i)=\det\|\lambda_j^{i-1}\|\,, \tag{4.109}$$

one obtains

$$Z(A^\dagger A)=\frac{1}{N!}\left(\prod_{k=1}^{N-1}\frac{k!}{2\pi}\right)\int\prod_i \mathrm{d}\phi_i\,\delta\left(\sum_i\phi_i+N\theta\right)\times\frac{|\Delta(\mathrm{e}^{\mathrm{i}\phi_1},\dots,\mathrm{e}^{\mathrm{i}\phi_N})|^2}{\Delta(2\sqrt{x_1},\dots,2\sqrt{x_N})\,\Delta(\cos\phi_1,\dots,\cos\phi_N)}\exp\left[2\sum_k\sqrt{x_k}\cos\phi_k\right], \tag{4.110}$$

or alternatively

$$Z(A^\dagger A)=\prod_{k=1}^{N-1}\frac{k!}{2\pi}\int\prod_i\mathrm{d}\phi_i\,\delta\left(\sum_i\phi_i+N\theta\right)\frac{\Delta(\sqrt{x_1}\mathrm{e}^{\mathrm{i}\phi_1},\dots,\sqrt{x_N}\mathrm{e}^{\mathrm{i}\phi_N})}{\Delta(x_1,\dots,x_N)}\exp\left[2\sum_k\sqrt{x_k}\cos\phi_k\right]. \tag{4.111}$$

The only difference between SU(N) and U(N) is due to the presence of the (periodic) delta function $\delta(\sum_i\phi_i+N\theta)$, introducing the dependence on θ corresponding to the constraint $\det U=1$. A formal solution is obtained by expanding in powers of $\mathrm{e}^{\mathrm{i}N\theta}$:

$$Z(A^\dagger A)=\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}Nm\theta}\det\|z_i^{j-1}I_{j-1-|m|}(2z_i)\|\left(\prod_{k=1}^{N-1}k!\right)\frac{1}{\Delta(z_1^2,\dots,z_N^2)}\,, \tag{4.112}$$

where $z_i=\sqrt{x_i}$. Eq. (4.112) in turn leads to the following representation of the free energy for the SU(N) single-link model:

$$F_N(\beta,\theta)=\log\sum_{m=-\infty}^{\infty}A_{m,N}(\beta)\,\mathrm{e}^{\mathrm{i}Nm\theta}\,, \tag{4.113}$$

where for convenience we have redefined the coupling: $\beta\to\beta\mathrm{e}^{\mathrm{i}\theta}$. Eq. (4.113) is useful for a large-N mean-field study [112], but it is certainly inconvenient at small N, where more specific integration techniques may be applied.

We mention that a large-N analysis of Eq. (4.113) for $\theta = 0$ leads to

$$F_N(\beta,0) = N^2\beta^2 + 2J_N(2N\beta) - 2J_{N-1}(2N\beta)\,J_{N+1}(2N\beta) - \sum_{k=1}^{\infty} kJ^2_{N+k}(2N\beta) + \mathrm{O}(\beta^{3N})\,. \quad (4.114)$$

It is also possible to establish a relationship between SU(N) and U(N) groups at the level of character coefficients. Thanks to the basic relationships

$$\chi_{\lambda_1+s,\dots,\lambda_N+s}(U) = (\det U)^s \chi_{\lambda_1,\dots,\lambda_N}(U)\,, \quad (4.115)$$

holding in U(N), one may impose the condition $\det U = 1$ in the integral representation of the character coefficients and obtain

$$z_{(r)} = \frac{\sum_{s=-\infty}^{\infty} \tilde{z}(r,s)}{\sum_{s=-\infty}^{\infty} \tilde{z}(0,s)}\,, \quad (4.116)$$

where, by definition, for U(N) groups

$$\tilde{z}(0,s) = \langle \det U^s \rangle, \qquad d_{(r)}\,\tilde{z}(r,s) = \langle \det U^s \chi_{(r)}(U) \rangle\,. \quad (4.117)$$

These relationships are the starting point for a systematic implementation of the corrections due to the SU(N) condition in the $1/N$ expansion of U(N) models [52,132]. A peculiarity of the SU(N) condition can be observed in the finite-N behavior of the eigenvalue density function $\rho(\phi, N)$, which shows a non-monotonic dependence on ϕ, characterized by the presence of N peaks. This is already apparent in the $\beta \to 0$ limit of the single-link integral, where [129]

$$\rho_{\mathrm{U}(N)}(\phi) \underset{\beta\to 0}{\to} \frac{1}{2\pi}, \qquad \rho_{\mathrm{SU}(N)}(\phi) \underset{\beta\to 0}{\to} \frac{1}{2\pi}\left(1 + (-1)^{N+1}\frac{2}{N}\cos N\phi\right). \quad (4.118)$$

5. Chiral chain models and gauge theories on polyhedra

5.1. Introduction

The use of the steepest-descent techniques allows to extend the number of the unitary-matrix models solved in the large-N limit to some few unitary-matrix systems. The interest for few-matrix models may arise for various reasons. Their large-N solutions may represent non-trivial benchmarks for new methods meant to investigate the large-N limit of more complex matrix models, such as QCD. Every matrix system may have a rôle in the context of two-dimensional quantum gravity; indeed, via the double scaling limit, its critical behavior is connected to two-dimensional models of matter coupled to gravity. Furthermore, every unitary-matrix model can be reinterpreted as the generating functional of a class of integrals over unitary groups, whose knowledge would be very useful for the strong-coupling expansion of many interesting models.

This section is dedicated to a class of finite-lattice chiral models termed chain models and defined by the partition function

$$Z_L = \int \prod_{i=1}^{L} \mathrm{d}U_i \exp\left[N\beta \sum_{i=1}^{L} \mathrm{Tr}(U_i U_{i+1}^\dagger + U_i^\dagger U_{i+1})\right], \quad (5.1)$$

where periodic boundary conditions are imposed: $U_{L+1} = U_1$.

Chiral chain models have interesting connections with gauge models. Fixing the gauge $A_0 = 0$, YM_2 on a $K \times L$ lattice (with free boundary conditions in the direction of size K) becomes equivalent to K decoupled chiral chains of length L.

Chiral chains with periodic boundary conditions enjoy another interesting equivalence with lattice gauge theories defined on the surface of polyhedra, where a link variable is assigned to each edge and a plaquette to each face. By choosing an appropriate gauge, lattice gauge theories on regular polyhedra like tetrahedron, cube, octahedron, etc., are equivalent respectively to periodic chiral chains with $L = 4, 6, 8$, etc. [70].

The thermodynamic properties of chiral chains can be derived by evaluating their partition functions. Free-energy density, internal energy, and specific heat are given respectively by

$$F_L = \frac{1}{LN^2} \log Z_L\,, \tag{5.2}$$

$$U_L = \frac{1}{2}\frac{\partial F_L}{\partial \beta}\,, \tag{5.3}$$

$$C_L = \beta^2 \frac{\partial U_L}{\partial \beta}\,. \tag{5.4}$$

When $L \to \infty$, Z_L can be reduced to the partition function of the Gross–Witten single-link model, and therefore shares the same thermodynamic properties. In particular, the free energy density at $N = \infty$ is piecewise analytic with a third-order transition at $\beta_c = \frac{1}{2}$ between the strong-coupling and weak-coupling domains. Furthermore, the behavior of C_∞ around β_c can be characterized by a specific heat critical exponent $\alpha = -1$. It is easy to see that the $L = 2$ chiral chain is also equivalent to the Gross–Witten model, but with β replaced by 2β; therefore $\beta_c = \frac{1}{4}$ and the critical properties are the same, e.g., $\alpha = -1$.

5.2. Saddle-point equation for chiral L-chains

The strategy used in Refs. [70,133] to compute the $N = \infty$ solutions for chiral chains with $L \le 4$ begins with group integrations in the partition function (Eq. (5.1)), with the help of the single-link integral, for all U_i except two. This leads to a representation for Z_L in the form

$$Z_L = \int \mathrm{d}U\,\mathrm{d}V \exp\left[N^2 S_{\mathrm{eff}}^{(L)}(UV^\dagger)\right] \tag{5.5}$$

suitable for a large-N steepest-descent analysis. Since the integral depends only on the combination $UV^\dagger$, changing variable to θ_j, $\mathrm{e}^{\mathrm{i}\theta_j}$ being the eigenvalues of $UV^\dagger$, leads to

$$Z_L \sim \int \prod_i \mathrm{d}\theta_i |\Delta(\theta_1, \ldots, \theta_N)|^2 \exp\left[N^2 S_{\mathrm{eff}}^{(L)}(\theta_k)\right]\,, \tag{5.6}$$

where $-\pi \le \theta_j \le \pi$, $\Delta(\theta_1, \ldots, \theta_N) = \det\|\Delta_{jk}\|$, $\Delta_{jk} = \mathrm{e}^{\mathrm{i}j\theta_k}$. In the large-$N$ limit, Z_L is determined by its stationary configuration, and the distribution of θ_j is specified by a density function $\rho_L(\theta)$, which is the solution of the equation

$$\fint \mathrm{d}\phi\, \rho_L(\phi) \cot\frac{\theta-\phi}{2} + \frac{\delta}{\delta\theta} S^{(L)}_{\mathrm{eff}}(\theta, \rho_L) = 0\,, \tag{5.7}$$

with the normalization condition

$$\int_{-\pi}^{\pi} \rho_L(\theta)\, \mathrm{d}\theta = 1\,. \tag{5.8}$$

For $L = 2$, Z_2 is already in the desired form with

$$S^{(2)}_{\mathrm{eff}} = 2\beta \frac{1}{N} \mathrm{Tr}(U_1 U_2^\dagger + U_1^\dagger U_2)\,, \tag{5.9}$$

and the large-N eigenvalue density $\rho_2(\theta)$ of the matrix $U_1 U_2^\dagger$ satisfies the Gross–Witten equation

$$\fint \mathrm{d}\phi\, \rho_2(\phi) \cot\frac{\theta-\phi}{2} - 4\beta \sin\theta = 0\,, \tag{5.10}$$

which differs from that of the infinite-chain model only in replacing β by 2β.

5.3. The large-N limit of the three-link chiral chain

In the $L = 3$ chain model, setting $U = U_1$ and $V = U_2$, $S^{(3)}_{\mathrm{eff}}$ is given by

$$\exp[N^2 S^{(3)}_{\mathrm{eff}}] = \exp[2N\beta\, \mathrm{Re}\,\mathrm{Tr}\, UV^\dagger] \int \mathrm{d}U_3 \exp[2N\beta\, \mathrm{Re}\,\mathrm{Tr}\, AU_3^\dagger]\,, \tag{5.11}$$

where $A = U + V$. Recognizing in the r.h.s. of (Eq. (5.11)) a single-link integral, one can deduce that the large-N limit of the spectral density $\rho_3(\theta)$ of the matrix $UV^\dagger$ satisfies the equation

$$2\beta(\sin\theta + \sin\tfrac{1}{2}\theta) - \fint \mathrm{d}\phi\, \rho_3(\phi) \left[\cot\frac{\theta-\phi}{2} + \frac{1}{2}\frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta + \cos\frac{1}{2}\phi}\right] = 0\,, \tag{5.12}$$

with the normalization condition $\int \rho_3(\theta)\, \mathrm{d}\theta = 1$. In order to find a solution for the above equation, one must distinguish between strong-coupling and weak-coupling regions.

In the weak-coupling region the solution of Eq. (5.12) is

$$\rho_3(\theta) = \frac{\beta}{\pi} \cos\frac{\theta}{4} \left[2\cos\frac{\theta}{2} + \sqrt{1 - \frac{1}{3\beta}}\right] \left[2\cos\frac{\theta}{2} - 2\sqrt{1 - \frac{1}{3\beta}}\right]^{1/2} \tag{5.13}$$

for

$$|\theta| \le \theta_{\mathrm{c}} = 2 \arccos \sqrt{1 - \frac{1}{3\beta}} \tag{5.14}$$

and $\rho_3(\theta)=0$ for $\theta_c \le |\theta| \le \pi$. This solution is valid for $\beta \ge \beta_c = \frac{1}{3}$, indicating that a critical point exists at $\beta_c = \frac{1}{3}$. Similarly one can calculate $\rho_3(\theta)$ in the strong-coupling domain $\beta \le \beta_c$ [70,133,134] finding:

$$\rho_3(\theta) = \frac{\beta}{2\pi}\left(y(\theta) + 1 - \frac{\sqrt{c}+\sqrt{4+c}}{2}\right)[(y(\theta)+\sqrt{c})(y(\theta)+\sqrt{4+c})]^{1/2}\,, \tag{5.15}$$

where

$$y(\theta) = \sqrt{4\cos^2\frac{\theta}{2} + c}\,, \tag{5.16}$$

and the parameter c is related to β by the equation

$$1 + \sqrt{c} + \tfrac{1}{2}c + (1 - \tfrac{1}{2}\sqrt{c})\sqrt{4+c} = \frac{1}{\beta}\,. \tag{5.17}$$

At $\beta = \beta_c$, $c = 0$ and therefore

$$\rho_3(\theta)_{\rm crit} = \frac{1}{3\pi}\left(2\cos\frac{\theta}{2}\right)^{3/2}\cos\frac{\theta}{4}\,, \tag{5.18}$$

in agreement with the critical limit of the weak-coupling solution (Eq. (5.13)).

Since $\rho_3(\pi) > 0$ for $\beta < \beta_c$ and $\rho_3(\pi) = 0$ for $\beta \ge \beta_c$, the critical point β_c can be also seen as the compactification point for the spectral density $\rho_3(\theta)$, similarly to what is observed in the Gross–Witten model.

5.4. *The large-N limit of the four-link chiral chain*

For $L = 4$, setting $U = U_1$ and $V = U_3$, $S^{(4)}_{\rm eff}$ is given by

$$\exp(N^2 S^{(4)}_{\rm eff}) = \int {\rm d}U_2 \exp(2N\beta\,{\rm Re\,Tr}\,AU_2^\dagger)\int {\rm d}U_4 \exp(2N\beta\,{\rm Re\,Tr}\,AU_4^\dagger)\,, \tag{5.19}$$

where again $A = U + V$. The large-N limit of the spectral density $\rho_4(\theta)$ of the matrix $UV^\dagger$ must be solution of the equation

$$4\beta\sin\tfrac{1}{2}\theta - \fint {\rm d}\phi\,\rho_4(\phi)\left[\cot\frac{\theta-\phi}{2} + \frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta + \cos\frac{1}{2}\phi}\right] = 0\,, \tag{5.20}$$

satisfying the normalization condition $\int\rho_4(\theta)\,{\rm d}\theta = 1$.

In order to solve Eq. (5.20) one must again separate weak- and strong-coupling domains. In the weak-coupling region the solution is

$$\begin{aligned} \rho_4(\theta) &= \frac{2\beta}{\pi}\sqrt{\sin^2\frac{\theta_c}{2} - \sin^2\frac{\theta}{2}} \qquad &&\text{for } 0 \le \theta \le \theta_c \le \pi\,, \\ \rho_4(\theta) &= 0 &&\text{for } \theta_c \le \theta \le \pi\,, \end{aligned} \tag{5.21}$$

with θ_c implicitly determined by the normalization condition $\int_{-\theta_c}^{\theta_c} \rho_4(\theta)\, d\theta = 1$. The solution (Eq. (5.21)) is valid for $\beta \geq \beta_c = \frac{1}{8}\pi$, since the normalization condition can be satisfied only in this region. $\frac{1}{8}\pi$ is then a point of non-analyticity representing the critical point for the transition from the weak to the strong-coupling domain.

In the strong-coupling domain $\beta < \beta_c = \frac{1}{8}\pi$ one finds

$$\rho_4(\theta) = \frac{\beta}{2}\sqrt{\lambda - \sin^2\frac{\theta}{2}}\,, \tag{5.22}$$

where λ is determined by the normalization condition $\int_{-\pi}^{\pi} \rho_4(\theta)\, d\theta = 1$. The strong- and weak-coupling expressions of $\rho_4(\theta)$ coincide at β_c:

$$\rho_4(\theta)_{\text{crit}} = \frac{\beta}{2}\sqrt{1 - \sin^2\frac{\theta}{2}}\,. \tag{5.23}$$

Notice that again the critical point $\beta_c = \frac{1}{8}\pi$ represents the compactification point of the spectral density $\rho_4(\theta)$; indeed $\rho_4(\pi) > 0$ for $\beta < \beta_c$, and $\rho_4(\pi) = 0$ for $\beta \geq \beta_c$.

5.5. *Critical properties of chiral chain models with $L \leq 4$*

In the following we derive the $N = \infty$ critical behavior of the specific heat in the models with $L = 3, 4$, using the exact results of Sections 5.3 and 5.4.

From the spectral density $\rho_3(\theta)$, the internal energy can be easily derived by $U_3 = \int d\theta\, \rho_3(\theta) \cos\theta$. One finds that U_3 is continuous at β_c. In the weak-coupling region $\beta \geq \beta_c = \frac{1}{3}$,

$$U_3 = \beta + \frac{1}{2} - \frac{1}{8\beta} - \beta\left(1 - \frac{1}{3\beta}\right)^{3/2},$$

$$C_3 = \beta^2 + \frac{1}{8} - \beta^2\left(1 + \frac{1}{6\beta}\right)\sqrt{1 - \frac{1}{3\beta}}\,. \tag{5.24}$$

Close to criticality, i.e., for $0 \leq \beta/\beta_c - 1 \ll 1$,

$$C_3 = \frac{17}{72} - \frac{1}{2\sqrt{3}}(\beta - \beta_c)^{1/2} + O(\beta - \beta_c)\,. \tag{5.25}$$

In the strong-coupling region, one finds

$$C_3 = \frac{17}{72} - \frac{1}{2\sqrt{3}}(\beta_c - \beta)^{1/2} + O(\beta_c - \beta) \tag{5.26}$$

for $0 \leq 1 - \beta/\beta_c \ll 1$. Then the weak- and strong-coupling expressions of C_3 show that the critical point $\beta_c = \frac{1}{3}$ is of the third order, and the critical exponent associated with the specific heat is $\alpha = -\frac{1}{2}$.

In the $L=4$ case, recalling that $\rho_4(\theta)$ is the spectral distribution of $U_1U_3^\dagger$, one writes

$$F_4 = \frac{1}{4}\left[8\beta\int d\theta\,\rho_4(\theta)\cos\frac{\theta}{2} - \int d\theta\,d\phi\,\rho_4(\theta)\rho_4(\phi)\log\left(\cos\frac{\theta}{2}+\cos\frac{\phi}{2}\right)\right.$$
$$\left. -\frac{3}{2} - \log 2\beta + \int\!\!\!\!-\, d\theta\,d\phi\,\rho_4(\theta)\rho_4(\phi)\log\sin^2\frac{\theta-\phi}{2}\right]. \tag{5.27}$$

Observing that, since $\rho_4(\theta)$ is a solution of the variational equation $\delta F_4/\delta\rho_4 = 0$, the following relation holds

$$\frac{dF_4}{d\beta} = \frac{\partial F_4}{\partial \beta}, \tag{5.28}$$

one can easily find that

$$U_4 = -\frac{1}{8\beta} + \int d\theta\,\rho_4(\theta)\cos\frac{\theta}{2}. \tag{5.29}$$

In this case, the study of the critical behavior around $\beta_c = \frac{1}{8}\pi$ is slightly subtler, since it requires the expansion of elliptic integrals $F(k)$ and $E(k)$ around $k=1$. Approaching criticality from the weak-coupling region, i.e., when $\beta\to\beta_c^+$, one obtains

$$C_4 = \frac{\pi^2}{32} + \frac{1}{8} - \frac{\pi^2}{16\log(4/\delta_w)} + O(\delta_w^2), \tag{5.30}$$

where $\delta_w^2 \sim \beta - \beta_c$, apart from logarithms. For $\beta\to\beta_c^-$

$$C_4 = \frac{\pi^2}{32} + \frac{1}{8} - \frac{\pi^2}{16\log(4/\delta_s)} + O(\delta_s^2), \tag{5.31}$$

where $\delta_s^2 \sim \beta_c - \beta$, apart from logarithms. A comparison of Eqs. (5.30) and (5.31) leads to the conclusion that the phase transition is again of the third order, with a specific heat critical exponent $\alpha = 0^-$.

In conclusion we have seen that chain models with $L = 2, 3, 4, \infty$ have a third-order phase transition at increasing values of the critical coupling, $\beta_c = \frac{1}{4}, \frac{1}{3}, \frac{1}{8}\pi, \frac{1}{2}$ respectively, with specific heat critical exponents $\alpha = -1, -\frac{1}{2}, 0^-, -1$ respectively. It is worth noticing that α increases when L goes from 2 to 4, reaching the limit of a third-order critical behavior, but in the large-L limit it returns to $\alpha = -1$.

The critical exponent ν, describing the double-scaling behavior for $N\to\infty$ and $\beta\to\beta_c$, can then be determined by the two-dimensional hyperscaling relationship $2\nu = 2-\alpha$. This relation has been proved to hold for the Gross–Witten problem, and therefore for the $L=2$ and $L=\infty$ chain models, where it is related to the equivalence of the corresponding double scaling limit with the continuum limit of a two-dimensional gravity model with central charge $c=-2$. It is then expected to hold in general for all values of L. At $L=4$, the value $\nu=1$ has been numerically verified, within a few per cent of uncertainty, by studying the scaling of the specific heat peak position at finite N. Notice that the exponents $\alpha = 0^-$, $\nu = 1$ found for $L=4$ correspond to a central charge $c=1$.

5.6. Strong-coupling expansion of chiral chain models

Strong-coupling series of the free energy density of chiral chain models can be generated by means of the character expansion, which leads to the result

$$F_L(\beta) = F(\beta) + \tilde{F}_L(\beta)\,, \tag{5.32}$$

where $F(\beta)$ is the free energy of the single unitary-matrix model,

$$\tilde{F}_L = \frac{1}{LN^2}\log\sum_{(r)} d^2_{(r)} z^L_{(r)}\,, \tag{5.33}$$

$\sum_{(r)}$ denotes the sum over all irreducible representations of U(N), and $d_{(r)}$ and $z_{(r)}(\beta)$ are the corresponding dimensions and character coefficients. The calculation of the strong-coupling series of $F_L(\beta)$ is considerably simplified in the large-N limit, due to the relationships (Eq. (4.108)) and

$$z_{(r)}(\beta) = \bar{z}_{(r)}\beta^n + \mathrm{O}(\beta^{2N})\,, \tag{5.34}$$

where $\bar{z}_{(r)}$ is independent of β and n is the order of the representation (r). Explicit expressions for $d_{(r)}$ and $\bar{z}_{(r)}$ were reported in Section 4.8. The large-N strong-coupling expansion of $\tilde{F}_L(\beta)$ is actually a series in β^L, i.e.,

$$\tilde{F}_L = \sum_n c(n,L)\beta^{nL}\,. \tag{5.35}$$

It is important to recall that the large-N character coefficients have jumps and singularities at $\beta = \frac{1}{2}$ [52], and therefore the relevant region for a strong-coupling character expansion is $\beta < \frac{1}{2}$.

Another interesting aspect of the large-N limit of chain models, studied by Green and Samuel using the strong-coupling character expansion [75], concerns the determinant channel, which should provide an order parameter for the phase transition. The quantity

$$\sigma = -\frac{1}{N}\log\langle \det U_i U^\dagger_{i+1}\rangle \tag{5.36}$$

is non-zero in the strong-coupling domain and zero in weak coupling at $N = \infty$. β_c may then be evaluated by determining where the strong-coupling evaluation of the order parameter σ vanishes. Like the free-energy, σ is calculable via a character expansion. Indeed

$$\langle \det U_i U^\dagger_{i+1}\rangle = \frac{\sum_{(r)} d_{(r)} z^{L-1}_{(r)} d_{(r,-1)} z_{(r,-1)}}{\sum_{(r)} d^2_{(r)} z^L_{(r)}}\,. \tag{5.37}$$

Green and Samuel evaluated a few orders of the above character expansion, obtaining estimates of β_c from the vanishing point of σ. Such estimates compare well with the exact results for $L = 3, 4$. In the cases where β_c is unknown, they found $\beta_c \simeq 0.44$ for $L = 5$, $\beta_c \simeq 0.47$ for $L = 6$, etc., with β_c monotonically approaching the value $\frac{1}{2}$ with increasing L.

In order to study the critical behavior of chain models for $L \geq 5$, one can also analyze the corresponding strong-coupling series of the free energy Eq. (5.32) [135]. An integral approximant analysis of the strong-coupling series of the specific heat led to the estimates $\beta_c \simeq 0.438$ for $L = 5$ and $\beta_c \simeq 0.474$ for $L = 6$, with small negative α, which could mimic an exponent $\alpha = 0^-$. For $L \geq 7$

a such strong-coupling analysis would lead to β_c larger than $\frac{1}{2}$, that is out of the region where a strong-coupling analysis can be predictive. Therefore something else must occur earlier, breaking the validity of the strong-coupling expansion. An example of this phenomenon is found in the Gross–Witten single-link model (recovered when $L \to \infty$), where the strong-coupling expansion of the $N = \infty$ free energy is just $F(\beta) = \beta^2$, an analytical function without any singularity; therefore, in this model, $\beta_c = \frac{1}{2}$ cannot be determined from a strong-coupling analysis of the free energy.

From such analysis one may hint at the following possible scenario: as for $L \le 4$, for $L = 5, 6$, that is when the estimate of β_c coming from the above strong-coupling analysis is smaller than $\frac{1}{2}$ and therefore acceptable. The term $\tilde{F}(\beta)$ in Eq. (5.32) should be the one relevant for the critical properties, determining the critical points and giving $\alpha \neq -1$ (maybe $\alpha = 0^-$ as in the $L = 4$ case). For $L \ge 7$ the critical point need not be a singular point of the free energy in strong or weak coupling, but just the point where weak-coupling and strong-coupling curves meet each other. This would cause a softer phase transition with $\alpha = -1$, as for the Gross–Witten single-link problem. We expect $\beta_c < \frac{1}{2}$ also for $L \ge 7$. This scenario is consistent with the results of the analysis of the character expansion of σ, defined in Eq. (5.36).

6. Simplicial chiral models

6.1. Definition of the models

Another interesting class of finite-lattice chiral models is obtained by considering the possibility that each of a finite number of unitary matrices may interact in a fully symmetric way with all other matrices, while preserving global chiral invariance; the resulting systems can be described as chiral models on $(d-1)$-dimensional simplexes, and thus termed "simplicial chiral models" [135,136].

The partition function for such a system is

$$Z_d = \int \prod_{i=1}^{d} \mathrm{d}U_i \exp\left[N\beta \sum_{i=1}^{d} \sum_{j=i+1}^{d} \mathrm{Tr}(U_i U_j^\dagger + U_j U_i^\dagger) \right]. \tag{6.1}$$

Eq. (6.1) encompasses as special cases a number of models that we have already introduced and solved; in particular, the chiral chains with $L \le 3$ correspond to the simplicial chiral models with $d \le 3$.

One of the most attractive features of these models is their relationship with higher-dimensional systems, with which they share the possibility of high coordination numbers. This relationship becomes exact in the large-d limit, where mean-field results are exact.

In the large-N limit and for arbitrary d a saddle-point equation can be derived, whose solution allows the evaluation of the large-N free energy

$$F_d = (1/N^2) \log Z_d \tag{6.2}$$

and of related thermodynamical quantities.

6.2. Saddle-point equation for simplicial chiral models

The strategy for the determination of the large-N saddle-point equation is based on the introduction of a single auxiliary variable A (a complex matrix), leading to the decoupling of the unitary matrix interaction:

$$Z_d = \tilde{Z}_d/\tilde{Z}_0 \,, \tag{6.3}$$

where

$$\tilde{Z}_d = \int \prod_{i=1}^{d} \mathrm{d}U_i \, \mathrm{d}A \exp\left[-N\beta \operatorname{Tr} AA^\dagger + N\beta \operatorname{Tr} A \sum_i U_i^\dagger + N\beta \operatorname{Tr} A^\dagger \sum_i U_i - N^2\beta d \right]. \tag{6.4}$$

We are now back to the single-link problem and, since we have solved it in Section 3 in terms of the function W, whose large-N limit is expressed by Eq. (3.15), we obtain

$$\tilde{Z}_d = \int \mathrm{d}A \exp[-N\beta \operatorname{Tr} AA^\dagger + NdW(\beta^2 AA^\dagger) - N^2\beta d] \,. \tag{6.5}$$

It is now convenient to express the result in terms of the eigenvalues x_i of the Hermitian semipositive-definite matrix $4\beta AA^\dagger$, obtaining

$$\tilde{Z}_d = \int \mathrm{d}\mu(x_i) \exp\left[-\frac{N}{4\beta} \sum_i x_i + N\, dW\left(\frac{x_i}{4}\right) - N^2\beta d \right]. \tag{6.6}$$

The angular integration can be performed, leading to

$$\mathrm{d}\mu(x_i) = \prod_i \mathrm{d}x_i \prod_{i>j} (x_i - x_j)^2 \,. \tag{6.7}$$

The saddle-point equation is therefore

$$\frac{\sqrt{r + x_i}}{2\beta} - d = \frac{1}{N} \sum_{i \neq j} \frac{(4-d)\sqrt{r + x_i} + d\sqrt{r + x_j}}{x_i - x_j} \,, \tag{6.8}$$

subject to the constraint (needed to define r)

$$\begin{cases} \dfrac{1}{N} \sum_i \dfrac{1}{\sqrt{r + x_i}} = 1 & \text{(strong coupling)}\,; \\ r = 0 & \text{(weak coupling)}\,. \end{cases} \tag{6.9}$$

The energy

$$U_d = \tfrac{1}{2} \partial F_d / \partial\beta \tag{6.10}$$

is easily expressed in terms of the eigenvalues:

$$d(d-1)U_d = \frac{1}{4\beta^2} \sum_i x_i - d - \frac{1}{\beta} \,. \tag{6.11}$$

In the large-N limit, after a change of variables to $z_i = \sqrt{r + x_i}$, we introduce as usual an eigenvalue density function $\rho(z)$, and turn Eq. (6.8) into the integral equation

$$\frac{z}{2\beta} - d = \int_a^b \mathrm{d}z' \, \rho(z') \left[\frac{2}{z - z'} - \frac{d-2}{z + z'} \right], \tag{6.12}$$

subject to the constraints

$$\int_a^b \rho(z')\,\mathrm{d}z' = 1\,, \tag{6.13}$$

$$\int_a^b \rho(z')\,\frac{\mathrm{d}z'}{z'} \le 1\,, \tag{6.14}$$

with equality holding in strong coupling, where $a = \sqrt{r}$. The easiest way of evaluating the free energy F_{d} is the integration of the large-N version of Eq. (6.11) with respect to β.

Very simple solutions are obtained for a few special values of d. When $d = 0$, the problem reduces to a Gaussian integration, and one easily finds that Eq. (6.12) is solved by

$$\rho(z) = \frac{z}{4\pi\beta}\frac{\sqrt{16\beta - (z^2 - a^2)}}{\sqrt{z^2 - a^2}} \tag{6.15}$$

and $\tilde{Z}_0 = \exp(N^2 \log\beta)$, independent of a as expected.

When $d = 2$ we obtain

$$\rho_{\mathrm{w}}(z) = \frac{1}{4\pi\beta}\sqrt{8\beta - (z - 4\beta)^2}, \quad \beta \ge \tfrac{1}{2}\,, \tag{6.16}$$

$$\rho_{\mathrm{s}}(z) = \frac{1}{4\pi\beta}z\sqrt{\frac{1 + 6\beta - z}{z - (1 - 2\beta)}}, \qquad r(\beta) = (1 - 2\beta)^2, \quad \beta \le \tfrac{1}{2}\,, \tag{6.17}$$

and these results are consistent with the reinterpretation of the model as a Gross–Witten one-plaquette system. Notice however that the matrix whose eigenvalue distribution has been evaluated is not the original unitary matrix, and corresponds to a different choice of physical degrees of freedom. This is the reason why, while knowing the solution for the free energy of the $d = 1$ system (trivial, non-interacting) and of the $d = 3$ system (three-link chiral chain), we cannot find easily explicit analytic forms for the corresponding eigenvalue densities.

The saddle-point equation (Eq. (6.12)) has been the subject of much study in recent times, because it is related to many different physical problems in the context of double-scaling limit investigations. In particular, in the range of values $0 \le d \le 4$, the same equation describes the behavior of O(n) spin models on random surfaces in the range $-2 \le n \le 2$, with the very simple mapping $n = d - 2$ [137]. In this range, the equation has been solved analytically in Refs. [138] and especially [139] in terms of θ-functions.

6.3. The large-N $d = 4$ simplicial chiral model

The chiral model on a tetrahedron is the first example within the family of simplicial chiral models which turns out to be really different from all the systems discussed in the previous sections. Explicit solutions were found for both the weak and the strong coupling phases, and they are best expressed in terms of a rescaled variable

$$\zeta = \sqrt{1 - z^2/b^2} \tag{6.18}$$

and of a dynamically determined parameter

$$k = \sqrt{1 - a^2/b^2}\,. \tag{6.19}$$

The resulting expressions, after defining $\beta\bar{\rho}(\zeta)\,\mathrm{d}\zeta \equiv \rho(z)\,\mathrm{d}z$, are

$$\bar{\rho}_{\mathrm{w}}(\zeta) = \frac{8}{E(k)^2}\left[\frac{\sqrt{k^2-\zeta^2}}{\sqrt{1-\zeta^2}}K(k) - \sqrt{k^2-\zeta^2}\sqrt{1-\zeta^2}\,\Pi(\zeta^2,k)\right], \tag{6.20}$$

$$\bar{\rho}_{\mathrm{s}}(\zeta) = \frac{8}{[E(k)-(1-k^2)K(k)]^2}\left[k^2\frac{\sqrt{1-\zeta^2}}{\sqrt{k^2-\zeta^2}}K(k) - \sqrt{k^2-\zeta^2}\sqrt{1-\zeta^2}\,\Pi(\zeta^2,k)\right], \tag{6.21}$$

where K, E and Π are the standard elliptic integrals, and $0 \le \zeta \le k$.

The complete solution is obtained by enforcing the normalization condition, which leads to a relationship between β and k, best expressed by the equation

$$\frac{1}{\beta} = \int_0^k \mathrm{d}\zeta\,\bar{\rho}(\zeta,k)\,. \tag{6.22}$$

Criticality corresponds to the limit $k \to 1$, and it is easy to recognize that both weak and strong coupling results lead in this limit to $\beta_{\mathrm{c}} = \frac{1}{4}$ and

$$\beta\bar{\rho}_{\mathrm{c}}(\zeta) = \zeta\log[(1+\zeta)/(1-\zeta)]\,. \tag{6.23}$$

Many interesting features of this model in the region around criticality can be studied analytically, and one may recognize that the critical behavior around $\beta_{\mathrm{c}} = \frac{1}{4}$ corresponds to a limiting case of a third-order phase transition with critical exponent of the specific heat $\alpha = 0^-$. In the double-scaling limit language this would correspond to a model with central charge $c = 1$ and logarithmic deviations from scaling. The critical behavior of the specific heat on both sides of criticality is described by

$$C \equiv \beta^2\frac{\partial U}{\partial\beta} \underset{k'\to 0}{\to} \frac{\pi^2+3}{36} - \frac{\pi^2}{12\log(4/k')} + O\left(\frac{1}{\log^2 k'}\right), \tag{6.24}$$

where $k' \equiv \sqrt{1-k^2}$.

6.4. The large-d limit

By introducing a function defined by

$$f(z) = \int_a^b \frac{\rho(z')}{z-z'}\,\mathrm{d}z', \qquad f(z) \underset{|z|\to\infty}{\to} \frac{1}{z}, \tag{6.25}$$

analytic in the complex z plane with the exception of a cut on the positive real axis in the interval $[a,b]$, we can turn the saddle-point equation (6.12) into the functional equation

$$\frac{z}{2\beta} - d = 2\,\mathrm{Re}f(z) + (d-2)f(-z)\,. \tag{6.26}$$

This equation can be the starting point of a systematic $1/d$ expansion, on whose details we shall not belabor, especially because its convergence for small values of d is very slow. It is however interesting to solve the large-d limit of Eq. (6.26) by the Ansatz

$$\rho(z) = \delta(z - \bar{z}) , \tag{6.27}$$

whose substitution into Eq. (6.25) leads to the solution

$$\bar{z} = \begin{cases} \beta d(1 + \sqrt{1 - 1/\beta d}), & \beta d \geq 1 , \\ 1, & \beta d \leq 1 . \end{cases} \tag{6.28}$$

The large-d limit predicts the location of the critical point $\beta_c = 1/d$, and shows complete equivalence with the mean-field solution of infinite-volume principal chiral models on a $d/2$-dimensional hypercubic lattice. The large-d prediction for the nature of criticality is that of a first-order phase transition, with

$$U = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{1}{\beta d}} - \frac{1}{4\beta d}, \quad \beta d \geq 1 . \tag{6.29}$$

6.5. The large-N criticality of simplicial models

The connection with the double-scaling limit problem naturally leads to the study of the finite-β critical behavior. In the regime $0 \leq d \leq 4$ one is helped by the equivalence with the solved problem of O(n) spin models on a random surface, which allows not only a determination of the critical value (found to satisfy the relationship $\beta_c d = 1$), but also an evaluation of the eigenvalue distribution at criticality [137]:

$$\rho_c(z) = \frac{2}{\pi\theta}\cos\frac{\pi\theta}{2}\frac{\sinh\theta u}{\cosh u} , \tag{6.30}$$

$$a_c = 0, \qquad b_c = \frac{2}{\theta}\tan\frac{\pi\theta}{2} , \tag{6.31}$$

where θ and u are defined by the parametrizations

$$4\cos^2\pi\theta/2 \equiv d = 1/\beta_c, \quad \cosh u \equiv b_c/z . \tag{6.32}$$

Unfortunately, the technique that was adopted in order to find the above solution does not apply to the regime $d > 4$, in which case one cannot choose $a_c = 0$. The saddle-point equation at criticality can however be solved numerically with very high accuracy, and one finds that the relationship

$$\beta_c d = 1 \tag{6.33}$$

is satisfied for all d, thus also matching the large-d predictions. The combinations $(a_c + b_c)/2$ and $a_c b_c$ admit a $1/d$ expansion, and the coefficients of the expansion are found numerically to be integer numbers up to order d^{-8}.

An analysis of criticality for $d > 4$ shows that its description is fully consistent with the existence of a first-order phase transition, with a discontinuity of the internal energy measured by $da_c^2/(4(d-1))$, again matching with the large-d (mean-field) predictions.

6.6. The strong-coupling expansion of simplicial models

There is nothing peculiar in performing the strong-coupling expansion of Eq. (6.1). There is however a substantial difference with respect to the case of chiral chains discussed in the previous section: because of the topology of simplexes, the strong-coupling configurations entering the calculation are no longer restricted to simple graphs whose vertices are joined by at most one link, and the full complexity of group integration on arbitrary graphs is now involved [130].

As a consequence, as far as the simplicial models can be solved by different techniques, they may also be used as generating functionals for these more involved group integrals, that enter in an essential way in all strong-coupling calculations in higher-dimensional standard chiral models and lattice gauge theories.

7. Asymptotically free matrix models

7.1. Two-dimensional principal chiral models

Two dimensional $\mathrm{SU}(N)\times\mathrm{SU}(N)$ principal chiral models, defined by the action

$$S=\frac{N}{T}\int d^2x\,\mathrm{Tr}\,\partial_\mu U(x)\partial_\mu U^\dagger(x)\,, \tag{7.1}$$

are the simplest asymptotically free field theories whose large-N limit is a sum over planar diagrams, like four dimensional $\mathrm{SU}(N)$ gauge theories. We assume, as usual, that the limit $N\to\infty$ is taken while keeping fixed the (rescaled) coupling $\beta=1/T$, and that the mass gap does not vanish even at $N=\infty$.

Using the existence of an infinite number of conservation laws and Bethe-Ansatz methods, the on-shell solution of the $\mathrm{SU}(N)\times\mathrm{SU}(N)$ chiral models has been proposed in terms of a factorized S-matrix [140,141]. The analysis of the corresponding bound states leads to the mass spectrum

$$M_r=M\frac{\sin(r\pi/N)}{\sin(\pi/N)},\quad 1\le r\le N-1\,, \tag{7.2}$$

where M_r is the mass of the r-particle bound state transforming as totally antisymmetric tensors of rank r. $M\equiv M_1$ is the mass of the fundamental state determining the Euclidean long-distance exponential behavior of the two-point Green's function

$$G(x)=\frac{1}{N}\langle\mathrm{Tr}\,U(0)U(x)^\dagger\rangle\,. \tag{7.3}$$

The mass-spectrum (7.2) has been verified numerically at $N=6$ by Monte Carlo simulations [142,143]: Monte Carlo data of the mass ratios M_2/M and M_3/M agree with formula (7.2) within statistical errors of about one per cent. Concerning the large-N limit of these models, it is important to notice that the S-matrix has a convergent expansion in powers of $1/N$, and becomes trivial, i.e., the S-matrix of free particles, in the large-N limit.

By using Bethe-Ansatz techniques, the mass/Λ-parameter ratio has also been computed, and the result is [144]

$$\frac{M}{\Lambda_{\overline{MS}}} = \sqrt{\frac{8\pi}{e}}\,\frac{\sin(\pi/N)}{\pi/N}\,, \tag{7.4}$$

which again enjoys a $1/N$ expansion with a finite radius of convergence. This exact but non-rigorous result has been substantially confirmed by Monte Carlo simulations at several values of N [132,145], and its large-N limit also by $N = \infty$ strong-coupling calculations [146,147].

While the on-shell physics of principal chiral models has been substantially solved, exact results of the off-shell physics are still missing, even in the large-N limit. When $N \to \infty$, principal chiral models should just reproduce a free-field theory in disguise. In other words, a local nonlinear mapping should exist between the Lagrangian fields U and some Gaussian variables [7]. However, the behavior of the two-point Green's function $G(x)$ of the Lagrangian field shows that such realization of a free-field theory is nontrivial. While at small Euclidean momenta, and therefore at large distance, there is a substantial numerical evidence for an essentially Gaussian behavior of $G(x)$ [132], at short distance renormalization group considerations lead to the asymptotic behavior

$$G(x) \sim [\log(1/x\Lambda)]^{\gamma_1/b_0}\,, \tag{7.5}$$

where Λ is a mass scale, and

$$\gamma_1/b_0 = 2(1 - 2/N^2) \underset{N\to\infty}{\to} 2\,. \tag{7.6}$$

b_0 and γ_1 are the first coefficients respectively of the β-function and of the anomalous dimension of the fundamental field. We recall that a free Gaussian Green's function behaves like $\log(1/x)$. Then at small distance $G(x)$ seems to describe the propagation of a composite object formed by two elementary Gaussian excitations, suggesting an interesting hadronization picture: in the large-N limit, the Lagrangian fields U, playing the rôle of non-interacting hadrons, are constituted by two confined particles, which appear free in the large momentum limit, due to asymptotic freedom.

We must mention that, according to Ref. [148], it is possible to take the large-N limit of principal chiral models in such a way that all the infinite states of the physical mass spectrum lay in a finite range of values. Since these states are equally spaced, the resulting spectrum does not show a mass gap. As a consequence, while exactly solvable, the corresponding theory is not the large-N version of the models whose exact S-matrix has been discussed above. The limit discussed in Ref. [148] cannot be reached by considering lattice versions of the models for larger and larger values of N, and exploring their critical region at fixed values of the conventionally rescaled coupling.

Numerical investigations by Monte Carlo simulations of lattice chiral models in the continuum limit show that the conventional large-N limit is rapidly approached, which confirms that the $1/N$ expansion, were it available, would be an effective predictive tool in the analysis of these models.

7.2. Principal chiral models on the lattice

In the persistent absence of an explicit solution, the large-N limit of two-dimensional chiral models has been investigated by applying analytical and numerical methods of lattice field theory, such as strong-coupling expansion and Monte Carlo simulations. In the following we describe the main results achieved by these studies.

A standard lattice version of the continuum action (7.1) is obtained by introducing a nearest-neighbor interaction, according to Eq. (2.10):

$$S_L = -2N\beta \sum_{x,\mu} \mathrm{Re}\,\mathrm{Tr}[U_x U^\dagger_{x+\mu}], \quad \beta = \frac{1}{T}. \tag{7.7}$$

SU(N) and U(N) lattice chiral models, obtained by constraining, respectively, $U_x \in \mathrm{SU}(N)$ and $U_x \in \mathrm{U}(N)$, are expected to have the same large-N limit at fixed β. In the continuum limit $\beta \to \infty$, SU(N) and U(N) lattice actions should describe the same theory even at finite N, since the additional U(1) degrees of freedom of U(N) models should decouple. In other words, the U(N) lattice theory represents a regularization of the SU(N)×SU(N) chiral field theory when restricting ourselves to its SU(N) degrees of freedom, i.e. when considering Green's functions of the field

$$\hat{U}_x = \frac{U_x}{(\det U_x)^{1/N}}, \tag{7.8}$$

e.g.,

$$G(x) \equiv (1/N)\langle \mathrm{Tr}\,\hat{U}_0 \hat{U}^\dagger_x \rangle, \tag{7.9}$$

whose large-distance behavior allows to define the fundamental mass M.

At finite N, while SU(N) lattice models should not have any singularity at finite β, U(N) lattice models should undergo a phase transition, driven by the U(1) degrees of freedom corresponding to the determinant of $U(x)$. The determinant two-point function

$$G_{\mathrm{d}}(x) \equiv \langle \det[U^\dagger(x)U(0)] \rangle^{1/N} \tag{7.10}$$

behaves like $x^{-f(\beta,N)}$ at large x in the weak-coupling region, with $f(\beta,N) \sim \mathrm{O}(1/N)$, but drops off exponentially in strong-coupling region, where $G_{\mathrm{d}}(x) \sim e^{-m_{\mathrm{d}}x}$ with [75]

$$m_{\mathrm{d}} = -\log\beta + \frac{1}{N}\log\frac{N!}{N^N} + \mathrm{O}(\beta^2). \tag{7.11}$$

This would indicate the existence of a phase transition at a finite β_{d} in U(N) lattice models. Such a transition, being driven by U(1) degrees of freedom, should be of the Kosterlitz–Thouless type: the mass propagating in the determinant channel m_{d} should vanish at the critical point β_{d} and stay zero for larger β. Hence for $\beta > \beta_{\mathrm{d}}$ this U(1) sector of the theory would decouple from the SU(N) degrees of freedom, which alone determine the continuum limit ($\beta \to \infty$) of principal chiral models.

The large-N limit of principal chiral models has been investigated by Monte Carlo simulations of SU(N) and U(N) models for several large values of N, studying their approach to the $N = \infty$ limit [132,129].

Many large-N strong-coupling calculations have been performed which allow a direct study of the $N = \infty$ limit. Within the nearest-neighbor formulation (7.7), the large-N strong-coupling expansion of the free energy has been calculated up to 18th order, and that of the fundamental Green's function $G(x)$ (defined in Eq. (7.3)) up to 15th order [75,130]. Large-N strong-coupling calculations have been performed also on the honeycomb lattice, within the corresponding nearest-neighbor formulation, which is expected to belong to the same class of universality with respect to the critical point $\beta = \infty$. On the honeycomb lattice the free energy has been computed up to $O(\beta^{26})$, and $G(x)$ up to $O(\beta^{20})$ [130].

Let us define the internal energy density

$$E \equiv 1 - U = 1 - \frac{1}{4}\frac{dF}{d\beta}, \qquad F = \frac{1}{N^2 V}\log Z\,, \tag{7.12}$$

and the specific heat

$$C = \frac{dE}{dT} = \frac{1}{4}\beta^2\frac{\partial^2 F}{\partial\beta^2}\,. \tag{7.13}$$

Monte Carlo simulations show that SU(N) and U(N) lattice chiral models have a peak in the specific heat which becomes sharper and sharper with increasing N, suggesting the presence of a critical phenomenon for $N = \infty$ at a finite β_c. In U(N) models the peak of C is observed in the region where the determinant degrees of freedom are massive, i.e., for $\beta < \beta_d$ (this feature characterizes also two-dimensional XY lattice models [149]). An estimate of the critical coupling β_c has been obtained by extrapolating the position $\beta_{\rm peak}(N)$ of the peak of the specific heat (at infinite volume) to $N \to \infty$ using a finite-N scaling Ansatz [129]

$$\beta_{\rm peak}(N) \simeq \beta_c + cN^{-\varepsilon}\,, \tag{7.14}$$

mimicking a finite-size scaling relationship. The above Ansatz arises from the idea that the parameter N may play a rôle quite analogous to the volume in the ordinary systems close to the criticality. This idea was already exploited in the study of one-matrix models [21,48,49], where the double scaling limit turns out to be very similar to finite-size scaling in a two-dimensional critical phenomenon. The finite-N scaling ansatz (7.14) has been verified in the similar context of the large-N Gross–Witten phase transition, as mentioned in Section 4.7. Since ε is supposed to be a critical exponent associated with the $N = \infty$ phase transition, it should be the same in the U(N) and SU(N) models.

The available U(N) and SU(N) Monte Carlo data (at $N = 9, 15, 21$ for U(N) and $N = 9, 15, 21, 30$ for SU(N)) fit very well the ansatz (7.14), and their extrapolation leads to the estimates $\beta_c = 0.3057(3)$ and $\varepsilon = 1.5(1)$. The interpretation of the exponent ε in this context is still an open problem. It is worth noticing that the value of the correlation length describing the propagation in the fundamental channel is finite at the phase transition: $\xi^{(c)} \simeq 2.8$.

The existence of this large-N phase transition is confirmed by an analysis of the $N = \infty$ 18th-order strong-coupling series of the free energy

$$F = 2\beta^2 + 2\beta^4 + 4\beta^6 + 19\beta^8 + 96\beta^{10} + 604\beta^{12} + 4036\beta^{14} + \frac{58471}{2}\beta^{16} + \frac{663184}{3}\beta^{18} + O(\beta^{20})\,, \tag{7.15}$$

which shows a second-order critical behavior:

$$C \sim |\beta - \beta_c|^{-\alpha}\,, \tag{7.16}$$

with $\beta_c = 0.3060(4)$ and $\alpha = 0.27(3)$, in agreement with the extrapolation of Monte Carlo data. The above estimates of β_c and α are slightly different from those given in Ref. [147]; they are obtained by a more refined analysis based on integral approximant techniques [150–152] and by the so-called critical point renormalization method [153].

Green and Samuel argued that the large-N phase transition of principal chiral models on the lattice is nothing but the large-N limit of the determinant phase transition present in U(N) lattice models [52,111]. According to this conjecture, β_d and β_{peak} should both converge to β_c in the large-N limit, and the order of the determinant phase transition would change from the infinite order of the Kosterlitz–Thouless mechanism to a second order with divergent specific heat. The available Monte Carlo data of U(N) lattice models at large N provide only a partial confirmation of this scenario; one can just get a hint that $\beta_d(N)$ is also approaching β_c with increasing N. The large-N phase transition of the SU(N) models could then be explained by the fact that the large-N limit of the SU(N) theory is the same as the large-N limit of the U(N) theory.

The large-N character expansion of the mass m_d propagating in the determinant channel has been calculated up to 6th order in the strong-coupling region, indicating a critical point (determined by the zero of the m_d series) slightly larger than our determination of β_c: $\beta_d(N = \infty) \simeq 0.324$ [52]. This discrepancy might be explained either by the shortness of the available character expansion of m_d or by the fact that such a determination of β_c relies on the absence of singular points before the strong-coupling series of m_d vanishes, and therefore a non-analyticity at $\beta_c \simeq 0.306$ would invalidate all strong-coupling predictions for $\beta > \beta_c$.

It is worth mentioning another feature of this large-N critical behavior which emerges from a numerical analysis of the phase distribution of the eigenvalues of the link operator

$$L = U_x U^\dagger_{x+\mu}: \tag{7.17}$$

the $N = \infty$ phase transition should be related to the compactification of the eigenvalues of L [129], like the Gross–Witten phase transition.

The existence of such a phase transition does not represent an obstruction to the use of strong-coupling expansion for the investigation of the continuum limit. Indeed large-N Monte Carlo data show scaling and asymptotic scaling (in the energy scheme) even for β smaller than the peak of the specific heat, suggesting an effective decoupling of the modes responsible for the large-N phase transition from those determining the physical continuum limit. This fact opens the road to tests of scaling and asymptotic scaling at $N = \infty$ based only on strong-coupling computations, given that the strong-coupling expansion should converge for $\beta < \beta_c$. (The strong-coupling analysis does not show evidence of singularities in the complex β-plane closer to the origin than β_c.)

In the continuum limit the dimensionless renormalization-group invariant function

$$A(p;\beta) \equiv \frac{\tilde{G}(0;\beta)}{\tilde{G}(p;\beta)} \tag{7.18}$$

turns into a function $A(y)$ of the ratio $y \equiv p^2/M_G^2$ only, where $M_G^2 \equiv 1/\xi_G^2$ and ξ_G is the second moment correlation length

$$\xi_G^2 \equiv \frac{1}{4}\frac{\sum_x x^2 G(x)}{\sum_x G(x)}\,. \tag{7.19}$$

$A(y)$ can be expanded in powers of y around $y=0$:

$$A(y) = 1 + y + \sum_{i=2}^{\infty} c_i y^i \,, \tag{7.20}$$

and the coefficients c_i parameterize the difference from a generalized Gaussian propagator. The zero y_0 of $A(y)$ closest to the origin is related to the ratio M^2/M_G^2, where M is the fundamental mass; indeed $y_0 = -M^2/M_G^2$. M^2/M_G^2 is in general different from one; it is one in Gaussian models (i.e. when $A(y) = 1 + y$).

Numerical simulations at large N, which allow an investigation of the region $y \geq 0$, have shown that the large-N limit of the function $A(y)$ is approached rapidly and that its behavior is essentially Gaussian for $y \lesssim 1$, indicating that $c_i \ll 1$ in Eq. (7.20) [142]. Important logarithmic corrections to the Gaussian behavior must eventually appear at sufficiently large momenta, as predicted by simple weak-coupling calculations supplemented by a renormalization group resummation:

$$\tilde{G}(p) \sim (\log p^2)/p^2 \tag{7.21}$$

for $p^2/M_G^2 \gg 1$ and in the large-N limit.

The approximate Gaussian behavior at small momentum is also confirmed by the direct estimate of the ratio M^2/M_G^2 obtained by extrapolating Monte Carlo data to $N = \infty$. The large-N limit of the ratio M^2/M_G^2 is rapidly approached, already at $N = 6$ within few per mille, leading to the estimate $M^2/M_G^2 = 0.982(2)$, which is very close to one [132]. Large-N strong-coupling computations of M^2/M_G^2 provide a quite stable curve for a large region of values of the correlation length, which agrees (within about 1%) with the continuum large-N value extrapolated by Monte Carlo data [147].

Monte Carlo simulations at large values of N ($N \geq 6$) also show that asymptotic scaling predictions applied to the fundamental mass are verified within a few per cent at relatively small values of the correlation length ($\xi \gtrsim 2$) and even before the peak of the specific heat in the so-called "energy scheme" [154]; the energy scheme is obtained by replacing T with a new temperature variable $T_E \propto E$. At $N = \infty$ a test of asymptotic scaling may be performed by using the large-N strong-coupling series of the fundamental mass. The two-loop renormalization group and a Bethe Ansatz evaluation of the mass/Λ-parameter ratio [144] lead to the following large-N asymptotic scaling prediction in the β_E scheme:

$$M \cong 16\sqrt{\pi/\mathrm{e}}\, \exp\left(\frac{\pi}{4}\right) \Lambda_{E,\,2l}(\beta_E)\,,$$

$$\Lambda_{E,\,2l}(\beta_E) = \sqrt{8\pi\beta_E}\, \exp(-8\pi\beta_E)\,,$$

$$\beta_E = \frac{1}{8E}\,. \tag{7.22}$$

Strong-coupling calculations, where the new coupling β_E is extracted from the strong-coupling series of E, show asymptotic scaling within about 5% in a relatively large region of values of the correlation length ($1.5 \lesssim \xi \lesssim 3$) [146,147].

The good behavior of the large-N β-function in the β_E scheme, and therefore the fact that physical quantities appear to be smooth functions of the energy, together with the critical behavior

(7.16), can be explained by the existence of a non-analytical zero at β_c of the β-function in the standard scheme:

$$\beta_L(T) \equiv a\,\mathrm{d}T/\mathrm{d}a \sim |\beta - \beta_c|^{\alpha} \tag{7.23}$$

around β_c, where α is the critical exponent of the specific heat. This is also confirmed by an analysis of the strong-coupling series of the magnetic susceptibility χ and M_G^2, which supports the relations

$$\frac{\mathrm{d}\log\chi}{\mathrm{d}\beta} \sim \frac{\mathrm{d}\log M_G^2}{\mathrm{d}\beta} \sim |\beta - \beta_c|^{-\alpha} \tag{7.24}$$

in the neighborhood of β_c, which are consequences of Eq. (7.23) [147].

The presence of a non-analytical zero (7.23) in the large-N β-function should imply that the asymptotic scaling regime (in the standard temperature variable) should be pushed to very large values of β. On the other hand, this singularity can be eliminated by changing the temperature variable to T_E, achieving a much faster approach to asymptotic scaling. The improvement obtained by using T_E can be observed in perturbation theory. For $N > 3$, the linear correction to the two-loop relation between the coupling and the Λ-parameter is considerably smaller in the β_E scheme. The Λ-parameter is defined by

$$\Lambda = \frac{1}{a(T)} \exp\left[- \int \frac{1}{\beta(T)} \mathrm{d}T \right], \tag{7.25}$$

where T is a generic temperature variable (or coupling); at small T,

$$\Lambda = \frac{1}{a(T)} (b_0 T)^{-b_1/b_0^2} \exp\left(- \frac{1}{b_0 T} \right) \left[1 + \frac{b_1^2 - b_0 b_2}{b_0^3} T + \mathrm{O}(T^2) \right], \tag{7.26}$$

where b_0, b_1, and b_2 are the first coefficients of the perturbative expansion of the β-function. Unlike b_0 and b_1, b_2 depends on the choice of the coupling. The coefficient of the linear correction in Eq. (7.26) is

$$\frac{b_1^2 - b_0 b_2}{b_0^3} \underset{N\to\infty}{\to} -0.00884 \tag{7.27}$$

in the energy scheme, and

$$\frac{b_1^2 - b_0 b_2}{b_0^3} \underset{N\to\infty}{\to} 0.06059 \tag{7.28}$$

in the standard scheme. This fact was overlooked in Ref. [132] (cf. [155]).

We finally mention that similar results have been obtained for two-dimensional chiral models on the honeycomb lattice by a large-N strong-coupling analysis. In fact an analysis of the 26th-order strong-coupling series of the free energy indicates the presence of a large-N phase transition, with specific heat exponent $\alpha \cong 0.17$, not far from that found on the square lattice (we have no reasons to expect that the large-N phase transition on the square and honeycomb lattices are in the same universality class). Furthermore, the mass-gap extracted from the 20th-order strong-coupling expansion of $G(x)$ allows to check the corresponding asymptotic scaling predictions in the energy scheme within about 10% [147].

7.3. The large-N limit of SU(N) lattice gauge theories

An overview of the large-N limit of the continuum formulation of QCD has been already presented in Section 2. In the following we report some results concerning the lattice approach.

Gauge models on the lattice have been mostly studied in their Wilson formulation

$$S_W = N\beta \sum_{x,\mu>\nu} \mathrm{Tr}[U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x) + \mathrm{h.c.}] . \tag{7.29}$$

In view of a large-N analysis one may consider both SU(N) and U(N) models, since they are expected to reproduce the same statistical theory in the limit $N \to \infty$ (at fixed β). As for two-dimensional chiral models, SU(N) and U(N) models should have the same continuum limit for any finite $N \geq 2$.

The phase diagram of statistical models defined by the Wilson action has been investigated by standard techniques, i.e., strong-coupling expansion, mean field [77], and Monte Carlo simulations [156–158]. These studies show the presence of a first-order phase transition in SU(N) models for $N \geq 4$, and in U(N) models for any finite N. A first-order phase transition is then expected also in the large-N limit at a finite value of β, which is estimated to be $\beta_c \approx 0.38$ by mean-field calculations and by extrapolation of Monte Carlo results. A review of these results can be found in Ref. [159]. Some speculations on the large-N phase diagram can be also found in Refs. [28,111]. The rôle of the determinant of Wilson loops in the phase transition of U(N) gauge models has been investigated in Ref. [111] by strong-coupling character expansion, and in Ref. [160] by Monte Carlo simulations.

Large-N mean-field calculations suggest the persistence of a first-order phase transition when an adjoint-representation coupling is added to the Wilson action [116,120].

The first-order phase transition of SU(N) lattice models at $N > 3$ can probably be avoided by choosing appropriate lattice actions closer to the renormalization group trajectory of the continuum limit, as shown in Ref. [161] for SU(5). In U(N) models the use of such improved actions should leave a residual transition, due to the extra U(1) degrees of freedom which should decouple at large β in order to reproduce the physical continuum limit of SU(N) gauge models.

It is worth mentioning two studies of confinement properties at large N, obtained essentially by strong-coupling arguments. In Ref. [162], the authors argue that deconfinement of heavy adjoint quarks by color screening is suppressed in the large-N limit. At $N = \infty$, the adjoint string tension is expected to be twice the fundamental string tension, as implied by factorization. In Ref. [23], strong-coupling based arguments point out that Wilson loops in O(N), U(N), and Sp(N) lattice gauge theories should have the same large-N limit, and therefore these theories should share the same confinement mechanism. Such results should be taken into account when studying confinement mechanisms.

Studies based on Monte Carlo simulations for $N > 3$ have not gone beyond an investigation of the phase diagram, so no results concerning the continuum limit of SU(N) lattice gauge theories with $N > 3$ have been produced. Estimates of the mass of the lightest glueball, obtained by a variational approach within a Hamiltonian lattice formulation, seem to indicate a rapid convergence of the $1/N$ expansion [163].

An important breakthrough for the study of the large-N limit of SU(N) gauge theories has been the introduction of the so-called reduced models. A quite complete review on this subject can be found in Ref. [6].

Eguchi and Kawai [63] pointed out that, as a consequence of the large-N factorization, one can construct one-site theories equivalent to lattice YM in the limit $N \to \infty$. The simplest example is given by the one-site matrix model obtained by replacing all link variables of the standard Wilson formulation with four SU(N) matrices according to the simple rule

$$U_\mu(x) \to U_\mu . \tag{7.30}$$

This leads to the reduced action

$$S_{\mathrm{EK}} = N\beta \sum_{\mu > \nu} \mathrm{Tr}[U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger + \mathrm{h.c.}] . \tag{7.31}$$

Reduced operators, and in particular reduced Wilson loops, can be constructed using the correspondence (Eq. (7.30)). In the large-N limit one can prove that expectation values of reduced Wilson loop operators satisfy the same Schwinger–Dyson equations as those in the Wilson formulation. Assuming that all features of the $N = \infty$ theory are captured by the Schwinger–Dyson equations of Wilson loops, the reduced model may provide a model equivalent to the standard Wilson theory at $N = \infty$. In the proof of this equivalence the residual symmetry of the reduced model

$$U_\mu \to Z_\mu U_\mu, \qquad Z_\mu \in Z_N , \tag{7.32}$$

where Z_N is the center of the SU(N) group, plays a crucial rôle. Therefore, the equivalence in the large-N limit of the Wilson formulation and the reduced model (Eq. (7.31)) is actually valid if the symmetry (Eq. (7.32)) is unbroken. This is verified only in the strong-coupling region; indeed in the weak-coupling region the Z_N^4 symmetry gets spontaneously broken and therefore the equivalence cannot be extended to weak coupling [164].

In order to avoid this unwanted phenomenon of symmetry breaking and to extend the equivalence to the most interesting region of the continuum limit, modifications of the original Eguchi–Kawai model have been proposed [64,164,165]. The most promising one for numerical simulation is the so-called twisted Eguchi–Kawai (TEK) model [64,165]. Instead of the correspondence Eq. (7.30), the twisted reduction prescription consists in replacing

$$U_\mu(x) \to T(x) U_\mu T(x)^\dagger , \tag{7.33}$$

where

$$T(x) = \prod_\mu (\Gamma_\mu)^{x_\mu} \tag{7.34}$$

and Γ_μ are traceless SU(N) matrices obeying the 't Hooft algebra

$$\Gamma_\nu \Gamma_\mu = Z_{\mu\nu} \Gamma_\mu \Gamma_\nu ; \tag{7.35}$$

$Z_{\mu\nu}$ is an element of the center of the group Z_N,

$$Z_{\mu\nu} = \exp\left(\mathrm{i}\frac{2\pi}{N} n_{\mu\nu}\right) , \tag{7.36}$$

where $n_{\mu\nu}$ is an antisymmetric tensor with $n_{\mu\nu} = 1$ for $\mu < \nu$. Γ_μ are the matrices implementing the translations by one lattice spacing in the μ direction (here it is crucial that the fields U_μ are in the adjoint representation). The twisted reduction applied to the Wilson action leads to the reduced action

$$S_{\mathrm{TEK}} = N\beta \sum_{\mu>\nu} \mathrm{Tr}[Z_{\mu\nu} U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger + \mathrm{h.c.}] . \tag{7.37}$$

The correspondence between correlation functions of the large-N pure gauge theory and those of the reduced twisted model is obtained as follows. Let $\mathscr{A}[U_\mu(x)]$ be any gauge invariant functional of the field $U_\mu(x)$, then

$$\langle \mathscr{A}[U_\mu(x)] \rangle_{[N=\infty,\mathrm{YM}]} = \langle \mathscr{A}[T(x) U_\mu T(x)^\dagger] \rangle_{[N=\infty,\mathrm{TEK}]} . \tag{7.38}$$

Once again the Schwinger–Dyson equations for the reduced Wilson loops, constructed using the correspondence Eq. (7.33), are identical to the loop equations in the Wilson formulation when $N \to \infty$. The residual symmetry Eq. (7.32), which is again crucial in the proof of the equivalence, should not be broken in the weak-coupling region, and therefore the equivalence should be complete in this case.

One can also show that

1. the reduced TEK model is equivalent to the corresponding field theory on a periodic box of size $L = \sqrt{N}$ [6];
2. in the large-N limit finite-N corrections are $O(1/N^2)$, just as in the SU(N) lattice gauge theory.

Moreover, since $N^2 = L^4$, finite-N corrections can be seen as finite-volume corrections. Therefore in twisted reduced models the large-N and thermodynamic limits are connected and approached simultaneously.

Monte Carlo studies of twisted reduced models at large N confirm the existence of a first-order phase transition at $N = \infty$ located at $\beta_c = 0.36(2)$ [166], which is consistent with the mean-field prediction $\beta_c \simeq 0.38$ [159]. This transition is a bulk transition, and it does not spoil confinement. The few and relatively old existing Monte Carlo results obtained in the weak-coupling region (cf. e.g. Refs. [166–168]) seem to support a rapid approach to the $N \to \infty$ limit of the physical quantities, and are relatively close to the corresponding results for SU(3) obtained by performing simulations within the Wilson formulation. This would indicate that $N = 3$ is sufficiently large to consider the large-N limit a good approximation of the theory.

We mention that hot twisted models can be constructed, which should be equivalent to QCD at finite temperature in the large-N limit (cf. Ref. [6] for details on this subject).

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