PHYSICAL REVIEW D VOLUME 23, NUMBER 8 15 APRIL 1981

Mirror symmetry and exact multimonopole solutions

M. K. Prasad
Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

P. Rossi*
Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 24 November 1980)

The newly found exact solutions for superimposed axially symmetric Yang-Mills-Higgs multimonopoles are shown to possess mirror symmetry. As a consequence, a new class of solutions of the Ernst equation of general relativity is also found. Moreover, the monopole solutions may be formulated in terms of a single real superpotential and their unitary gauge form derived.

A major breakthrough in our understanding of the monopole sector in classical Yang-Mills theories has recently come from the explicit construction of static multimonopole solutions.[1-3] The main purpose of this paper is to investigate the symmetry properties of these solutions. The construction described in Refs. 2 and 3 leads naturally to superimposed axially symmetric monopoles. As we shall show, these configurations also enjoy mirror symmetry. Two relevant by-products of this result are discussed: the natural emergence of real gauge solutions that are also solutions of the Ernst equation of general relativity, and the possibility of completely reformulating the results in terms of a single superpotential, explicitly determined, and such that the unitary gauge form of the solution is derived in a straightforward way.

I. REFORMULATION OF THE SOLUTIONS IN TERMS OF REAL FIELDS

Let us define the SU(2) gauge potentials A_μ^a in four-dimensional Euclidean space and the gauge field strength:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc} A^b_\mu A^c_\nu . \tag{1.1}$$

Solutions of the self-duality Bogomol'nyi equations[4]

$$F^a_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma} F^a_{\rho\sigma} \tag{1.2}$$

corresponding to multimonopoles with magnetic charge n may be found within the framework described in Refs. 2 and 3. Let us outline the steps and results that are relevant to our discussion.

Four complex variables are defined,

$$p = \frac{x_1 + ix_2}{\sqrt{2}}, \quad \bar{p} = \frac{x_1 - ix_2}{\sqrt{2}}, \qquad q = \frac{x_3 - ix_4}{\sqrt{2}}, \quad \bar{q} = \frac{x_3 + ix_4}{\sqrt{2}} \tag{1.3}$$

and Yang's R gauge[5] is assumed. In the R gauge the potentials $A_\mu = (\sigma^a/2i)A^a_\mu$ assume the form

$$A_u = \begin{pmatrix} -\dfrac{\phi_u}{2\phi} & 0 \\ \dfrac{\rho_u}{\phi} & \dfrac{\phi_u}{2\phi} \end{pmatrix}, \quad A_{\bar{u}} = \begin{pmatrix} \dfrac{\phi_{\bar{u}}}{2\phi} & -\dfrac{\bar{\rho}_{\bar{u}}}{\phi} \\ 0 & -\dfrac{\phi_{\bar{u}}}{2\phi} \end{pmatrix}, \tag{1.4}$$

where $u \equiv p, q$ and the subscript u denotes differentiation with respect to the corresponding variable.

It is found that the self-duality equations corresponding to $n(\geq 2)$ monopoles are solved by[6,7,1,2]

$$_n\phi = \frac{d_n^0}{d_{n-1}^0}, \quad {}_n\rho = (-1)^n \frac{d_n^{-1}}{d_{n-1}^0}, \quad {}_n\bar{\rho} = (-1)^{n+1}\frac{d_n^1}{d_{n-1}^0}, \tag{1.5}$$

where

$$d_n^0 = \begin{vmatrix} \Delta_0 & \cdots & \Delta_{-n+1} \\ \vdots & \ddots & \\ \Delta_{n-1} & & \Delta_0 \end{vmatrix}, \quad d_n^{-1} = \begin{vmatrix} \Delta_{-1} & \cdots & \Delta_{-n} \\ \vdots & \ddots & \\ \Delta_{n-2} & & \Delta_{-1} \end{vmatrix}, \tag{1.6}$$

$$d_n^1 = \begin{vmatrix} \Delta_1 & \cdots & \Delta_{-n+2} \\ \vdots & \ddots & \\ \Delta_n & & \Delta_1 \end{vmatrix}$$

and

$$\Delta_0 = {}_1\phi = e^{ix_4}\Lambda_0 , \tag{1.7a}$$

$$\Delta_{-n} = (-1)^n e^{ix_4}(\sqrt{2}p)^{-n}(1+\partial_3)^n \Lambda_n , \tag{1.7b}$$

$$\Delta_n = (-1)^n e^{ix_4}(\sqrt{2}\,p)^{-n}(1-\partial_3)^n \Lambda_n , \tag{1.7c}$$

where

$$\Lambda_n = \bar{p}^{-1}\partial_p \Lambda_{n+1} = p^{-1}\partial_{\bar{p}}\Lambda_{n+1} \tag{1.8}$$

and

$$\Lambda_0 = \sum_{K=1}^{n} \alpha_K \frac{\sinh R_K}{R_K}, \quad R_K^2 = x_1^2 + x_2^2 + (x_3 - z_K)^2 . \tag{1.9}$$

Solutions in this form are in general complex valued. The reality of the gauge field in some gauge is ensured if one can find matrices $\bar{V}(p,q)$ and $V(p,q)$ such that, defining

$$R = \begin{bmatrix} \frac{1}{\sqrt{\phi}} & 0 \\ \frac{\rho}{\sqrt{\phi}} & \sqrt{\phi} \end{bmatrix}, \quad \bar{R} = \begin{bmatrix} \sqrt{\phi} & -\frac{\bar{\rho}}{\sqrt{\phi}} \\ 0 & \frac{1}{\sqrt{\phi}} \end{bmatrix}, \tag{1.10}$$

$\bar{V}R\bar{R}^{-1}V$ turns out to be a positive-definite Hermitian matrix. When this happens we may define a gauge-transformation matrix L such that

$$LL^\dagger = (R^\dagger \bar{V}^\dagger V^{-1}\bar{R})^{-1} \tag{1.11}$$

and the gauge transformation

$$A_\mu \to L^{-1}A_\mu L + L^{-1}\partial_\mu L \tag{1.12}$$

will make the gauge fields real. L is defined only up to a multiplication by an arbitrary unitary matrix: this reflects the (unitary) gauge arbitrariness of the real fields.

It is apparent from the analysis presented in Refs. 2 and 3 that the general form of the matrices $V, \bar{V}$ for the superimposed monopole solution is

$$\bar{V} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & -\gamma^{-1}(\sqrt{2}\,p)^{-n} \\ \gamma(\sqrt{2}\,p)^n & 0 \end{bmatrix}, \tag{1.13}$$

where γ is a real constant. One can then easily find that

$$LL^\dagger = \begin{bmatrix} \gamma\,{}_n\bar{K} & \gamma e^{-in\theta}{}_nQ \\ \gamma e^{in\theta}{}_nQ & -\gamma\,{}_nK \end{bmatrix}, \tag{1.14}$$

where we have defined

$$(\sqrt{2}\,p) = re^{i\theta}, \quad (\sqrt{2}\,\bar{p}) = re^{-i\theta}, \quad r = (x_1^2 + x_2^2)^{1/2} \tag{1.15}$$

and

$$\begin{aligned} &{}_n\bar{K} = (\sqrt{2}\,p)^n\,{}_n\bar{\rho}e^{-ix_4}, \quad {}_nK = (\sqrt{2}\,p)^n\,{}_n\rho^* e^{ix_4}, \\ &{}_nQ = r^n({}_n\phi\,{}_n\phi^*)^{1/2}, \quad {}_n\bar{K}\,{}_nK + {}_nQ^2 = -\frac{1}{\gamma^2}, \end{aligned} \tag{1.16}$$

where ${}_nK$, ${}_n\bar{K}$, and ${}_nQ$ are real functions of r and $z = x_3$. We may remove the arbitrariness in the definition of L by the choice that the resulting gauge transformation must preserve the R-gauge form of the solution.

It is a property of triangular matrices that this will happen if we choose the triangular square root of $LL^\dagger$:

$$L = \begin{bmatrix} (\gamma\,{}_n\bar{K})^{1/2} & 0 \\ e^{in\theta}\sqrt{\gamma}\,\frac{{}_nQ}{({}_n\bar{K})^{1/2}} & \frac{1}{(\gamma\,{}_n\bar{K})^{1/2}} \end{bmatrix},$$

$$L^\dagger = \begin{bmatrix} (\gamma\,{}_n\bar{K})^{1/2} & e^{-in\theta}\sqrt{\gamma}\,\frac{{}_nQ}{({}_n\bar{K})^{1/2}} \\ 0 & \frac{1}{(\gamma\,{}_n\bar{K})^{1/2}} \end{bmatrix}. \tag{1.17}$$

It requires long and tedious algebra to check that the gauge fields are

$$\tilde{A}_p = \frac{e^{-i\theta}}{2\sqrt{2}} \begin{bmatrix} \frac{n}{r} + \frac{{}_n\bar{K}_r}{{}_n\bar{K}} - \frac{{}_nQ_r}{{}_nQ} & 0 \\ 2e^{in\theta}\frac{{}_n\bar{K}_r}{\gamma\,{}_n\bar{K}\,{}_nQ} & -\frac{n}{r} - \frac{{}_n\bar{K}_r}{{}_n\bar{K}} + \frac{{}_nQ_r}{{}_nQ} \end{bmatrix}, \quad \tilde{A}_{\bar{p}} = -(\tilde{A}_p)^\dagger \tag{1.18}$$

$$\tilde{A}_q = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 + \frac{{}_n\bar{K}_z}{{}_n\bar{K}} - \frac{{}_nQ_z}{{}_nQ} & 0 \\ 2e^{in\theta}\frac{{}_n\bar{K} + {}_n\bar{K}_z}{\gamma\,{}_n\bar{K}\,{}_nQ} & -1 - \frac{{}_n\bar{K}_z}{{}_n\bar{K}} + \frac{{}_nQ_z}{{}_nQ} \end{bmatrix}, \quad \tilde{A}_{\bar{q}} = -(\tilde{A}_q)^\dagger . \tag{1.19}$$

We may now define

$${}_n\tilde{\phi} = \frac{{}_nQ}{{}_n\bar{K}}e^{-z}e^{-in\theta}, \quad {}_n\tilde{\rho} = -\frac{1}{\gamma\,{}_n\bar{K}}e^{-z} \tag{1.20}$$

and check that, in agreement with Eq. (1.4),

$$\tilde{A}_u = \begin{bmatrix} -\frac{{}_n\tilde{\phi}_u}{2\,{}_n\tilde{\phi}} & 0 \\ \frac{{}_n\tilde{\rho}_u}{{}_n\tilde{\phi}} & \frac{{}_n\tilde{\phi}_u}{2\,{}_n\tilde{\phi}} \end{bmatrix}, \quad \tilde{A}_{\bar{u}} = -(\tilde{A}_u)^\dagger . \tag{1.21}$$

Let us observe that the only dependence on the azimuthal coordinate θ is absorbed into the phase factor $e^{-in\theta}$, as we expect to happen in the case of axial symmetry. Moreover, the dependence on the axial coordinates r, z happens to enter only through real functions. As we shall see in the following section, this phenomenon is strictly related to the existence of a further, discrete symmetry of the solution, usually referred to as "mirror symmetry."

II. PROPERTIES OF AXIALLY SYMMETRIC MONOPOLES

As discussed in Ref. 8, a simple way to describe axially symmetric gauge-field configurations is obtained by introducing sets of orthonormal vectors,

$$\begin{aligned} u_1^a &= (\cos n\theta, \sin n\theta, 0)\,, \\ u_2^a &= (0,0,1)\,, \\ u_3^a &= (\sin n\theta, -\cos n\theta, 0)\,, \end{aligned} \tag{2.1}$$

$$\begin{aligned} u_1^\mu &= (\cos\theta, \sin\theta, 0, 0)\,, \\ u_2^\mu &= (0,0,1,0)\,, \\ u_3^\mu &= (\sin\theta, -\cos\theta, 0, 0)\,, \\ u_4^\mu &= (0,0,0,1)\,, \end{aligned} \tag{2.2}$$

and expanding the fields in the form

$$A_\mu^a = u_\nu^\mu u_b^a W_\nu^b(r,z)\,. \tag{2.3}$$

We may reexpress the A_p, A_q fields in terms of the new variables W_ν^b thus obtaining

$$A_p = \frac{e^{-i\theta}}{2\sqrt{2}} \begin{pmatrix} W_3^2 - iW_1^2 & e^{-in\theta}(W_1^3 + W_3^1 + iW_3^3 - iW_1^1) \\ e^{in\theta}(W_3^1 - W_1^3 - iW_3^3 - iW_1^1) & -W_3^2 + iW_1^2 \end{pmatrix}, \quad A_{\bar{p}} = -(A_p)^\dagger \tag{2.4}$$

$$A_q = \frac{1}{2\sqrt{2}} \begin{pmatrix} W_4^2 - iW_2^2 & e^{-in\theta}(W_3^3 + W_4^1 + iW_4^3 - iW_2^1) \\ e^{in\theta}(W_4^1 - W_2^3 - iW_4^3 - iW_2^1) & -W_4^2 + iW_2^2 \end{pmatrix}, \quad A_{\bar{q}} = -(A_q)^\dagger\,. \tag{2.5}$$

This is the most general form real axisymmetric fields may assume.

Let us, however, consider the possibility of requiring mirror symmetry. A mirror transformation M is defined as a reflection through a plane containing the symmetry axis z. Under such a transformation

$$W_i(r,z,\theta) \to W_i(r,z,-\theta)\,, \quad i=1,2,4 \tag{2.6a}$$

$$W_3(r,z,\theta) \to -W_3(r,z,-\theta)\,. \tag{2.6b}$$

However, this transformation alone would change the sign of the magnetic charge, a pseudoscalar quantity; therefore it cannot possibly be a symmetry of monopoles, and we must associate to it a Z transformation (magnetic charge conjugation) defined by

$$W_i(r,z,\theta) \to W_i(r,z,\theta)\,, \quad i=1,2,3 \tag{2.7a}$$

$$W_4(r,z,\theta) \to -W_4(r,z,\theta)\,. \tag{2.7b}$$

Since we are dealing with gauge theories, the MZ transformation is completed by a gauge transformation that must be consistent with the request that

$$(MZ)^2 = I\,. \tag{2.8}$$

$e^{i\pi T_3}$ is such a transformation: T_3 generates gauge rotations leaving the third gauge component of W_ν^b unaltered.[9] We can define the complete mirror transformation:

$$\mathfrak{M} = MZ \, \exp(i\pi T_3)\,. \tag{2.9}$$

We then immediately find that $\mathfrak{M}$ invariance implies[10,11]

$$\begin{aligned} &W_1^1 = W_2^1 = W_1^2 = W_2^2 = 0\,, \\ &W_3^3 = W_4^3 = 0 \end{aligned} \tag{2.10}$$

and the mirror-invariant axisymmetric fields are in the form

$$A_p = \frac{e^{-i\theta}}{2\sqrt{2}} \begin{pmatrix} W_3^2 & e^{-in\theta}(W_1^3 + W_3^1) \\ e^{in\theta}(W_3^1 - W_1^3) & -W_3^2 \end{pmatrix}, \quad A_{\bar{p}} = -(A_p)^\dagger \tag{2.11a}$$

$$A_q = \frac{1}{2\sqrt{2}} \begin{pmatrix} W_4^2 & e^{-in\theta}(W_2^3 + W_4^1) \\ e^{in\theta}(W_4^1 - W_2^3) & -W_4^2 \end{pmatrix}, \quad A_{\bar{q}} = -(A_q)^\dagger\,. \tag{2.11b}$$

Comparison with Eqs. (1.18) and (1.19) is now straightforward. It makes apparent that the multimonopole solutions are indeed mirror symmetric, and it leads to the identifications

$$W_3^1 = \frac{{}_n\bar{K}_r}{r\,{}_n\bar{K}\,{}_nQ}\,, \quad W_4^1 = \frac{{}_n\bar{K}_z + {}_n\bar{K}}{r\,{}_n\bar{K}\,{}_nQ}\,, \tag{2.12}$$

$$W_3^2 = \frac{n}{r} + \frac{{}_n\bar{K}_r}{{}_n\bar{K}} - \frac{{}_nQ_r}{{}_nQ}\,, \quad W_4^2 = 1 + \frac{{}_n\bar{K}_z}{{}_n\bar{K}} - \frac{{}_nQ_z}{{}_nQ}\,, \tag{2.13}$$

$$W_1^3 + W_3^1 = 0\,, \quad W_2^3 + W_4^1 = 0\,. \tag{2.14}$$

Further insight is obtained by performing the (singular) gauge transformation

$$L=\begin{pmatrix} e^{-in\theta/2} & \\ & e^{in\theta/2}\end{pmatrix}. \tag{2.15}$$

This transformation removes $e^{\pm in\theta}$ factors from Eqs. (2.4) and (2.5) while affecting only the W_3^2 component in a nontrivial way:

$$\tilde{W}_3^2=W_3^2-\frac{n}{r},\quad \tilde{W}_\nu^b=W_\nu^b \text{ otherwise.} \tag{2.16}$$

For the purpose of comparison with previous notation,[8,11] let us define

$$\frac{\eta^\alpha}{r}=-\tilde{W}_3^\alpha\ ,\quad \varphi^\alpha=\tilde{W}_4^\alpha\,. \tag{2.17}$$

The Bogomol'nyi self-duality equations then assume the form

$$\frac{1}{r}\varepsilon^{\mu\nu}(\partial_\mu\eta^\alpha+\epsilon^{\alpha\beta\gamma}\tilde{W}_\mu^\beta\eta^\gamma)=\partial_\nu\varphi^\alpha+\epsilon^{\alpha\beta\gamma}\tilde{W}_\nu^\beta\varphi^\gamma\,,\quad \mu,\nu=1,2 \tag{2.18}$$

$$\tfrac{1}{2}\varepsilon^{\mu\nu}(\partial_\mu\tilde{W}_\nu^\alpha-\partial_\nu\tilde{W}_\mu^\beta+\epsilon^{\alpha\beta\gamma}\tilde{W}_\mu^\beta\tilde{W}_\nu^\gamma)=\frac{1}{r}\epsilon^{\alpha\beta\gamma}\varphi^\beta\eta^\gamma\,. \tag{2.19}$$

It is easy to show that Eqs. (2.18) imply the existence of a superpotential $\tau(r,z)$ such that quantities that are gauge invariant under transformations preserving axial symmetry may be reexpressed in terms of τ:

$$\varphi^\alpha\varphi^\alpha=1-\nabla^2\tau\,, \tag{2.20}$$

$$\eta^\alpha\eta^\alpha=n^2-r^2\left(\partial_r\partial_r-\frac{1}{r}\partial_r+\partial_z\partial_z\right)\tau\,, \tag{2.21}$$

$$\eta^\alpha\varphi^\alpha=-\partial_z\tau\,. \tag{2.22}$$

Let us now consider the consequences of the transformation Eq. (2.15) on the mirror-symmetric multimonopole solutions. First of all, since the R gauge is once more preserved, we may define rotated potentials

$$_n\phi^1=\frac{_nQ}{_n\overline{K}}e^{-z}\,,\quad {}_n\rho^1=-\frac{1}{r\,_n\overline{K}}e^{-z} \tag{2.23}$$

that are now real functions.

The field components reduce to

$$\varphi^1=\frac{\partial_z\,{}_n\rho^1}{_n\phi^1}\,,\quad \varphi^2=-\frac{\partial_z\,{}_n\phi^1}{_n\phi^1}\,, \tag{2.24}$$

$$\frac{\eta^1}{r}=-\frac{\partial_r\,{}_n\rho^1}{_n\phi^1}\,,\quad \frac{\eta^2}{r}=\frac{\partial_r\,{}_n\phi^1}{_n\phi^1}\,, \tag{2.25}$$

$$\tilde{W}_r^3=\frac{\eta^1}{r}\,,\quad \tilde{W}_z^3=-\varphi^1\,. \tag{2.26}$$

This form of the multimonopole solutions is especially interesting. Indeed it has been shown[12,13] that all solutions of the stationary axially symmetric Einstein equations correspond to solutions of the self-dual mirror-symmetric SU(2) gauge fields satisfying the special conditions Eqs. (2.26).

As a consequence, the R-gauge multimonopole solutions, automatically satisfying Eqs. (2.26), naturally provide solutions to the Ernst equation,

$$\mathrm{Re}\,\epsilon\nabla^2\epsilon-(\nabla\epsilon)^2=0\,, \tag{2.27}$$

by the identification

$$\epsilon\equiv{}_n\phi^1+i\,{}_n\rho^1=\left({}_nQ-\frac{i}{r}\right)\frac{e^{-z}}{_n\overline{K}}\,. \tag{2.28}$$

Finally let us observe that the gauge-invariant property of multimonopole solutions,[2]

$$A_4^aA_4^a\equiv\varphi^a\varphi^a=1-\nabla^2\sum_{K=1}^{n}\ln{}_K\phi\,, \tag{2.29}$$

leads to an immediate identification of the superpotential:

$$\tau=\sum_K\ln{}_K\phi=\ln d_n^0\,. \tag{2.30}$$

Moreover, as shown in Ref. 8, when the axially symmetric gauge fields are also mirror symmetric τ turns out to be a solution of the nonlinear fourth-order partial differential equation:

$$\frac{1}{r}hE-\partial_\mu\left(\frac{r\partial_\mu h+\epsilon_{\mu\nu}\partial_\nu\psi}{E}\right)=0\,, \tag{2.31}$$

where

$$h^2\equiv\varphi^\alpha\varphi^\alpha\,,\quad E^2+\psi^2\equiv\eta^\alpha\eta^\alpha\,,$$
$$h\psi\equiv\eta^\alpha\varphi^\alpha\,,\quad hE\equiv\varepsilon^{\alpha\beta}\varphi^\alpha\eta^\beta \tag{2.32}$$

are implicitly defined in terms of τ through Eqs. (2.20), (2.21), and (2.22).

All fields may now be reexpressed as functions of τ, so that we are able to write down the solution in any arbitrary real mirror-symmetric gauge:

$$\varphi^1=h\sin\alpha\,,\quad \varphi^2=h\cos\alpha\,,$$
$$\eta^1=\psi\sin\alpha+E\cos\alpha\,,\quad \eta^2=\psi\cos\alpha-E\sin\alpha\,, \tag{2.33}$$
$$\tilde{W}_\mu^3=\frac{r\epsilon_{\mu\nu}\partial_\nu h-\partial_\mu\psi}{E}+\partial_\mu\alpha\,,$$

where $\alpha(r,z)$ is an arbitrary real (Abelian) gauge function and $\alpha=0$ is the unitary gauge.

ACKNOWLEDGMENTS

We would like to thank Nick Manton for an illuminating discussion on discrete symmetries in gauge theories. This work was supported in part through funds provided by the U. S. Department of Energy under Contract No. DE-AC02-76ERO-3069.

*On leave of absence from the Scuola Normale Superiore, Pisa, Italy.

[1]R. S. Ward, Commun. Math. Phys. (to be published).
[2]M. K. Prasad, Commun. Math. Phys. (to be published).
[3]M. K. Prasad, A. Sinha, and L. L. C. Wang, Phys. Rev. D (to be published).
[4]E. Bogomol'nyi, Yad. Fiz. 24, 861 (1976) [Sov. J. Nucl. Phys. 24, 449 (1976)].
[5]C. N. Yang, Phys. Rev. Lett. 38, 1377 (1977).
[6]M. F. Atiyah and R. S. Ward, Commun. Math. Phys. 55, 117 (1977).
[7]E. F. Corrigan, D. B. Fairlie, R. G Yates, and P. Goddard, Commun. Math. Phys. 58, 223 (1978).
[8]C. Rebbi and P. Rossi, Phys. Rev. D 22, 2010 (1980).
[9]N. S. Manton (unpublished).
[10]P. Houston and L. O'Raifeartaigh, *Proceedings of the Seminar on Group Theoretical Methods in Physics, Moscow, 1979* (Zvenigorod, Moscow, 1979).
[11]N. S. Manton, Nucl. Phys. B135, 319 (1978).
[12]L. Witten, Phys. Rev. D 19, 718 (1979).
[13]P. Forgacs, Z. Horvath, and L. Palla, Phys. Rev. Lett. 45, 505 (1980).