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Quantum chromodynamics on a tetrahedron at $N_c = \infty$

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We show how the $N_c = \infty$ limit for the single-link integral can be extracted from the finite-N_c solution both in the weak- and in the strong-coupling regimes. This result is used to provide a complete $N_c = \infty$ solution for quantum chromodynamics on a tetrahedron, thus implicitly providing a solution to the Makeenko-Migdal equation for a *three*-dimensional (albeit elementary) lattice system.

I. INTRODUCTION

The use of steepest-descent techniques has recently led to many interesting studies of lattice gauge theories in the large-N_c limit. Although this limit can greatly reduce the degree of freedom of the gauge systems due to the absence of fluctuations, extreme complexity still remains so that, till now, only a few truncated models have been solved. In this approach, one central step involves the evaluation of the single-link integral

$$Z_0(A^\dagger A,\beta)=\int dU \exp[N\beta \,\mathrm{Tr}(A^\dagger U + AU^\dagger)]\ , \quad (1.1)$$

where U is an element of the group $U(N)$ and A is an arbitrary $N\times N$ matrix. We have recently shown[1] that $Z_0(A^\dagger A,\beta)$ can be explicitly expressed in terms of elementary functions, and, for N finite, it is entire in β. It has also been shown that the large-N limit of Z_0 must admit two distinct branches, referred to as the strong- and the weak-coupling (small and large β, respectively) regimes.

In this paper, we show how the $N_c=\infty$ limit for the single-link integral can be extracted from the finite-N_c solution both in the weak- and in the strong-coupling regimes, and how this result can be used to study finite lattice quantum chromodynamics (QCD) at $N_c=\infty$. In particular, the complete $N_c=\infty$ solution for QCD on a tetrahedron is obtained (as well as for other simpler configurations), thus implicitly providing for the first time an exact solution to the Makeenko-Migdal equation[2] for a three-dimensional (albeit elementary) lattice system.

The large-N limit of Eq. (1.1) in the weak-coupling regime was first obtained by Brower and Nauenberg[3] by directly solving a partial differential equation for Z_0. Recently, Brezin and Gross[4] extended the analysis to include the strong-coupling regime and showed that, at the "strong-weak" crossover point,

$$\lim_{N\to\infty} \frac{1}{N}\,\mathrm{Tr}((AA^\dagger)^{-1/2})=2\beta_c\ , \quad (1.2)$$

the third derivative of $F\equiv(1/N^2)\ln Z_0$ had a discontinuity, symptomatic of a "third-order" transition. In Sec. II we show how Eq. (1.2) can be obtained from our knowledge of the exact finite-N solution for Z_0.

QCD on a tetrahedron is defined by the action

$$S_4=\frac{1}{g^2}\sum_f \mathrm{Tr}(U_p(f)+U_p^\dagger(f)) \quad (1.3)$$

where the plaquette sum is labeled by vertices of four triangle faces (123), (234), (341), and (412), [Fig. 1(a)] and the plaquette product is

$$U_p(ijk)=U(ij)U(jk)U(ki)\ , \quad (1.4)$$

with $U(ij)$ the unitary $N\times N$ matrix variable for the link from the ith to the jth vertices. By gauging $U(42)$ and $U(13)$ to identity, the action (1.3) becomes that of an L-link periodic chiral chain [Fig. 1(b)],

$$S_L=N\beta\sum_{l=1}^{L} \mathrm{Tr}(U_l U_{l+1}^\dagger + U_{l+1}U_l^\dagger)\ , \quad (1.5)$$

where $L=4$, $\beta=(Ng^2)^{-1}$, $U_1=U_5=U(12)$, $U_2^\dagger=$ $=U(23)$, $U_3=U(34)$, and $U_4^\dagger=U(41)$. We have previously shown[1] that, in general, gauge theories on polyhedra are equivalent to chiral fields on a periodic lattice, and have solved them exactly at $N_c=\infty$ in the weak-coupling region for periods $L=2$, 3, 4, and ∞. In Sec. III we obtain corresponding solutions at $N_c=\infty$ in the strong-coupling

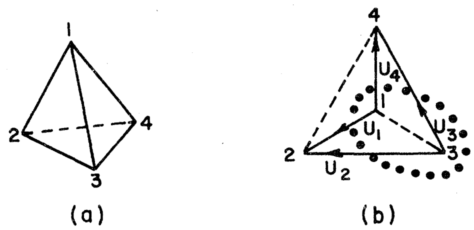

FIG. 1. (a) A tetrahedron with link variable $U(ij)$ between vertices. (b) Tetrahedron flattened to plane. Dashed lines have their link variables gauged to $U=1$, leaving a four-link chiral chain. A closed path (dashed) covers the surface by entering and exiting the triangle faces across the nongauged lines.

limit. [The strong-coupling solutions for $L=2$ and ∞ were implicitly known from the analysis of two-dimensional QCD (QCD_2) by Gross and Witten.[5] Friedan[6] has previously obtained a strong-coupling solution for the case $L=3$. We show in Appendix A that our solution for $L=3$, although arrived at using a completely different technique, agrees with that of Friedan, thus confirming its uniqueness.] Our result completes the effort, began in Ref. 1, in establishing the fact that, for $L=2$, 3, 4, and ∞, a third-order crossover singularity occurs for $N_c=\infty$ at $\beta_c=\frac{1}{4}$, $\frac{1}{3}$, $\pi/8$, and $\frac{1}{2}$, respectively (Fig. 2).

It is by now clear from the analysis of the single-link integral (1.1) that the original singularity found by Gross and Witten in QCD_2 cannot be purely kinematical since the condition (1.2) depends on the structure of the external sources A and $A^\dagger$. However, it is not obvious whether or not the subsequent integrals over the sources will smear out the singularity so that no hint of the Gross-Witten singularity persists in QCD_4.

Our analysis clearly indicates that whereas the integration over sources does remove the original Gross-Witten singularity at $\beta_c=\frac{1}{2}$, it does not remove it, but rather moves it to a new location. Unlike a conventional phase transition which usually requires infinite spatial extension, this crossover singularity comes about due primarily to the large fluctuation in the infinite group volume at $N_c=\infty$ while the spatial correlations remain short ranged. Therefore, our result strongly suggests that this singular phenomenon persists for QCD_4 at $N_c=\infty$.

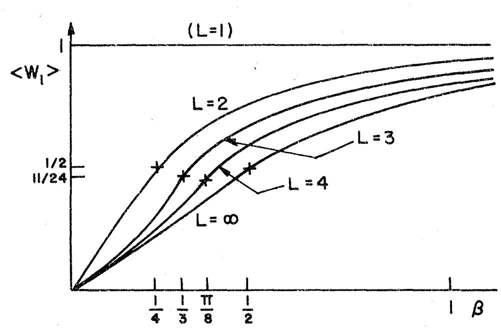

FIG. 2. The behavior of the single-plaquette expectation value as a function of β and L. At the critical point (marked by a cross) $\partial\langle W_1\rangle\partial\beta$ is still continuous, but its derivative shows a discontinuity.

Whereas this singularity might be generic for gauge theories at $N_c=\infty$, its location in β does depend on the local topology. Recent efforts[7] in estimating the crossover point at $N_c=\infty$ from series expansions indicate a value for β_c at a little less than $\frac{1}{2}$. It is amazing that this is in good agreement with our result for the "minimum" three-dimensional finite lattice system, the tetrahedron, where $\beta_c=\pi/8\simeq 0.395$.

II. $N_c=\infty$ LIMIT OF SINGLE-LINK INTEGRAL

We have shown in Ref. 1 that at finite N the single-link integral Eq. (1.1) is proportional to the wave function for a system of N independent fermions and the Slater determinant can be rearranged in terms of modified Bessel functions I_{k-1}, $k=1,2,\ldots,N$,

$$Z_0(A^\dagger A,\beta)=\det[(z_j{}^2)^{k-1}\psi_{k-1}(z_j)]/\det[(z_j{}^2)^{k-1}]\ , \quad (2.1)$$

$$\psi_k(z)=(k!)(2/z)^k I_k(z)\ , \quad (2.2)$$

where $z_j=2N\beta\sqrt{x_j}$, $j=1,2,\ldots,N$ are eigenvalues of the matrix $2N\beta\sqrt{AA^\dagger}$. Since x_j's are bounded, the large-N limit of (2.1) can in principle be obtained from the known asymptotic limit

$$\psi_{k-1}(z)\underset{z\to\infty}{\simeq}[\tfrac{1}{2}(1+(1+z^2/k^2)^{1/2})]^{1-k}(1+z^2/k^2)^{-1/4}$$
$$\times\exp[(k^2+z^2)^{1/2}-k]\ . \quad (2.3)$$

Owing to the explicit dependence on k in (2.3), the asymptotic limit of (2.1) cannot be explicitly carried out. However, Eq. (2.3) suggests an ansatz that, in the large-N limit, the dependence on the index k under the square roots in (2.3) can be replaced by a "mean" value $\bar{k}$, so that, up to a numerical constant,

$$Z_0\propto\frac{\det[(z_j{}^2+\bar{k}^2)^{(k-1)/2}]}{\det[(z_j{}^2+\bar{k}^2)^{k-1}]}$$
$$\times\prod_{j=1}^{N}\{\exp[(z_j{}^2+\bar{k}^2)^{1/2}]/(z_j{}^2+\bar{k}^2)^{1/4}\}\ . \quad (2.4)$$

Replacing z_j by $2N\beta\sqrt{x_j}$ and defining $r\equiv(\bar{k}/N)^2$, where $0\le r\le 1$, we obtain at large N

$$F \equiv \frac{1}{N^2} \ln Z_0$$
$$\simeq \frac{1}{N}\sum_j (r+4\beta^2 x_j)^{1/2} - \frac{1}{2N^2}\sum_{i,j} \ln\left[\frac{(r+4\beta^2 x_i)^{1/2}+(r+4\beta^2 x_j)^{1/2}}{2}\right] - \frac{r}{4} - \frac{3}{4}, \quad (2.5)$$

where the constant is determined by making use of the known limit $\beta \to 0$.

Since F must satisfy the differential equation[3]

$$\frac{1}{N}\sum_k x_k \frac{\partial F}{\partial x_k} + \frac{1}{N^2}\sum_k x_k^2\left(\frac{\partial F}{\partial x_k}\right)^2 + \frac{1}{N^2}\sum_{s\neq k}\frac{x_s - x_k}{x_k - x_s}\left(\frac{\partial F}{\partial x_k} - \frac{\partial F}{\partial x_s}\right) = \beta^2\left(\sum_k x_k\right)F, \quad (2.6)$$

this allows us to check on the consistency of our ansatz (2.4). Cumbersome but straightforward algebra then turns Eq. (2.6) into the condition

$$(r)\left[\frac{1}{N}\sum_j (r+4\beta^2 x_j)^{-1/2} - 1\right] = 0, \quad (2.7)$$

which determines r (or the mean value $\bar{k}$).

Equation (2.7) allows two possibilities:

$$\text{(i)} \quad r = 0 \quad (2.8)$$

and

$$\text{(ii)} \quad \frac{1}{N}\sum_j (r+4\beta^2 x_j)^{-1/2} = 1. \quad (2.9)$$

When substituted back into Eq. (2.5), Eq. (2.8) leads to the weak-coupling solution of Brower and Nauenberg[3] and Eq. (2.9) leads to the strong-coupling solution of Brezin and Gross.[4]

We note that, from (2.8) and (2.9), r can take on values only in the range [0, 1], consistent with our original ansatz. The case $r \to 1$ corresponds to the extreme strong-coupling limit where only a finite number of eigenvalues are different from zero and it reproduces the known solutions for this limiting case.[3,8] On the other hand, the condition (2.8) occurs when

$$\lim_{N\to\infty} \frac{1}{N}\sum_j (x_j^{-1/2}) = 2\beta_c \quad (2.10)$$

is first obtained, i.e., Eq. (1.2), and (2.8) persists throughout the entire weak-coupling region $\beta < \beta_c$. Equation (2.10) defines a strong-weak crossover point where the third derivative of F is discontinuous.[4]

III. QCD ON TETRAHEDRON

Given (1.5), the partition function for a four-link chiral chain can be expressed as

$$Z_4 = \int dU\, dV \exp[N^2 S_{\text{eff}}(UV^\dagger)], \quad (3.1)$$

where U, V can be any pair of nonadjacent link variables and the integrand is simply the square of a single-link integral Eq. (1.1), with $A = U + V$. We have previously explained in Ref. 1 that this representation, Eq. (3.1), together with the result of Sec. II, is directly suitable for a large-N steepest-descent analysis. By changing variables to θ_j, $e^{i\theta_j}$ being eigenvalues of $UV^\dagger$, the large-N limit of Z_4 is dominated by a stationary configuration, specified by a spectral density $\rho(\theta)$ which can be obtained by solving the equation

$$P\int d\phi\, \rho(\phi)\cot\left(\frac{\theta-\phi}{2}\right) + \frac{\delta}{\delta\theta} S_{\text{eff}}[\theta,\rho] = 0. \quad (3.2)$$

It follows from Eq. (2.5) that, for $-\theta_c \leq \theta \leq \theta_c$, Eq. (3.2) can be written as

$$\frac{4\beta^2 \sin\theta}{g(\theta)} = P\int_{-\theta_c}^{\theta_c} d\phi\, \rho(\phi)\left[\cot\left(\frac{\theta-\phi}{2}\right) - \frac{1}{2}\left(\frac{g'(\theta)}{g(\theta)+g(\phi)}\right)\right], \quad (3.3)$$

where

$$g(\theta) = \tfrac{1}{2}[8\beta^2(1+\cos\theta) + r]^{1/2} \quad (3.4)$$

and $\rho(\phi)$ satisfies a normalization condition

$$\int_{-\theta_c}^{\theta_c} d\phi\, \rho(\phi) = 1. \quad (3.5)$$

Whereas $0 < \theta_c < \pi$ in the weak-coupling region, the limit in the strong-coupling region spans the entire range $[-\pi, \pi]$ and the condition Eq. (2.9) in a continuum notation becomes

$$\int_{-\pi}^{\pi} d\phi[\rho(\phi)/g(\phi)] = 2. \quad (3.6)$$

The spectral density $\rho(\theta)$ can now be found by the following procedure. In terms of the variable $z = e^{i\theta}$, the right-hand side of Eq. (3.3) is the principal part of an analytic function

$$F(z) = \oint \frac{dz'}{z'}\rho(z')\left[\frac{g(z')}{g(z)}\right]\left[\frac{z+z'}{z-z'}\right], \quad (3.7)$$

where the integration contour is along the unit circle. Note that the product $g(z)F(z)$ is analytic in the interior of the unit circle, which can be mapped into the entire complex plane by a variable change

$$x \equiv 2(1+\cos\theta) = 2 + z + 1/z. \quad (3.8)$$

A careful analysis then indicates that a real analytic function $H(x)$ can be constructed,

$$H(x) \equiv -\frac{g(\theta)F(z)}{[x(4-x)]^{1/2}} = \frac{1}{\pi}\int_{x_c}^{4} \frac{dx'}{x'-x}\,\mathrm{Im}H(x')\ , \quad (3.9)$$

where $x_c \equiv 2(1+\cos\theta_c)$. When approaching the branch cut from above, we find

$$\mathrm{Im}H(x) = \frac{2\pi g(\theta)}{[x(4-x)]^{1/2}}\,\rho(\theta) > 0, \quad \mathrm{Re}H(x) = 2\beta^2\ . \quad (3.10)$$

Equations (3.9) and (3.10) allow us to solve for $\rho(\theta)$ uniquely.

A. Weak-coupling limit

This limit has been solved previously in Ref. 1; we repeat the analysis here for completeness. With $r=0$, $g(\theta)$ reduces to $\beta\sqrt{x}$ so that $\mathrm{Im}H = 2\pi\beta\rho(\theta)/(4-x)^{1/2} > 0$ for $x_c \leq x \leq 4$. Since $\rho(0)$ is in general nonzero, we immediately obtained a unique function $H(x)$ satisfying (3.9) and (3.10),

$$H(x) = 2\beta^2\{1 - [(x-x_c)]^{1/2}/[(x-4)]^{1/2}\}\ , \quad (3.11)$$

so that the spectral density is

$$\rho(\theta) = \left(\frac{2\beta}{\pi}\right)\left(\sin^2\frac{\theta_c}{2} - \sin^2\frac{\theta}{2}\right)^{1/2}\ , \quad (3.12)$$

where $0 \leq \theta_c \leq \pi$ is determined by the normalization, Eq. (3.5). This, in turn, indicates that the weak-coupling solution Eq. (3.12) is valid only for $\beta \geq \beta_c = \pi/8$. When $\beta = \beta_c$, $\theta_c = \pi$.

B. Strong-coupling limit

In terms of $H(x)$, the strong-coupling condition (3.6) or (2.9) can be reduced to

$$H(-r/4\beta^2) = 2\beta\ . \quad (3.13)$$

Since $\theta_c = \pi$ in this region, and both $\rho(0)$ and $\rho(\pi)$ are in general nonzero, a unique solution for $H(x)$ satisfying (3.9), (3.10), and (3.13) is

$$H(x) = 2\beta^2\left\{1 - \frac{x + r/4\beta^2}{[x(x-4)]^{1/2}}\right\} = 2\beta^2\left\{1 - \frac{g^2(\theta)/\beta^2}{[x(x-4)]^{1/2}}\right\}\ , \quad (3.14)$$

so that, from (3.10),

$$\rho(\theta) = \frac{g(\theta)}{\pi} = \frac{2\beta}{4}\left(\lambda - \sin^2\frac{\theta}{2}\right)^{1/2} \quad (3.15)$$

($\lambda \equiv 1 + r/16\beta^2$). It is clear either from the construction of $H(x)$ or directly from (3.15) that $\rho(\theta)$ satisfies the strong-coupling condition (3.6) automatically. The constant r, $0 \leq r \leq 1$, is again determined by the normalization condition (3.5), i.e.,

$$\frac{2\beta}{\pi}\int_{-\pi}^{\pi} d\phi\left[\left(1 + \frac{r}{16\beta^2}\right) - \sin^2\frac{\phi}{2}\right]^{1/2} = 1\ , \quad (3.16)$$

which is possible only for $\beta \leq \beta_c = \pi/8$. As $\beta \to \beta_c^-$, r vanishes; Eq. (3.15) then agrees with the limit of Eq. (3.12) with $\beta \to \beta_c^+$.

It is interesting to note that in both the weak- and the strong-coupling regions, our spectral density (3.12) and (3.15) coincides in form with that found by Jevicki and Sakita[9] for an apparently unrelated problem: the Hamiltonian problem for a single propagating plaquette. There is a vague hint of some sort of universality such that a few common features (four independent matrix variables with a dominant purely kinetic interaction) seem to determine the $N=\infty$ behavior independent of the details of the model.

For completeness, we also solve in Appendix A the strong-coupling three-link chiral chain previously considered by Friedan.[6] Our analysis has the advantage of being much less involved than that of Friedan. Although our result agrees with his, our analysis bears no apparent relation to his in intermediate steps.

Finally, we would like to comment on the relationship of our approach to the Makeenko-Migdal (MM) equation for loop averages. As shown by Paffuti and Rossi[2] and by Friedan[6] for two- and three-plaquette problems, our steepest-descent analysis for determining the spectral density $\rho(\theta)$ is equivalent to a purely algebraic approach obtained by restricting the MM equation on the lattice to an appropriately chosen subset of loops on which the equation closes. This can also be established for the four-plaquette problem. An algorithm can then be found so that a general loop average can always be expressed algebraically in terms of the generating functional for the subset of loops appropriate for the density $\rho(\theta)$ (which we have obtained without directly using the MM loop equation).

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APPENDIX A: THREE-LINK CHIRAL CHAIN IN STRONG-COUPLING REGION

For the $L=3$ chiral chain, the spectral density satisfies

$$(2\beta\sin\theta)\left(1 + \frac{\beta}{g(\theta)}\right) = P\int d\phi\,\rho_3(\phi)\left[\cot\left(\frac{\theta-\phi}{2}\right) - \frac{g'(\theta)}{g(\theta)+g(\phi)}\right]\ , \quad \text{(A1)}$$

where $g(\theta)$ is given by (3.4). In terms of the variable $z=e^{i\theta}$, the right-hand side of (A1) becomes the principal part of an analytic function

$$F_3(z)\equiv\frac{1}{2}\oint\frac{dz'}{z'}\,\rho_3(z')\left[1+\frac{g(z')}{g(z)}\right]\left[\frac{z+z'}{z-z'}\right] \tag{A2}$$

$$\equiv H_1(z)+\frac{1}{g(z)}H_2(z)\ , \tag{A3}$$

where H_1 and H_2 are analytic in the interior of the unit circle.

To exploit the analyticity structure, we first change the variable from z to x, Eq. (3.8), then to

$$y\equiv(x+c)^{1/2}=\frac{g(\theta)}{\beta}\ , \tag{A4}$$

where $c=r/4\beta^2$. One can then show that

$$H_3(y)\equiv-\frac{(x+c)^{1/2}}{[x(4-x)]^{1/2}}\,F_3(z) \tag{A5}$$

is real analytic in y, with branch cut between $\sqrt{c}$ and $(4+c)^{1/2}$, and on the cut,

$$\mathrm{Re}H_3=\beta(1+y)\ . \tag{A6}$$

In terms of $H_3(y)$, the strong-coupling condition (3.6) becomes

$$H_3(0)=\beta \tag{A7}$$

and the normalization condition (3.5) becomes an asymptotic condition

$$H_3(y)\underset{y\to\infty}{\sim}\frac{-1}{2y}\ . \tag{A8}$$

Equations (A6) and (A7) allows us to determine $H_3(y)$ uniquely. We find

$$H_3(y)=\beta(1+y)-\frac{\beta y(y+b)}{[(y-\sqrt{c}\,)(y-(4+c)^{1/2})]^{1/2}}\ , \tag{A9}$$

where

$$b\equiv(1+\delta^{-1})\equiv1-\frac{\sqrt{c}+(4+c)^{1/2}}{2}\ . \tag{A10}$$

Furthermore, by enforcing (A8), one obtains

$$\delta^2-\delta^{-1}=\beta^{-1}\ , \tag{A11}$$

which relates c (thus r and $\bar{k}$) to β. It then follows that the spectral density in the strong-coupling domain is

$$\rho_3(\theta)=\frac{\beta}{2\pi}(y+b)[(y+\sqrt{c}\,)(y+\sqrt{4+c})]^{1/2}\ . \tag{A12}$$

The crossover point occurs at $c=r=0$, which, from (A10), corresponds to $\delta=-1$, $b=0$, and from (A11), $\beta_c=\frac{1}{3}$, as expected.[1,6] At $\beta=\frac{1}{3}$, we obtain

$$\rho_3(\theta)=\frac{1}{3\pi}\left(\cos\frac{\theta}{4}\right)\left(2\cos\frac{\theta}{2}\right)^{3/2} \tag{A13}$$

in agreement with the weak-coupling limit result obtained in Ref. 1.

Lastly, let us compare our result with that of Friedan. From (A2) and (A3), it can be seen that $H_1(z)$ is related to the $D(z)$ function of Friedan by

$$H_1(z)=iD(z)-i\beta(z-z^{-1})\ . \tag{A14}$$

From (A3) and (A5), we can show that

$$P(z)\equiv\beta^{-1}z^2D^2(z)=z^3+a(\beta)\beta^{-2}z^2+\beta^{-1}z+1\ , \tag{A15}$$

which is the key result of Friedan, obtained by a somewhat involved reasoning.

With the help of (A11), the coefficient $a(\beta)$ in (A15) can be expressed as

$$a(\beta)=\beta^2(\delta^{-2}-2\delta)\ , \tag{A16}$$

which can be shown to be a function of β^3 only, therefore satisfying the strong-coupling expansion constraint. Finally, by identifying δ with z_2 of Friedan, the demand that $P(z)$ has a double zero at z_2, $|z_2|\leq1$, leads to

$$z_2^{\,3}-\beta^{-1}z_2-2=0\ . \tag{A17}$$

Alas, this mysterious condition of Friedan can now be understood as the normalization condition (A11), where

$$z_2=\delta=-\left(\frac{(4+c)^{1/2}-\sqrt{c}}{2}\right)\ . \tag{A18}$$

Note that $|z_2|\leq1$ precisely corresponds to our strong-coupling condition $c\geq0$. Therefore, Friedan's result agrees with our strong-coupling solution, (A12).

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[1] R. C. Brower, P. Rossi, and C.-I Tan, preceding paper, Phys. Rev. D 23, 942 (1981).

[2] Yu. M. Makeenko and A. A. Migdal, Phys. Lett. 88B, 135 (1979). In this connection, see also G. Paffuti and P. Rossi, Phys. Lett. 92B, 321 (1980).

[3] R. C. Brower and M. Nauenberg, University of California, Santa Cruz report (unpublished).

[4] E. Brezin and D. Gross, Santa Barbara Report No. NSF-ITP-80-42 (unpublished).

[5] D. Gross and E. Witten, Phys. Rev. D 21, 446 (1980).

[6]D. Friedan, Berkeley report (unpublished).
[7]J. Kogut and J. Shigemitsu, Phys. Rev. Lett. 45, 410 (1980); R. Brower, G. Cristofano, and J. Rudnick, University of California, Santa Cruz report (unpublished).
[8]I. Bars and F. Green, Phys. Rev. D 20, 3311 (1979).
[9]A. Jevicki and B. Sakita, Phys. Rev. D 22, 467 (1980); R. Wadia, Phys. Lett. 93B, 403 (1980).