Simplicial chiral models

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Principal chiral models on a $(d-1)$-dimensional simplex are introduced and studied analytically in the large N limit. The $d = 0,\ 2,\ 4,$ and ∞ models are explicitly solved. The relationship with standard lattice models and with few-matrix systems in the double scaling limit is discussed.

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The importance of understanding the large N limit of matrix-valued field models cannot be overestimated. Not only is this the basic ingredient of the $1/N$ expansion in the physically relevant case of QCD, but also the existence of large N criticalities for finite values of the coupling is the starting point for the approach to two-dimensional quantum gravity known as the double scaling limit. Moreover, we note that solutions of few-matrix systems may have a direct application to more complex systems in the context of strong-coupling expansion, since they may be reinterpreted as generating functionals for classes of group integrals that are required in strong-coupling calculations [1]. Unfortunately, our knowledge of exact solutions for the large N limit of unitary matrix models is still impressively poor. After the solution of Gross and Witten [2] of the single link problem, exact results were obtained only for the external field problem and a few toy models ($L = 3, 4$ chiral chains) [3,4].

In this Brief Report we introduce a new class of lattice chiral models, whose large N behavior can be analyzed by solving an integral equation for the eigenvalue distribution of a single Hermitian semidefinite positive matrix. The models we are going to study are principal chiral models, with a global $\mathrm{U}(N)\times\mathrm{U}(N)$ symmetry, defined on a $(d-1)$-dimensional simplex formed by connecting in a fully symmetric way d vertices by $(d-1)(d-2)/2$ links. The partition function for such a system is defined to be

$$Z_d = \int \prod_{i=1}^{d} dU_i \exp\left(N\beta \sum_{i>j=1}^{d} \mathrm{Tr}[U_i U_j^\dagger + U_j U_i^\dagger]\right). \quad (1)$$

Despite its apparent simplicity, this class of models includes most of the previously known solvable systems. As a function of the parameter d, which specifies the coordination number of lattice sites, it interpolates between the two-dimensional Gross-Witten model (with third order phase transition) and the infinite-dimensional mean field solution (showing a first order phase transition) of standard infinite-volume lattice models. It also includes a case which, in the double scaling limit, corresponds to that of a $c = 1$ conformal field theory.

It is possible to eliminate the direct interactions among the unitary matrices U_i by introducing an identity in the form

$$1 = \frac{\int dA \exp\left[-N\beta\,\mathrm{Tr}\left(A - \sum_{i=1}^{d} U_i\right)\left(A^\dagger - \sum_{j=1}^{d} U_j^\dagger\right)\right]}{\int dA \exp(-N\beta\,\mathrm{Tr}AA^\dagger)}, \quad (2)$$

where A is an $N\times N$ complex matrix. As a consequence we obtain $Z_d = \tilde{Z}_d/\tilde{Z}_0$ where

$$\tilde{Z}_d = \int \prod_{i=1}^{d} dU_i dA \exp\left\{-N\beta\,\mathrm{Tr}AA^\dagger + N\beta\,\mathrm{Tr}A\sum_i U_i^\dagger + N\beta\,\mathrm{Tr}A^\dagger \sum_i U_i - N^2\beta d\right\}. \quad (3)$$

We now introduce the function

$$F[BB^\dagger] = \frac{1}{N^2}\ln \int dU \exp\left(\frac{N}{2}\mathrm{Tr}[BU^\dagger + UB^\dagger]\right). \quad (4)$$

F is a known function of the eigenvalues x_i of the Hermitian semipositive definite matrix $BB^\dagger$. More specifically, in the large N limit, we know that [5,6]

$$F(x_i) = \frac{1}{N}\sum_i (r + x_i)^{1/2} - \frac{1}{2N^2}\sum_{i,j}\ln\left[\frac{(r+x_i)^{1/2} + (r+x_j)^{1/2}}{2}\right] - \frac{r}{4} - \frac{3}{4}, \quad (5)$$

and there are two distinct phases: (a) weak coupling $r = 0$; (b) strong coupling $(1/N)\sum_i (r + x_i)^{-1/2} = 1$. Up to irrelevant factors, it follows that

$$\tilde{Z}_d = \int dB \exp\left\{-\frac{N}{4\beta}\mathrm{Tr}BB^\dagger + N^2 dF(BB^\dagger) - N^2\beta d\right\}, \quad (6)$$

where B replaces $2\beta A$.

Morris [7] has shown that the angular integration can be performed in the case of complex matrices, and, again up to irrelevant factors, we may replace Eq. (6) by

$$\tilde{Z}_d = \int_0^\infty dx_i \prod_{i>j}(x_i - x_j)^2 \exp\left\{ -\frac{N}{4\beta}\sum_i x_i + N^2 dF(x_i) - N^2\beta \right\}. \tag{7}$$

In the large N limit it is legitimate to evaluate this integral by a saddle-point method. The saddle-point equation resulting from Eqs. (5) and (7) is

$$\frac{\sqrt{r+x_i}}{2\beta} - d = \frac{1}{N}\sum_{j\neq i}\frac{(4-d)\sqrt{r+x_i} + d\sqrt{r+x_j}}{x_i - x_j}. \tag{8}$$

We introduce a new variable $z_i \equiv \sqrt{r+x_i}$ and, in the large N limit, we assume that these eigenvalues lie in a single interval $[a, b]$. Denoting the large N eigenvalue density by $\rho(z)$, Eq. (8) becomes an integral equation for $\rho(z)$,

$$\frac{z}{2\beta} - d = \fint_a^b dz'\rho(z')\left[\frac{2}{z-z'} - \frac{(d-2)}{z+z'}\right], \tag{9}$$

where the integration region is restricted by the condition $0 \leq a \leq b$, with a and b determined dynamically. In particular, the normalization condition

$$\int_a^b \rho(z')dz' = 1 \tag{10}$$

must always be satisfied. Furthermore, one has the constraint

$$\int_a^b \rho(z')\frac{dz'}{z'} \leq 1, \tag{11}$$

with the equality holding exactly in the strong-coupling region where $a = \sqrt{r}$.

Let us begin by first discussing several simple cases where Eq. (9) can be solved readily. For $d = 0$, the problem reduces to one with a pure Gaussian interaction, and, by a more or less standard technique, one finds that

$$\rho(z) = \frac{1}{4\pi\beta}\sqrt{16\beta - z^2}. \tag{12}$$

For $d = 0$, there is no weak-coupling phase. Note also that, since the F term in Eq. (6) vanishes for $d = 0$, Eq. (12) is obtained with $a = 0$. As a consequence, up to a constant,

$$\tilde{Z}_0 = \exp(N^2 \ln\beta), \tag{13}$$

as expected.

When $d = 2$ we obtain

$$\rho_w(z) = \frac{1}{4\pi\beta}\sqrt{8\beta - (z-4\beta)^2}, \quad \beta \geq \tfrac{1}{2},$$

$$\rho_s(z) = \frac{1}{4\pi\beta}z\sqrt{\frac{(1+6\beta)-z}{z-(1-2\beta)}}, \quad r(\beta) = (1-2\beta)^2, \quad \beta \leq \tfrac{1}{2}, \tag{14}$$

and one may show that all results are consistent with a reinterpretation of the model as a Gross-Witten [2] one-plaquette model, with $\beta_c = \frac{1}{2}$.

When $d = 3$ the model can be mapped into the three-link chiral chain, which is known to possess a third order phase transition at $\beta_c = \frac{1}{3}$ [3].

The first nontrivial and new situation begins at $d = 4$. We have explicitly solved the $d = 4$ model, both in the weak- and in the strong-coupling phase. The eigenvalue density may be expressed in terms of elliptic integrals, and supplementary conditions allow for the determination of a and b.

Introducing the variable $k(\beta) = \sqrt{1 - a^2/b^2}$, in weak coupling we obtain, in terms of standard elliptic integrals K, Π, and E,

$$\rho_w(z) = \frac{b}{2\pi^2\beta}(z^2 - a^2)^{1/2}(b^2 - z^2)^{-1/2} \times \left[K(k) - \frac{z^2}{b^2}\Pi\left(1 - \frac{z^2}{b^2}, k\right)\right], \tag{15}$$

with the condition $4\pi\beta = bE(k)$. In strong coupling we have

$$\rho_s(z) = \frac{1}{2\pi^2\beta}\frac{z^2}{b}(z^2 - a^2)^{-1/2}(b^2 - z^2)^{-1/2} \times \left[(b^2 - a^2)K(k) - (z^2 - a^2)\Pi\left(1 - \frac{z^2}{b^2}, k\right)\right], \tag{16}$$

with the condition $4\pi\beta = b[E(k) - (a^2/b^2)K(k)]$. In both regimes, Eq. (10) must also be satisfied. Closed form solutions for the constraints may be obtained at criticality: when $\beta = \beta_c = \frac{1}{4}$, we get $a = 0$, $b = \pi$, and

$$\rho_c(z) = \frac{z}{\pi^2}\ln\frac{1 + \sqrt{1 - z^2/\pi^2}}{1 - \sqrt{1 - z^2/\pi^2}}. \tag{17}$$

Let us finally observe that a large d solution of Eq. (9) may easily be found by assuming $\rho(z) \to \delta(z - \bar{z})$. The weak-coupling solution is

$$\bar{z} = \beta d\left[1 + \sqrt{1 - \frac{1}{\beta d}}\right], \quad \beta \geq \beta_c = \frac{1}{d}, \tag{18}$$

and for strong coupling $\bar{z} = 0$. Amusingly enough, this solution turns out to coincide with the large $D \equiv d/2$ mean field solution [8] of infinite-volume principal chiral models on D-dimensional hypercubic lattices with the same coordination number as our corresponding models.

We would like to add a few comments. Solving Eq. (9) is certainly a well defined problem for any value of d, and in particular we expect to be able to find explicit solutions for simple cases, such as $d = 1$ and $d = 3$. It is also possible to analyze Eq. (9) numerically; details of our analytical and numerical techniques will be reported elsewhere; we only mention that for sufficiently large $\beta > \beta_c$ we can get the eigenvalue distribution with desired accuracy, while near criticality convergence is slow: however within 1% accuracy we have evidence that $\beta_c = 1/d$ for all integer values of d [9]. It would be quite interesting to achieve more information, both qualitative and quantitative, on the d dependence of the phase transition.

The thermodynamical quantity whose computation is easiest is the internal energy per unit link, w_1, which may be obtained from

$$d(d-1)w_1 = \frac{1}{4\beta^2}\int_a^b dz'\rho(z')(z'^2 - r) - d - \frac{1}{\beta}. \quad (19)$$

One may then extract, in the vicinity of β_c, the critical exponent for the specific heat, α. At present we know that when $d = 2, \alpha = -1$, when $d = 3, \alpha = -\frac{1}{2}$, when $d = 4, \alpha = 0$, and for sufficiently large d the transition is first order, i.e., $\alpha = 1$.

It is worth observing in this context that a more general model involving four unitary matrices and three couplings, interpolating between our $d = 4$ case and the four-link chiral chain, can be reexpressed as a model of two coupled complex matrices and admits many solvable limits, all characterized by $\alpha = 0$, which corresponds to a $c = 1$ conformal field theory.

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