

k -string tensions in $SU(N)$ gauge theoriesLuigi Del Debbio,^{1,*} Haralambos Panagopoulos,^{2,†} Paolo Rossi,^{1,‡} and Ettore Vicari^{1,§}¹*Dipartimento di Fisica dell'Università di Pisa and I.N.F.N., I-56127 Pisa, Italy*²*Department of Physics, University of Cyprus, Nicosia CY-1678, Cyprus*

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In the context of four-dimensional $SU(N)$ gauge theories, we study the spectrum of the confining strings. We compute, for the $SU(6)$ gauge theory formulated on a lattice, the three independent string tensions σ_k related to sources with Z_N charge $k=1,2,3$, using Monte Carlo simulations. Our results, whose uncertainty is approximately 2% for $k=2$ and 4% for $k=3$, are consistent with the sine formula $\sigma_k/\sigma = \sin(k\pi/N)/\sin(\pi/N)$ for the ratio between σ_k and the standard string tension σ , and show deviations from the Casimir scaling. The sine formula is known to emerge in supersymmetric $SU(N)$ gauge theories and in M theory. We comment on an analogous behavior exhibited by two-dimensional $SU(N) \times SU(N)$ chiral models.

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Quantum chromodynamics is a non-Abelian gauge theory based on the gauge group $SU(3)$. The mechanisms underlying many of its fundamental properties, such as confinement, chiral symmetry, topological effects and the axial anomaly, are under active investigation; they are being studied by different approaches, including numerical simulations of the theory formulated on the lattice, several models of the vacuum, as well as some recent proposals derived from M theory and the AdS conformal field theory (CFT) correspondence. Many features of QCD can be better understood by extending the study to $SU(N)$ gauge theories with N larger than three and in particular by examining the large- N limit.

Four-dimensional gauge theories exhibit confinement; i.e. static sources in the fundamental representation develop a linear potential characterized by a string tension σ . As pointed out in many studies, it is important to investigate the behavior of the system in the presence of static sources in representations higher than the fundamental one. This may provide useful hints on the mechanism responsible for confinement, helping to identify the most appropriate models of the QCD vacuum and to select among the various confinement hypotheses. Among the latter, the so-called Casimir scaling hypothesis for the potential between heavy-quark sources in different representations has attracted much interest (see e.g. the recent publications [1–5]).

$SU(N)$ gauge theories confine by means of chromoelectric flux tubes carrying a charge in the center Z_N of the gauge group. A chromoelectric source of charge k with respect to Z_N is confined by a k string with string tension σ_k ($\sigma_1 \equiv \sigma$ is the string tension related to the fundamental representation). If $\sigma_k < k\sigma$, then a string with charge k is stable against decay to k strings of charge one. Charge conjugation implies $\sigma_k = \sigma_{N-k}$. Therefore $SU(3)$ has only one independent string tension determining the large distance behavior of the potential for $k \neq 0$. One must consider larger values of N to look for distinct k strings.

As pointed out in Ref. [6], it is interesting to compare the k -string tension ratios

$$R(k,N) \equiv \frac{\sigma_k}{\sigma} \quad (1)$$

in different theories. The idea is that such ratios may reveal a universal behavior within a large class of models characterized by $SU(N)$ symmetry, such as $SU(N)$ gauge theories and their supersymmetric extensions. It has been noted that stable k strings are related to the totally antisymmetric representations of rank k , and that in various realizations of supersymmetric $SU(N)$ gauge theories $R(k,N)$ satisfies the sine formula $R(k,N) = S(k,N)$ where

$$S(k,N) \equiv \frac{\sin(k\pi/N)}{\sin(\pi/N)}. \quad (2)$$

$R(k,N)$ has been computed for the $\mathcal{N}=2$ supersymmetric $SU(N)$ gauge theory softly broken to $\mathcal{N}=1$ [7,8], obtaining Eq. (2). The same result is found also in the context of M theory, and extended to the case of large breaking of the $\mathcal{N}=2$ supersymmetric theory [8]. An interesting question is whether the sine formula holds in nonsupersymmetric $SU(N)$ gauge theories. The M-theory approach to nonsupersymmetric QCD, although it is still at a rather speculative stage, suggests that it may be so [9,8]. However, as discussed in Refs. [8,6], corrections from various sources cannot be excluded, so that this prediction cannot be considered robust.

Another interesting and suggestive hypothesis is that the k -string tension ratio satisfies the so-called Casimir scaling law [10], i.e. $R(k,N) = C(k,N)$ where

$$C(k,N) \equiv \frac{k(N-k)}{N-1} \quad (3)$$

is the ratio between the values of the quadratic Casimir operators in the rank- k antisymmetric and in the fundamental representations. The Casimir ratio is satisfied on the one hand by the strong-coupling limit of the lattice Hamiltonian formulation of $SU(N)$ gauge theories [11], and on the other hand by the small-distance behavior of the potential between

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two static charges in different representations, as shown by perturbation theory up to two loops [12]. Interest in Casimir scaling was recently revived [1–5]; it has been triggered by numerical studies of SU(3) lattice gauge theory [3,4], which indicate that Monte Carlo data for the potential between charges in different representations are consistent with Casimir scaling up to a relatively large distance, $r \approx 1$ fm.

The Casimir scaling law holds exactly in two-dimensional QCD. In higher dimensions no strong arguments exist in favor of a mechanism preserving Casimir scaling across the roughening transition, from strong to weak coupling; nor from small distance (essentially perturbative, characterized by a Coulombic potential) to large distance (characterized by a string tension for sources carrying Z_N charge). We have shown explicitly [13] that Casimir scaling does not survive the next-to-leading order calculation of the ratios $R(k, N)$ in the strong-coupling lattice Hamiltonian approach.

It is interesting to note that the sine formula (2) emerges also in the context of the two-dimensional $SU(N) \times SU(N)$ chiral models. As amply discussed in the literature (see e.g., Refs. [14,15] and references therein), d -dimensional chiral models and $2d$ -dimensional lattice gauge theories manifest deep analogies in the continuum and on the lattice. Indeed, one may establish the following correspondence table for the lattice formulations:

<i>Chiral Models</i>	<i>Gauge Models</i>
site, link	link, plaquette
loop	surface
length	area
mass M	string tension σ
two-point correlation	Wilson loop

One may also add to this table the bound state masses M_k of chiral models and the k -string tensions σ_k of gauge theories. In particular, in the case $d=1$ the relation is exact, and one can prove that Casimir scaling holds for the masses of the bound states. In analogy to four-dimensional SU(N) gauge theories, in two-dimensional $SU(N) \times SU(N)$ chiral models the Casimir scaling law holds for the strong-coupling limit of the corresponding lattice Hamiltonians (but it is not satisfied by the corrections) and for the small-distance behavior of the correlation functions related to different representations. On the other hand, the exact S matrix [16], derived using the existence of an infinite number of conservation laws and Bethe-ansatz methods, dictates that all bound states belong to the rank- k antisymmetric representations and satisfy the sine formula, $M_k = M \sin(k\pi/N)/\sin(\pi/N)$, where M_k is the mass of the k -particle bound state. The question arises again: does this result extend to four-dimensional SU(N) gauge theories?

This issue can be investigated numerically using the lattice formulation of SU(N) gauge theories. Recent numerical results for $R(2, N)$, obtained for $N=4,5$ [17,18], show that $R(2, N) < 2$; thus, $\sigma_2 < 2\sigma$, indicating that flux tubes attract each other. However, the error estimates on $R(2, N)$ do not allow any exclusion of the two above-mentioned hypotheses. Indeed the sine and Casimir formulas give numerically close predictions for $k=2$, so that high accuracy is necessary to distinguish them.

TABLE I. MC results for the k -string tensions.

γ	Lattice	$a\sqrt{\sigma}$	σ_2/σ	σ_3/σ
0.342	$8^3 \times 20$	0.3151(6)	1.65(2)	1.91(3)
	$12^3 \times 24$	0.3239(8)	1.66(3)	1.91(9)
0.344	$12^3 \times 24$	0.2973(5)	1.73(2)	1.95(5)
	$10^3 \times 20$	0.2534(6)	1.73(3)	2.08(10)
0.348	$12^3 \times 24$	0.2535(6)	1.71(4)	2.06(11)
	$12^3 \times 24$	0.2380(6)	1.72(3)	1.95(9)
0.354	$12^3 \times 24$	0.2103(5)	1.73(3)	2.04(6)

In this work we further investigate the spectrum of the string states. We present results from Monte Carlo (MC) simulations of the four-dimensional SU(6) lattice gauge theory using the Wilson formulation. For $N=6$ there are two nontrivial k -string tensions besides the fundamental one, thus providing a stringent test of the various ideas discussed above. We anticipate here our final results for the two independent k -string tension ratios:

$$R(2,6) = 1.72 \pm 0.03, \quad (4)$$

$$R(3,6) = 1.99 \pm 0.07. \quad (5)$$

They are both consistent with the predictions of the sine formula (2), which are $S(2,6) = 1.732 \dots$ and $S(3,6) = 2$, respectively. On the other hand, our results show deviations from Casimir scaling; the predictions in this case, $C(2,6) = 1.6$ and $C(3,6) = 1.8$, are off by approximately four and three error bars, respectively.

In our simulations we employed the Cabibbo-Marinari algorithm [19] to upgrade SU(N) matrices by updating their SU(2) subgroups (we selected 15 subgroups). This was done by alternating microcanonical over-relaxation and heat-bath steps, typically in a 4:1 ratio. Table I contains some information on our MC runs: The coupling values [20] $\gamma \equiv \beta/(2N^2)$, lattice sizes, and the results for the k -string tensions. The number of sweeps per run was typically above 500k, and measurements were taken every 10–20 sweeps. The values of γ were chosen to lie beyond the first order phase transition which occurs in the Wilson formulation of SU(N) gauge theories for N sufficiently large (see e.g. Refs. [21,13]). In order to determine the value γ_c where the first order transition occurs, we performed simulations starting from hot and cold configurations to display hysteresis, and from mixed-phase configurations, obtaining the estimate $\gamma_c = 0.3389(4)$. We used asymmetric lattices ($L^3 \times T$) with a larger time size. For some values of γ we performed simulations for two lattice sizes, to check for finite size effects. The lattice sizes L were chosen so that $L\sqrt{\sigma} \geq 2.5$, and for most of them $L\sqrt{\sigma} \approx 3$. This requirement ensures that finite size effects on k -string ratios are negligible, as can be seen by comparison of the results for different sizes (see also Refs. [17]). Further confirmation comes from preliminary results (≈ 200 k sweeps) on a $16^3 \times 32$ for $\gamma = 0.350$.

In our simulations we have also measured the topological charge Q by a cooling technique. A severe form of critical slowing down is observed in this case: The autocorrelation

time τ_Q for Q appears to increase exponentially, $\tau_Q \propto \exp(c/\sigma^{1/2})$ with $c \approx 2.4$. As a consequence, the run for the largest value of γ considered, $\gamma = 0.354$, did not correctly sample Q , presumably because it was not sufficiently long ($\approx 300k$ sweeps). This dramatic effect was not observed in the correlations used to determine the k -string tensions (a blocking analysis did not show significant time correlations in measurements taken every 10–20 sweeps), suggesting an approximate decoupling between the topological and nontopological modes. This suggestion is also supported by the fact that string tension results for $\gamma = 0.354$, extracted from a simulation which did not sample correctly Q , turn out to be in agreement with those for smaller γ , for which Q was sampled correctly. We note incidentally that this phenomenon has already been observed in simulations of two-dimensional CP^{N-1} models [22].

The k -string tensions are extracted from the large time behavior of correlators of strings in the antisymmetric representations, closed through periodic boundary conditions (see e.g. Refs. [23,17]):

$$C_r(t) = \sum_{x_1, x_2} \langle \chi_r[P(0;0)] \chi_r[P(x_1, x_2; t)] \rangle \quad (6)$$

where $P(x_1, x_2; t) = \prod_{x_3} U_3(x_1, x_2, x_3; t)$, $U(\mathbf{x}; t)$ are the usual link variables, and χ_r is the character of the representation r . In particular, $\chi_f[P] = \text{Tr } P$ for the fundamental representation, and

$$\chi_{k=2}[P] = \text{Tr } P^2 - (\text{Tr } P)^2, \quad (7)$$

$$\chi_{k=3}[P] = 2\text{Tr } P^3 - 3\text{Tr } P^2 \text{Tr } P + (\text{Tr } P)^3 \quad (8)$$

for the antisymmetric representations of rank $k=2$ and $k=3$, respectively.

These correlators decay exponentially as $\exp(-m_k t)$ where m_k is the mass of the lightest state in the corresponding representation. For a k -loop of size L , the k -string tension is obtained using the relation [23]

$$m_k = \sigma_k L - \frac{\pi}{3L}. \quad (9)$$

The last term in Eq. (9) is conjectured to be a universal correction, and it is related to the universal critical behavior of the flux excitations described by a free bosonic string [24,23]. In order to improve the efficiency of the measurements we used smearing and blocking procedures (see e.g. Refs. [25]) to construct new operators with a better overlap with the lightest state. The masses m_k were obtained by fitting the time dependence of the correlations. The fitting range is the source of systematic error; we have checked that all reasonable choices of this range yield consistent results within the quoted errors. More details on the Monte Carlo simulations and the analysis of the data will be reported elsewhere [13].

The results for the ratios $R(2,6)$ and $R(3,6)$ are presented in Table I, and plotted in Figs. 1 and 2 versus $a^2\sigma$ to evidence possible scaling corrections for the case $k=2$ and k

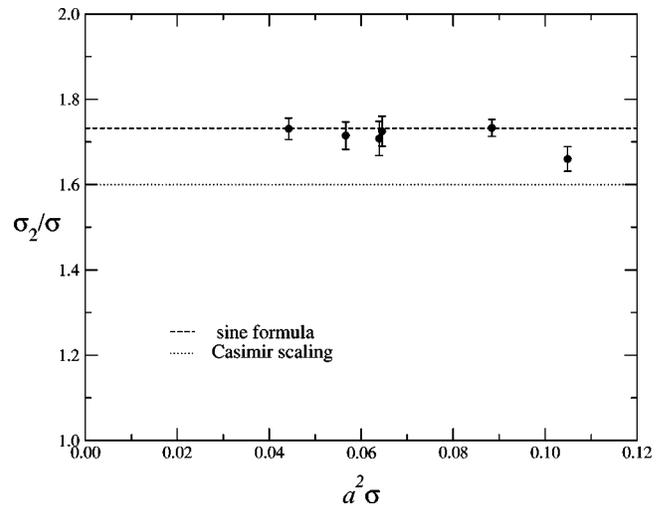


FIG. 1. The scaling ratio $R(2,6)$ as a function of $a^2\sigma$.

$=3$ respectively, together with the sine formula (2) and the Casimir scaling predictions. The ratio $R(2,6)$ shows good scaling for $\gamma \geq 0.344$. Scaling deviations are observed only for $\gamma = 0.342$, and this may be due to the vicinity of the phase transition. The data for $\gamma \geq 0.344$ are consistent with a constant, thus we do not attempt to fit the dependence of our result on the lattice spacing a . Our final value for $R(2,6)$ is obtained by combining the results at $\gamma = 0.348$ (for the largest lattice) and $\gamma = 0.350$. The error we report is given by the typical error of each single point. Of course, this estimate assumes that the scaling corrections are small and negligible for $a^2\sigma \approx 0.05$. The data for smaller γ , and in particular the one for $\gamma = 0.344$, are essentially used to check this fact. They suggest that the scaling corrections are at most of the same size of the error we report. MC runs at the largest value of γ , i.e. $\gamma = 0.354$, show the aforementioned decoupling of the string tensions from the topological degrees of freedom; however, given the poor sampling of the topological charge in those runs, we do not include them in the final estimate of the string tension ratio. Similar comments apply to the $R(3,6)$ ratio.

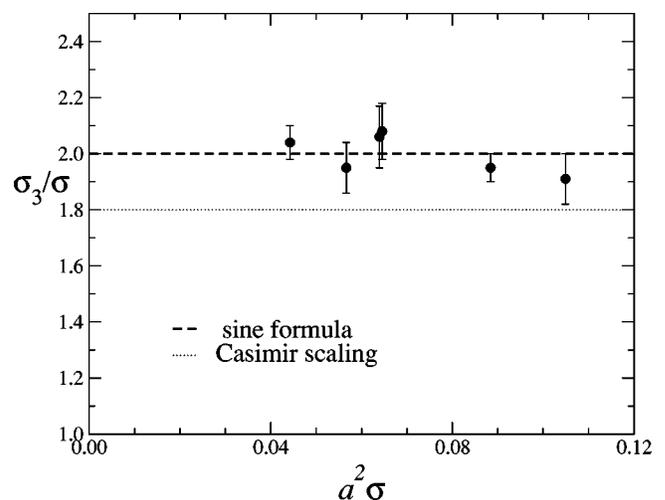


FIG. 2. The scaling ratio $R(3,6)$ as a function of $a^2\sigma$.

We have also explored correlators in the symmetric rank-2 representation, finding no evidence for stable bound states, in accordance with general arguments and with the spectrum of chiral models.

Our final estimates have been reported in Eqs. (4) and (5). They show deviations from Casimir scaling [26]. It is worthwhile to emphasize that such corrections are to be expected, as mentioned previously. This fact is further confirmed by the computation of the ratio σ_k/σ to $O(g^{-8})$ in the strong-coupling expansion of the lattice Hamiltonian formulation of d -dimensional $SU(N)$ gauge theories. We obtained [13]

$$\frac{\sigma_k}{\sigma} = \frac{k(N-k)}{N-1} \left[1 + \frac{(d-2)f(k,N)}{(g^2N)^4} + \dots \right] \quad (10)$$

where $f(k,N)$ is explicitly k dependent. In particular $f(2,N) = 6/N + O(1/N^2)$.

In conclusion, we claim that our numerical results for the four-dimensional $SU(6)$ gauge theory are consistent with the sine formula and the universality hypothesis that is behind it. Of course, they do not prove that it holds exactly. But they

put a stringent bound on the size of the possible corrections. On the other hand, our results show a clear evidence of deviations from the Casimir scaling. This fact should be relevant for the recent debate on confinement models, such as those discussed in Refs. [1,2,5]. However, Casimir scaling may still be considered as a reasonable approximation, since the largest deviations we observed were about 10%.

One last remark regards the large- N behavior of the sine formula: $S(k,\infty) = k + O(1/N^2)$. In this respect the sine formula is peculiar because there are no *a priori* reasons for the k -string tension ratio to be even in $1/N$. The same observation applies to the two-dimensional $SU(N) \times SU(N)$ chiral models, but there we know that the sine formula holds and it comes from the structure of the S matrix, which is essentially determined by the existence of an infinite number of conservation laws.

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