25th-order high-temperature expansion results for three-dimensional Ising-like systems on the simple-cubic lattice

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(Received 14 January 2002; published 27 June 2002)

25th-order high-temperature series are computed for a general nearest-neighbor three-dimensional Ising model with arbitrary potential on the simple cubic lattice. In particular, we consider three improved potentials characterized by suppressed leading scaling corrections. Critical exponents are extracted from high-temperature series specialized to improved potentials, obtaining $\gamma=1.2373(2)$, $\nu=0.63012(16)$, $\alpha=0.1096(5)$, $\eta=0.036\,39(15)$, $\beta=0.326\,53(10)$, and $\delta=4.78\,93(8)$. Moreover, biased analyses of the 25th-order series of the standard Ising model provide the estimate $\Delta=0.52(3)$ for the exponent associated with the leading scaling corrections. By the same technique, we study the small-magnetization expansion of the Helmholtz free energy. The results are then applied to the construction of parametric representations of the critical equation of state, using a systematic approach based on a global stationarity condition. Accurate estimates of several universal amplitude ratios are also presented.

DOI: 10.1103/PhysRevE.65.066127 PACS number(s): 05.70.Jk, 64.60.Fr, 64.10.+h, 75.10.Hk

I. INTRODUCTION

The Ising model is one of the most studied systems in the theory of phase transitions, not only because it is the simplest nontrivial model that has a critical behavior with nonclassical exponents, but also because it describes the critical behavior of many physical systems. Indeed, many systems characterized by short-range interactions and a scalar order parameter undergo a continuous phase transition belonging to the Ising universality class. We mention the liquid-vapor transition in simple fluids and the critical transitions in multicomponent fluid mixtures, in uniaxial antiferromagnetic materials, and in micellar systems. Continuous transitions belonging to the three-dimensional Ising universality class are also expected in high-energy physics, for instance in the electroweak theory at finite temperature and in the theory of strong interactions at finite temperature and finite baryon-number chemical potential. For a recent review, see, e.g., Ref. [1].

The high-temperature (HT) expansion is one of the most efficient approaches to the study of critical phenomena. Very precise results have been obtained by performing careful extrapolations to the critical point, by using several different methods, see, e.g., Ref. [2]. For moderately long series, such as those available for models in the three-dimensional Ising universality class, the nonanalytic confluent corrections are the main source of systematic errors. For instance, according to renormalization-group theory, the critical behavior of the magnetic susceptibility is given by the Wegner expansion,

$$\chi = Ct^{-\gamma}(1 + a_\chi t^\Delta + a_2 t^{2\Delta} + \cdots + b t^{\Delta_2} + \cdots + e_1 t + e_2 t^2 + \cdots), \tag{1}$$

where $t \equiv (T - T_c)/T_c$ is the reduced temperature and Δ is a noninteger exponent, $\Delta \approx 0.5$ in the Ising case. In the analysis of HT expansions these nonanalytic terms introduce large and dangerously undetectable systematic deviations in the results.

In order to obtain precise estimates of the critical parameters, the approximants of the HT series should properly allow for the confluent nonanalytic corrections [3–9]. However, the extensive numerical work that has been done shows that in practice, with the series of moderate length that are available today, no unbiased analysis is able to take effectively into account nonanalytic correction-to-scaling terms. In order to treat them properly, one should use biased methods in which the presence of the leading nonanalytic term with exponent Δ is imposed (see, e.g., Refs. [10–17]). An alternative approach to this problem consists in considering models—we call them *improved*—that do not couple the leading irrelevant operator that gives rise to the confluent correction of order t^Δ. Therefore, such correction does not appear in the expansion of *any* thermodynamic quantity near the critical point: for instance, $a_\chi = 0$ in Eq. (1). In this case, we expect standard analysis techniques to be much more effective, since the main source of systematic error should have been eliminated. There are no methods that allow us to determine exactly improved models, and one must therefore use numerical techniques. One may use HT expansions, but in this case the improved model is determined with a relatively large error [1,6,8,9,18,19] so that the final results do not significantly improve the estimates obtained from standard analyses using biased approximants. Recently [19–27], it has been realized that Monte Carlo (MC) simulations using finite-size scaling techniques are very effective for this purpose, obtaining accurate determinations of several improved models in the Ising, XY, and O(3) universality classes.

As shown in Refs. [19,25,27–29], analyses of the HT series for the improved models lead to a significant improvement in the estimates of the critical exponents and of other

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infinite-volume HT quantities. Our working hypothesis is that, with the series of current length, the systematic errors, i.e., the systematic deviations that are not taken into account in the analysis, are largely due to the leading confluent correction, so that improved models give results with smaller and, more importantly, reliable error estimates. This hypothesis can be checked by comparing the results obtained using different improved models: if correct, they should agree within error bars. In the following we shall report results that confirm our hypothesis. Indeed, the estimates obtained from three different improved Hamiltonians are perfectly consistent. Moreover, they are very stable with respect to the order of the series considered in the analysis, without showing dangerous trends, but only an apparent reduction of the error. The results obtained in Ref. [19] using 20th-order series are fully consistent with the 25th-order analysis that we present.

We consider scalar models on a simple-cubic lattice with Hamiltonian

$$\mathcal{H} = -\beta \sum_{\langle i,j \rangle} \phi_i \phi_j + \sum_i V(\phi_i^2), \tag{2}$$

where $\beta \equiv 1/T$, $\langle i,j \rangle$ indicates nearest-neighbor sites, ϕ_i are real variables, and $V(\phi^2)$ is a generic potential satisfying appropriate stability constraints. These models are expected to have either a critical transition belonging to the Ising universality class or a first-order transition between a disordered and an ordered phase, apart from special cases that correspond to multicritical points. Using the linked-cluster expansion technique, we compute, for an arbitrary potential, the HT expansion of the two-point correlation function to 25th order on a simple-cubic lattice. These results extend those of Ref. [19] that reported the two-point function to 20th order [30]. In particular, we consider three classes of models depending on an irrelevant parameter, which is fixed by requiring the absence of the leading scaling correction. The first one is the ϕ^4 lattice model with potential

$$V(\phi^2) = \phi^2 + \lambda_4 (\phi^2 - 1)^2. \tag{3}$$

MC simulations using finite-size scaling techniques have shown that the model is improved for [31]

$$\lambda_4 = \lambda_4^* = 1.10(2). \tag{4}$$

A consistent but less precise estimate can be obtained from the HT expansion [19]. The second class of models is the ϕ^6 lattice model with potential

$$V(\phi^2) = \phi^2 + \lambda_4 (\phi^2 - 1)^2 + \lambda_6 (\phi^2 - 1)^3. \tag{5}$$

Fixing $\lambda_6 = 1$, the ϕ^6 Hamiltonian is improved for [19]

$$\lambda_4 = \lambda_4^* = 1.90(4). \tag{6}$$

Finally, we consider the spin-1 (or Blume-Capel) Hamiltonian

$$\mathcal{H} = -\beta \sum_{\langle i,j \rangle} s_i s_j + D \sum_i s_i^2, \tag{7}$$

where the variables s_i take the values $0, \pm 1$. An improved spin-1 model is obtained for [32]

$$D = D^* = 0.641(8). \tag{8}$$

The comparison of the results obtained using the above-mentioned improved Hamiltonians represents a strong check of the expected reduction of systematic errors in the HT results, and provides an estimate of the residual errors due to the subleading confluent corrections to scaling.

We also extend the HT expansion of the zero-momentum n-point correlation functions χ_n. In particular, we compute χ_4, χ_6, and χ_8 to 21st, 19th, and 17th order, respectively. The analysis of such series provides information on the small-magnetization expansion of the Helmholtz free energy in the HT phase. These results are used to determine approximate representations of the equation of state that are valid in the critical regime in the whole (t,H) plane. For this purpose, following Ref. [19], we use a systematic approximation scheme based on polynomial parametric representations and on a global stationarity condition. This approach allows us to obtain an accurate determination of the critical equation of state in the whole critical region up to the coexistence curve.

In Table I we anticipate most of the results that we shall obtain in this paper. We report HT estimates of the critical exponents and of the coefficients parametrizing the small-magnetization expansion of the Helmholtz free energy: they are denoted by IHT, where the ''I'' stresses the fact that we are using improved models. Then, we report several amplitude ratios (definitions are given in Sec. V). Those appearing in the column IHT-PR are obtained from an approximate representation of the equation of state that uses the HT results as inputs, those labeled by LT are obtained from the analysis of low-temperature expansions, while those reported under IHT-PR+LT are obtained combining the IHT-PR and LT results. The comparison with the corresponding Table XIII of Ref. [19] shows that the estimates obtained from the 25th-order series are essentially identical to those obtained by using the shorter 20th-order series. However, the longer series allows us to give error bars that are smaller by a factor of 1.5–2, depending on the observable. The estimates reported in Table I are in substantial agreement with, and substantially more precise than, the best theoretical and experimental results that have been previously obtained [10,20–23,35–50]. For a comprehensive recent review of theoretical and experimental results, see Ref. [1]. On the experimental side, we mention the planned experiments in microgravity environment described in Ref. [51], which may substantially improve the experimental determinations of the critical quantities and make the comparison with the theoretical computations more stringent.

After completion of this work, the study reported in Ref. [17] appeared, where analyses of 25th-order series for spin-S models are reported. Results for the critical exponents are obtained by means of biased analyses, essentially by fixing Δ. Comparing Ref. [17] with Refs. [10,13], where 21st- and 23rd-order series are analyzed, a trend appears towards better agreement with improved Hamiltonian results (Ref. [19] and present paper). The latest results of the authors of Ref. [17]

TABLE I. Summary of the results obtained in this paper (unless a reference is explicitly cited) by our high-temperature calculations (IHT), by using the parametric representation of the equation of state (IHT-PR), by analyzing the low-temperature expansion (LT), and by combining the results of the two approaches (IHT-PR+LT). The estimates of critical exponents marked by an asterisk have been obtained using scaling and hyperscaling relations.

		IHT	IHT-PR	LT	IHT-PR+LT
Critical	γ	1.2373(2)			
exponents	ν	0.63012(16)			
	α	0.110(2),*0.1096(5)			
	η	*0.03639(15)			
	β	*0.32653(10)			
	δ	*4.7893(8)			
	Δ	0.52(3)			
	ω	0.83(5)			
	$\omega_{\rm NR}$	2.0208(12) [19,33]			
Small-magnetization	g_4^+	23.56(2)			
expansion of	r_6	2.056(5)			
the free-energy	r_8	2.3(1)			
in the HT phase	r_{10}	$-13(4)$ [19]	$-10.6(1.8)$		
Universal	U_0		0.532(3)		
amplitude	U_2		4.76(2)		
ratios;	U_4		$-9.0(2)$		
see Sec. V	R_c^+		0.0567(3)		
for notations	R_c^-		0.02242(12)		
	R_4^+		7.81(2)		
	v_3		6.050(13)		
	R_4^-		93.6(6)		
	v_4		16.17(10)		
	R_χ		1.660(4)		
	w^2			4.75(4) [14]	
	U_ξ				1.956(7)
	Q^+		0.01880(8)		
	R_ξ^+		0.2659(4)		
	Q^-				0.00472(5)
	Q_c		0.3315(10)		
	g_3^-				13.19(6)
	g_4^-				76.8(8)
	Q_ξ^+	1.000200(3)			
	Q_ξ^-			1.032(4) [34]	
	$U_{\xi_{\rm gap}}$				1.896(10)
	Q_ξ^c				1.024(4)
	Q_2				1.195(10)
	P_m		1.2498(6)		
	P_c		0.3933(7)		
	R_p		1.9665(10)		

are in full agreement with our estimates.

The paper is organized as follows. In Sec. II we report on the HT expansions. Section III reports on the results of our analysis of the HT series for the critical exponents. In Sec. IV we determine approximate representations of the critical equation of state. In Sec. IV A we give the definitions, in Sec. IV B we give estimates of the zero-momentum four-point coupling and of the first few coefficients of the small-magnetization expansion of the equation of state, in Sec. IV C we explain the method, and in Sec. IV D we give the final results. In Sec. V we present estimates of several universal amplitude ratios. In Sec. VI we determine the low-momentum behavior of the two-point function in the HT phase.

TABLE II. Coefficients of the HT series for the improved models. Lower-order coefficients appear in Ref. [19].

n	ϕ^4: $\lambda_4=1.10$	ϕ^6: $\lambda_6=1,\lambda_4=1.90$	spin-1: $D=0.641$
		χ	
21	958465949.119795229380125	55356759.0258594943774739	521863527.549747127784405
22	2581828793.17418316658592	130996257.131383657648562	1367254366.70256684609648
23	6953921835.10625772660286	309956395.981892002096689	3581814299.63029965928082
24	18716342130.2600278822297	732873665.558914443007657	9376338630.49601545283933
25	50369768053.5367726030130	1732674465.68758001711514	24543094928.9205155990856
		m_2	
21	32990320251.5660972216018	1900950559.23375555678011	17908950773.4801706544197
22	94071328367.8146359923071	4762044317.91673448231502	49684326561.5439542757331
23	267461898855.689392585599	11894571003.1970044574018	137433163639.457494472451
24	758423675496.642760823002	29631147101.2512233682029	379139772127.101469600055
25	2145329356955.42924803892	73634162230.2093808561076	1043350926215.22611634874
		m_4	
20	541141652908.631074719231	35399348720.3598637148375	299758906549.791610350073
21	1643345014677.80358819408	94444621918.7858920241050	885976701269.736104292700
22	4961021084766.33884428748	250485298262.046958470064	2603026564263.78069815384
23	14895796670810.3387628037	660748522303.208118944668	7606210964865.32821158574
24	44504475774409.2126174407	1734347627024.93369651634	22115153167519.1984380502
25	132362288688779.709839376	4531641133142.45499870752	64005596692608.8036008995
		χ_4	
19	−141558376231.985023846408	−9210343000.40488445467068	−77210883309.3840433243811
20	−440895445559.088001425635	−25206881115.0765162521666	−234263398532.544236218037
21	−1363771989486.31756523825	−68511054288.5805997438372	−705801443484.646787710146
		χ_6	
18	25922773662329.4681285982	1657400403425.39611029038	13110582140461.8241625980
19	93214547843378.1420243052	5239283130720.37310719268	46080008679021.7095625364
		χ_8	
17	−3021378127745877.943411840	−188904527250502.5683919596	−1360671334948122.792253527

II. HIGH-TEMPERATURE EXPANSION

We considered a simple-cubic lattice and computed the HT expansion of several quantities for a generic lattice model defined by the Hamiltonian (2), using the vertex- and edge-renormalized linked-cluster expansion technique, developed in Refs. [18,52] and described in detail in Ref. [53]. Some technical points that allowed us to extend the computation of Ref. [53] will be reported in a forthcoming publication. We computed the 25th-order HT expansion of the two-point function

$$G(x)=\langle\phi(0)\phi(x)\rangle. \tag{9}$$

In the present context we consider its moments,

$$m_{2j}=\sum_x |x|^{2j}G(x), \tag{10}$$

and therefore, the magnetic susceptibility $\chi\equiv m_0$ and the second-moment correlation length $\xi^2=m_2/(6\chi)$.

We also calculated the HT expansion of the zero-momentum connected $2j$-point correlation functions χ_{2j},

$$\chi_{2j}=\sum_{x_2,\ldots,x_{2j}}\langle\phi(0)\phi(x_2)\cdots\phi(x_{2j-1})\phi(x_{2j})\rangle_c \tag{11}$$

($\chi=\chi_2$). More precisely, we computed χ_4 to 21st order, χ_6 to 19th order, χ_8 to 17th order. The correlation function χ_{10} was computed to 15th order in Ref. [19].

It would be pointless to present here the full results for an arbitrary potential: the resulting expressions are only fit for further computer manipulation. They are available on request. In Table II we give the new coefficients only for the three improved models we have considered, i.e., for the ϕ^4 model at $\lambda_4=1.10$, for the ϕ^6 model at $\lambda_6=1$ and $\lambda_4=1.90$, and for the spin-1 model at $D=0.641$.

For the standard Ising model, we give below the coefficients of the terms that extend the expansions presented in Refs. [15,53] for χ, m_2, and χ_4:

$$\chi = \cdots + 18\,554\,916\,271\,112\,254 v^{24} + 85\,923\,704\,942\,057\,238 v^{25} + O(v^{26}),$$

$$m_2 = \cdots + 977\,496\,788\,431\,483\,776 v^{24} + 4\,767\,378\,698\,515\,169\,334 v^{25} + O(v^{26}),$$

$$\chi_4 = \cdots - 6\,306\,916\,133\,817\,628 v^{18} - 34\,120\,335\,459\,595\,728 v^{19} - 183\,166\,058\,308\,506\,108 v^{20} - 976\,373\,577\,976\,196\,368 v^{21} + O(v^{22}), \quad (12)$$

where $v \equiv \tanh \beta$.

III. THE CRITICAL EXPONENTS

In this section we shall report three different analyses for the critical exponents. In Sec. III A we shall use integral approximants and derive the estimates reported in Table I. In Secs. III B and III C we shall use two other methods that have been recently used in the literature [10,16,17] to confirm the integral-approximant results.

A. Analysis using integral approximants

In order to estimate γ and ν, we analyze the HT series of the magnetic susceptibility and of the second-moment correlation length, respectively. We follow closely Appendix B of Ref. [19], to which the reader is referred for more details.

We use integral approximants (IA's) of first, second, and third order (see Ref. [2] for a review). Given a nth-order series $f(\beta) = \Sigma_{i=0}^{n} c_i \beta^i$, its kth-order integral approximant $[m_k/m_{k-1}/\cdots/m_0/l]$ IAk is a solution of the inhomogeneous kth-order linear differential equation

$$P_k(\beta) f^{(k)}(\beta) + P_{k-1}(\beta) f^{(k-1)}(\beta) + \cdots + P_1(\beta) f^{(1)}(\beta) + P_0(\beta) f(\beta) + R(\beta) = 0, \quad (13)$$

where the functions $P_i(\beta)$ and $R(\beta)$ are polynomials of order m_i and l, respectively, which are determined by the known nth-order small-β expansion of $f(\beta)$. Following Fisher and Chen [9], we also consider integral approximants, FCIAk's, in which $P_k(\beta)$ is a polynomial in β^2. FCIAk's allow for the presence of the antiferromagnetic singularity at $\beta_c^{\rm af} = -\beta_c$ [54]. In our analyses we consider diagonal or quasidiagonal approximants, since they are expected to give the most accurate results. For each set of IAk's we determine the average of the values corresponding to all nondefective IAk's. The error bar from each class of IA's is essentially the spread of the results, and it is given by the standard deviation of the results obtained from all nondefective IA's. In most cases the nondefective IA's are more than 90%.

All IA's considered give perfectly consistent results. Moreover, the results turn out to be very stable with respect to the number of terms of the series, so that there is no need to perform problematic extrapolations in the number of terms in order to obtain the final estimates. In Fig. 1 we show the estimates of γ obtained by analyzing the series of χ for the ϕ^4 model at $\lambda_4 = 1.10$ by using IA1's, IA2's, IA3's, and FCIA2's, as a function of order n of the series considered in the analysis. Perfect agreement is also found among the results for the three improved Hamiltonians. This is shown in Fig. 2, where the results of the IA2 analyses for the three improved Hamiltonians are reported versus n. In Fig. 2 we also show the results of the IA2 analysis applied to the series of χ for the standard Ising spin-1/2 model. The corresponding results disagree with those obtained by using improved Hamiltonians: clearly, there is a large error that is not taken into account by the spread of the approximants. The results for the Ising model improve if one biases the analysis by using the very accurate MC estimate of β_c [47]: $\beta_c = 0.221\,654\,59(10)$. Indeed, γ drops from 1.245 to $\gamma = 1.2400(5)$. However, the error obtained from the spread of the approximants is still incorrect. Results that are closer to those obtained by using the improved Hamiltonians (and substantially compatible with them) are only obtained by additionally biasing the series, allowing for $O(t^{\Delta})$ confluent corrections, see, e.g., Ref. [10].

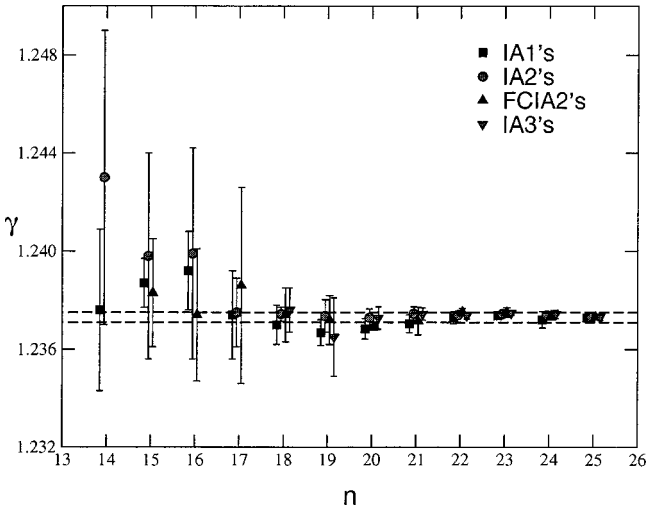

FIG. 1. Estimates of γ as obtained by analyzing the HT series of χ for the ϕ^4 model at $\lambda_4 = 1.10$ versus the order n of the series considered in the analysis. Several approximants (defined in the text) are considered.

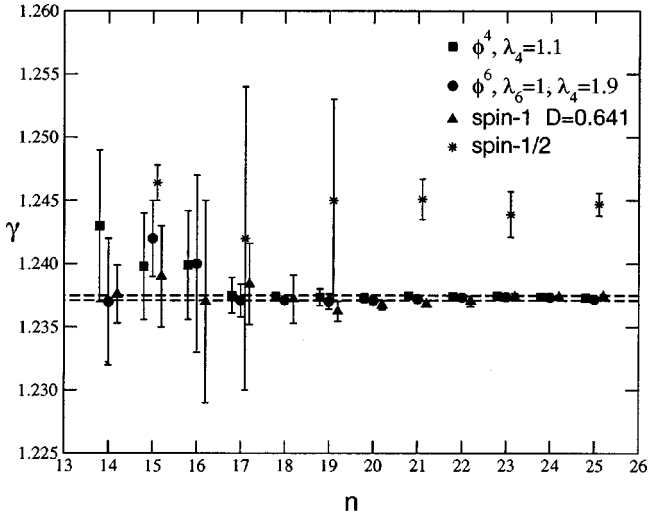

FIG. 2. Estimates of γ as obtained by analyzing the HT series of χ for the improved models and for the spin-1/2 model, versus the order n of the series considered in the analysis. IA2's are considered.

For the ϕ^4 lattice model we obtained

$$\beta_c(\lambda_4=1.10)=0.375\,097\,5(5), \tag{14}$$

$$\gamma_e(\lambda_4)=1.237\,32(10)+0.006(\lambda_4-1.10), \tag{15}$$

where $\gamma_e(\lambda_4)$ is the effective critical exponent obtained in the IA analysis, which has a small but nonvanishing dependence on λ_4 around the favorite value $\lambda_4=1.10$. (Here and in the following, we write explicitly the dependence on λ_4 and equivalent couplings: should a better estimate of λ_4^* become available, it can be immediately used to improve our results.) The number between parentheses is basically the spread of the approximants at $\lambda_4=1.10$. The λ_4 dependence is estimated by determining the variation of the results when changing λ_4 around $\lambda_4=1.10$. The best estimate of γ should be obtained at $\lambda_4=\lambda_4^*$. Thus, using the MC estimate of λ_4^*, i.e., $\lambda_4^*=1.10(2)$, and taking into account its uncertainty, we obtain the estimate $\gamma=1.237\,32(10)[12]$ (which is also reported in Table III), where the error in brackets is related to the uncertainty on λ_4^*. As final error we consider, prudentially, the sum of these two numbers. The estimate (14) is in substantial agreement with the MC estimate of β_c [23] obtained using finite-size scaling techniques, $\beta_c(\lambda_4=1.10)=0.375\,096\,6(4)$.

Similarly, for the ϕ^6 lattice model we obtain

$$\beta_c(\lambda_4=1.90,\lambda_6=1)=0.426\,979\,1(5), \tag{16}$$

$$\gamma_e(\lambda_4,\lambda_6=1)=1.237\,26(10)+0.0055(\lambda_4-1.90), \tag{17}$$

and, using the MC result $\lambda_4^*=1.90(4)$, the estimate $\gamma=1.237\,26(10)$[22]; for the spin-1 model

$$\beta_c(D=0.641)=0.385\,671\,7(10), \tag{18}$$

$$\gamma_e(D)=1.237\,25(20)-0.012(D-0.641), \tag{19}$$

and therefore, using $D^*=0.641(8)$, $\gamma=1.237\,25(20)$[10].

Our final estimate of γ is obtained by combining the results of the three improved Hamiltonians: as an estimate we take the weighted average of the three results, and as estimate of the uncertainty the smallest of the three errors. According to this rather subjective but reasonable procedure, we obtain

$$\gamma=1.2373(2). \tag{20}$$

A direct estimate of the specific-heat exponent α is obtained from the singular behavior of χ at the antiferromagnetic critical point $\beta_c^{\rm af}=-\beta_c$, since [54]

$$\chi=c_0+c_1(\beta-\beta_c^{\rm af})^{\theta_{\rm af}}+\cdots \tag{21}$$

where

$$\theta_{\rm af}=1-\alpha. \tag{22}$$

FCIAk's provide rather precise estimates of $\theta_{\rm af}$. The corresponding results for α are reported in the second line of Table III. No error in brackets is reported since the dependence on λ_4, D is negligible. As the final estimate we give

$$\alpha=0.110(2). \tag{23}$$

The exponent ν is obtained from the series of the second-moment correlation length ξ, since $\xi^2\sim(\beta_c-\beta)^{-2\nu}$. Unbiased analyses of the 24th-order series of ξ^2/β provide the results reported in the third line of Table III. The corresponding estimates of β_c are consistent with those derived from χ,

TABLE III. Critical exponents obtained from the HT analysis. In parentheses we report the approximant error at λ^* or D^*, in brackets the uncertainty due to the error on λ^* or D^*, in braces the uncertainty due to the error on β_c.

		ϕ^4	ϕ^6	spin-1
γ	χ-series	1.23732(10)[12]	1.23726(10)[22]	1.23725(20)[10]
α	χ-series at $\beta_c^{\rm af}$	0.110(2)	0.110(2)	0.112(5)
ν	ξ^2-series	0.6302(2)[1]	0.6301(3)[3]	0.6300(2)[1]
ν	ξ^2-series (β_c-biased)	0.63014(1){6}[9]	0.63009(1){16}[16]	0.63010(1){10}[9]
$\eta\nu$	χ,ξ^2-series (CPRM)	0.02294(3)[6]	0.02291(2)[10]	0.02294(8)[4]

although less precise. For instance, for the ϕ^4 model at $\lambda_4 = 1.10$ we found $\beta_c = 0.375\,098(2)$.

In order to get a more precise estimate of ν, we follow the procedure suggested in Ref. [2], i.e., we use the estimate of β_c obtained from χ to bias the analysis of ξ^2. For this purpose we use IA's that have a singularity at a fixed value of β_c, or, in order to take into account the antiferromagnetic singularity, a pair of singularities at $\pm\beta_c$; the two choices give equivalent results. This analysis provides the following effective exponents for the three classes of models. For $\lambda_4 \approx \lambda_4^*$,

$$\nu_e(\lambda_4) = 0.630\,14(1)\{6\} + 0.0045(\lambda_4 - 1.10) \tag{24}$$

for the ϕ^4 model, where the number in braces gives the variation of the estimate when β_c varies within one error bar;

$$\nu_e(\lambda_4, \lambda_6 = 1) = 0.630\,09(1)\{16\} + 0.004(\lambda_4 - 1.90) \tag{25}$$

for the ϕ^6 model;

$$\nu_e(D) = 0.630\,10(1)\{10\} - 0.011(D - 0.641) \tag{26}$$

for the spin-1 model. Then, using the MC estimates of λ_4^*, D^*, one obtains the results reported in Table III, where the error due to the uncertainty on λ_4^* and D^* is reported between brackets. They are perfectly consistent with the results of the unbiased analysis, but more precise. Combining the results of Table III as we did for γ, we obtain

$$\nu = 0.630\,12(16). \tag{27}$$

Using the hyperscaling relation $\alpha = 2 - 3\nu$, we derive

$$\alpha = 0.1096(5), \tag{28}$$

which is fully consistent with, but more precise than, the direct estimate (23).

Using the above-reported results for γ and ν and the scaling relation $\gamma = (2 - \eta)\nu$, we obtain $\eta = 0.0364(6)$, where the error is estimated by considering the errors on γ and ν as independent, which is of course not true. We can obtain an estimate of η with a smaller, yet reliable, error by applying the so-called critical-point renormalization method (CPRM) [55] to the series of χ and ξ^2. This method provides an estimate for the combination $\eta\nu$. Proceeding as before, we obtain

$$[\eta\nu]_e(\lambda_4) = 0.022\,94(3) + 0.003(\lambda_4 - 1.10) \tag{29}$$

for the ϕ^4 model,

$$[\eta\nu]_e(\lambda_4, \lambda_6 = 1) = 0.022\,91(2) + 0.0025(\lambda_4 - 1.90) \tag{30}$$

for the ϕ^6 model, and

$$[\eta\nu]_e(D) = 0.022\,94(8) - 0.005(D - 0.641) \tag{31}$$

for the spin-1 model. We then obtain the results reported in Table III, which lead to an estimate of η with a considerably smaller error:

$$\eta = 0.036\,39(15). \tag{32}$$

Then, by using the scaling relations we obtain

$$\delta = \frac{5 - \eta}{1 + \eta} = 4.7893(8), \tag{33}$$

$$\beta = \frac{\nu}{2}(1 + \eta) = 0.326\,53(10), \tag{34}$$

where the error on β has been estimated by considering the errors of ν and η as independent.

Finally, we estimate the exponent Δ. For this purpose, we analyze the HT expansion of $t^\gamma\chi$ that behaves like

$$t^\gamma\chi = C^+(1 + a_\chi t^\Delta + \cdots), \tag{35}$$

for $t \equiv 1 - \beta/\beta_c \to 0$. We consider the spin-1/2 model—here improved models are not useful since $a_\chi \approx 0$—fix the exponent γ to our best estimate, $\gamma = 1.2373$, and use biased IA's that are singular at $\beta_c = 0.221\,654\,59(10)$, which is the most precise MC estimate of the critical point [47]. We obtain

$$\Delta = 0.52(3), \tag{36}$$

where the error takes into account the uncertainty on β_c and γ. Correspondingly, we obtain $\omega = \Delta/\nu = 0.83(5)$. Consistent results are obtained from the analysis of the series of $t^{2\nu}\xi^2$, fixing ν and β_c.

B. The ratio method

In order to check the above-reported results, we consider the ratio method proposed by Zinn-Justin in Ref. [3] (also see Ref. [2]). Such a method has been recently employed in Refs. [10,17] to analyze the 25th-order HT expansions of spin-S models on the simple cubic and on the body-centered-cubic lattice.

According to this method, given a quantity

$$S = \sum_n c_n\beta^n \approx A_S(\beta_c - \beta)^{-\zeta}[1 + a_S(\beta_c - \beta)^\epsilon + \cdots], \tag{37}$$

one considers the sequences

$$\beta_c^{(n)} = \left(\frac{c_{n-2}c_{n-3}}{c_n c_{n-1}}\right)^{1/4} \exp\left[\frac{s_n + s_{n-2}}{2s_n(s_n - s_{n-2})}\right], \tag{38}$$

$$\zeta^{(n)} = 1 + 2\frac{s_n + s_{n-2}}{(s_n - s_{n-2})^2}, \tag{39}$$

where

$$s_n = -\frac{1}{2}\left[\frac{1}{\ln(c_n c_{n-4}/c_{n-2}^2)} + \frac{1}{\ln(c_{n-1}c_{n-5}/c_{n-3}^2)}\right]. \tag{40}$$

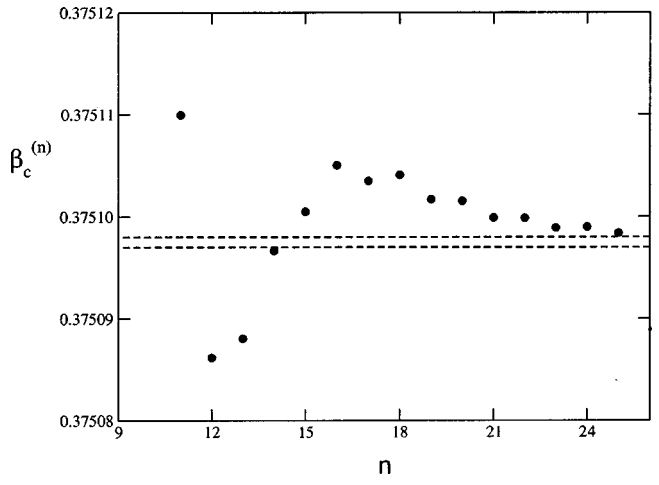

FIG. 3. Sequence $\beta_c^{(n)}$ for the ϕ^4 model at $\lambda_4=1.10$ using the series for χ. The dashed lines indicate the IA estimate of β_c.

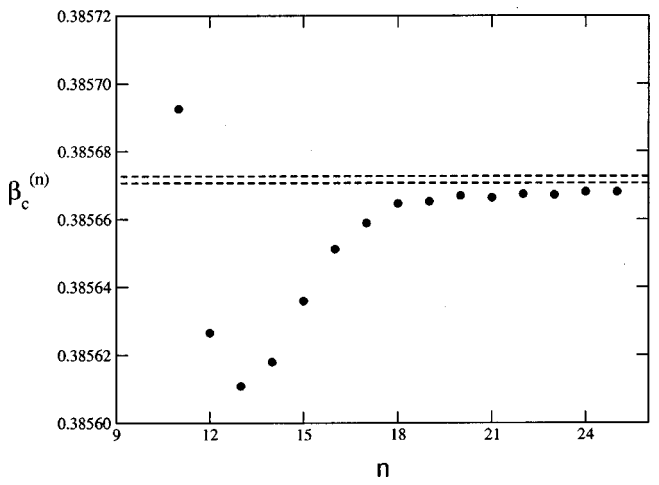

FIG. 4. Sequence $\beta_c^{(n)}$ for the spin-1 model at $D=0.641$ using the series for χ. The dashed lines indicate the IA estimate of β_c.

Asymptotically, the two sequences $\beta_c^{(n)}$ and $\zeta^{(n)}$ approach β_c and ζ, with corrections of $O(1/n^{1+\epsilon})$ and $O(1/n^{\epsilon})$, respectively. More precisely, if

$$c_n \approx \beta_c^{-n} n^{\zeta-1}(A_0 + A_\epsilon n^{-\epsilon}) \tag{41}$$

for $n\to\infty$, then

$$\beta_c^{(n)} \approx \beta_c\left[1 + \frac{A_\epsilon}{2A_0}\epsilon^2(\epsilon-1)\frac{1}{n^{1+\epsilon}}\right], \tag{42}$$

$$\zeta^{(n)} \approx \zeta\left[1 + \frac{A_\epsilon}{\zeta A_0}\epsilon(\epsilon^2-1)\frac{1}{n^{\epsilon}}\right]. \tag{43}$$

Note that, if only analytic corrections are present, i.e., $\epsilon=1$, the convergence is faster with corrections of order n^{-3} and n^{-2} for β_c and ζ:

$$\beta_c^{(n)} \approx \beta_c\left[1 - \left(\frac{A_1^2}{A_0^2}\frac{\zeta-2}{\zeta-1} + \frac{7}{12}(\zeta-1)\right)\frac{1}{n^3}\right], \tag{44}$$

$$\zeta^{(n)} \approx \zeta\left[1 - \left(3\frac{A_1^2}{A_0^2}\frac{\zeta-2}{\zeta-1} + \frac{3}{4}(\zeta-1)\right)\frac{1}{\zeta n^2}\right]. \tag{45}$$

In Figs. 3 and 4 we show, for the ϕ^4 model at $\lambda_4=1.10$ and for the spin-1 model at $D=0.641$, respectively, the sequence $\beta_c^{(n)}$ obtained using $S=\chi$. The sequence clearly approaches the IA estimate. For the ϕ^4 model the agreement is quite good and indeed $\beta_c^{(n)}$ differs from the IA estimate (14) by 15×10^{-7} and 9×10^{-7} for $n=24,25$ (note that the error on the IA estimate of β_c is 5×10^{-7}). In principle, one could try to extrapolate the sequence $\beta_c^{(n)}$ to get a better estimate of β_c. For this purpose, we have tried to fit $\beta_c^{(n)}$ assuming a behavior of the form

$$\beta_c^{(n)} = a + b n^{-\sigma}, \tag{46}$$

where a, b, and σ are free parameters. If we interpolate $\beta_c^{(n)}$ for $n=21,23,25$ with Eq. (46), we obtain

$$\beta_c^{(n)} = 0.375\,097\,7(-5) + 3.0(+4)\times10^{-6}\left(\frac{n}{20}\right)^{-6.6(-1.5)}, \tag{47}$$

where the ''errors'' show the variation of the parameters between the interpolations with $n=21,23,25$ and $n=19,21,23$. Analogously, the even sequence $n=20,22,24$ gives

$$\beta_c^{(n)} = 0.375\,096\,8(-25) + 4(+2)\times10^{-6}\left(\frac{n}{20}\right)^{-6(-2)}. \tag{48}$$

The extrapolated values are in perfect agreement with Eq. (14), but it is quite difficult to interpret the results for σ. Indeed, in an improved model the leading corrections in the coefficients c_n are of order $n^{-\Delta_2}$, n^{-1}, with [56] $\Delta_2\approx1$. The analytic term gives a contribution of order n^{-3}, while the nonanalytic one gives a correction of order $n^{-\Delta_2-1}$. However, its amplitude is of order Δ_2-1, and thus, since $\Delta_2\approx1$, it could be very small. The next correction terms are of order $n^{-\Delta_3}$, $n^{-1-\Delta}$, and give rise to corrections of order $n^{-1-\Delta_3}$, $n^{-2-\Delta}$. Inclusion of corrections with $2<\sigma\lesssim5/2$ does not improve the fit [57]. Clearly, we are not yet sufficiently asymptotic to be able to extrapolate using the leading asymptotic behavior. At the values of n we are considering, several corrections are still important and apparently conspire to give a uniformly small correction.

The same behavior is observed in the ϕ^6 model, where both odd and even points extrapolate to 0.426 978 7, with effective exponent $\sigma\approx12, 8$. The agreement with the IA estimate (16) is quite good. We finally analyze the spin-1 results. Even points show again a very fast convergence with $\sigma\approx9$ and extrapolate to 0.385 666 2. Odd points instead are well fitted by assuming corrections of order n^{-2} or $n^{-5/2}$. Fixing $\sigma=-2$, we obtain 0.385 673 0, while for $\sigma=-5/2$ we have 0.385 671 9. Again, the IA result (18) is very well confirmed.

For comparison, in Fig. 5 we plot the sequence $\beta_c^{(n)}$ for the spin-1/2 model versus $1/n^{3/2}$, which should be approximately the leading correction. The higher-n results have ap-

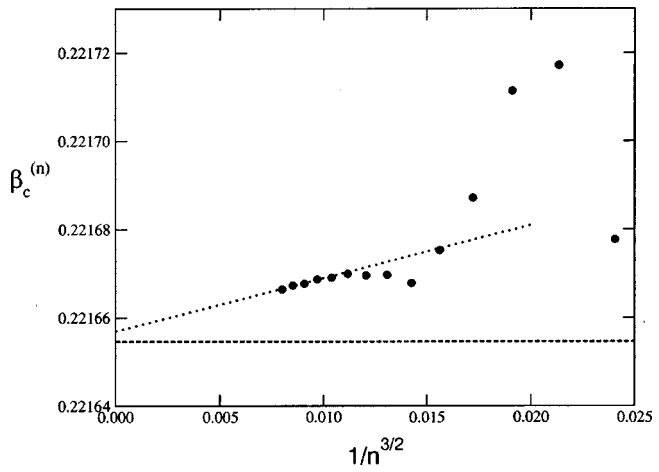

FIG. 5. Sequence $\beta_c^{(n)}$ for the spin-1/2 model using the series for χ. The dashed lines indicate the MC estimate of β_c, while the dotted line corresponds to a $n^{-3/2}$ extrapolation of the four points with $n=22,23,24,25$.

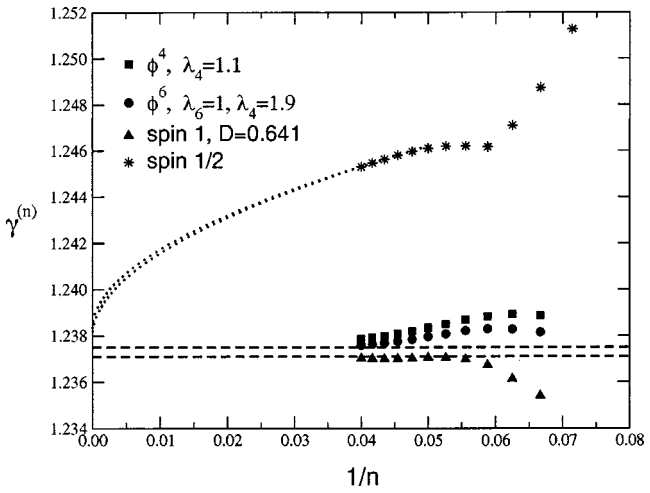

FIG. 6. Sequences $\gamma^{(n)}$ for the ϕ^4 model at $\lambda_4=1.10$, the ϕ^6 model at $\lambda_4=1.90$, the spin-1 model at $D=0.641$, and the standard Ising model. The dashed lines indicate the IA estimate of γ.

parently the predicted $O(n^{-3/2})$ behavior, and indeed an extrapolation with Eq. (46) and $\sigma=3/2$ gives results that are close to the MC estimate of β_c. The odd (even) points extrapolate to 0.221 656 86 (0.221 657 17): they are close to the MC estimate [47] 0.221 654 59(10). However, it is hard to go beyond a relative precision of 10^{-5}.

In Fig. 6 we show the sequence $\gamma^{(n)}$ as obtained from the series of χ for the three improved models and for the standard Ising model. The improved results clearly approach our best estimate $\gamma=1.2373(2)$, the ϕ^4 and ϕ^6 models from above and the spin-1 model from below. Note that the results are flat and no extrapolation is needed. We also report the sequence $\gamma^{(n)}$ for the Ising model. If we extrapolate the results assuming a behavior of the form $a+bn^{-\Delta}$, with $\Delta=0.52$, we obtain $\gamma=1.238\,57$, 1.238 32, and 1.238 01 using pairs $n=(21,23)$, (22,24), and (23,25). Clearly, the estimates converge towards the IA estimate $\gamma=1.2373(2)$.

In Fig. 7 we show the sequence $[2\nu]^{(n)}$ obtained from the series of ξ^2. Again, the improved models show a very good convergence to the IA estimate, in spite of the fact that the analysis is unbiased—the value of β_c is not fixed. The Ising results are sensibly higher and steadily decreasing, reaching $\nu\approx 0.638$ for $n=25$. Results that are closer to the IA estimate are obtained by an extrapolation. Assuming a behavior of the form $a+bn^{-\Delta}$, we obtain $\nu=0.6290$ and 0.6284 from even and odd sequences, respectively. Again, the agreement is satisfactory.

In conclusion, this analysis based on the variant of the ratio method proposed by Zinn-Justin [3] supports the IA estimate obtained in Sec. III A.

C. Matching the coefficients with their asymptotic form

In the preceding section we have determined the critical exponents and β_c by generating sequences that converge to the asymptotic value for $n\to\infty$. In this section, following Ref. [16], we wish to perform a more straightforward analysis, both conceptually and practically. The idea is to generate sequences of estimates by fitting the expansion coefficients with their asymptotic form. By adding a sufficiently large number of terms we can make the convergence as fast as possible, although of course the procedure becomes unstable if the number of terms included is too large compared to the number of available terms. In practice, one should include those terms that give rise to the maximal stability of the results. In some sense, the variant ratio method of the previous section corresponds to considering the leading singular behavior and the first analytic correction—and also the leading nonanalytic term if we further extrapolate the sequence.

On the cubic lattice, the large-order behavior is dictated by the singularities at $\pm\beta_c$. Indeed, given an observable S with expansion $S=\Sigma_n c_n\beta^n$, for $n\to\infty$ the expansion coefficients behave like

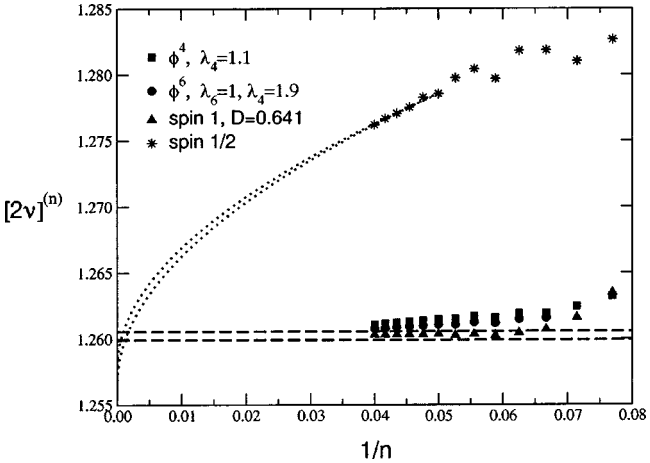

FIG. 7. Sequences $[2\nu]^{(n)}$ for the ϕ^4 model at $\lambda_4=1.10$, the ϕ^6 model at $\lambda_4=1.90$, the spin-1 model at $D=0.641$, and the standard Ising model. The dashed lines indicate the IA estimate of 2ν.

$$\beta_c^n c_n = n^{\zeta-1}\left(A_0 + \frac{A_1}{n^\Delta} + \frac{A_2}{n} + \frac{A_3}{n^{1+\Delta}} + \frac{A_4}{n^2} + \cdots\right) + (-1)^n n^{-(\theta_{\rm af}+1)}\left(B_0 + \frac{B_1}{n} + \frac{B_2}{n^2} + \frac{B_{\Delta_{\rm af}}}{n^{\Delta_{\rm af}}} + \cdots\right). \tag{49}$$

Here, we have neglected all subleading exponents except the first one Δ; in particular, we neglected the first subdominant Δ_2. However, since $\Delta_2 \approx 1$ [56], for all practical purposes a term $n^{-\Delta_2}$ cannot be distinguished from a purely analytic correction. Also, we do not write terms of order $n^{-k\Delta}$ since they cannot be distinguished from the analytic terms and corrections of order $n^{-m-\Delta}$. Note also the presence of the parity-dependent corrections with exponent $\theta_{\rm af}$ and the subleading corrections with exponent $\Delta_{\rm af}$. For the susceptibility χ, it is known [54] that $\theta_{\rm af} = 1-\alpha$. The argument can be generalized to all moments m_{2k} and thus in all cases we predict $\theta_{\rm af} = 1-\alpha$. We have tested this prediction for χ, cf. Sec. III A, m_2, and m_4. By analyzing the expansion of m_2 with biased IAk's that have a pair of singularities in $\pm\beta_c$, we obtain $\theta_{\rm af} = 0.884(12)$, while from the expansion of m_4 we obtain $\theta_{\rm af} = 0.90(9)$. These results are clearly compatible with the prediction $\theta_{\rm af} = 1-\alpha = 0.8904(5)$. For the exponent $\Delta_{\rm af}$ nothing is known. However, the results appear to be quite insensitive to the choice of $\Delta_{\rm af}$. For this reason, in the following we only report the results corresponding to purely analytic corrections, i.e., we set $B_{\Delta_{\rm af}} = 0$. We checked that the choice $\Delta_{\rm af} = 1/2$ gives equivalent results.

Note that this method allows us to determine the nonuniversal amplitudes A_0, A_Δ, ..., and consequently the amplitudes a_i appearing in the expansion of S for $\beta \to \beta_c$. If

$$S = A_S(\beta_c - \beta)^{-\zeta}[1 + a_S(\beta_c - \beta)^\Delta] \tag{50}$$

then

$$A_S = \Gamma(\zeta) A_0, \tag{51}$$

$$a_S = \frac{\Gamma(\zeta-\Delta) A_1}{\Gamma(\zeta) A_0}. \tag{52}$$

In the following, we shall perform two different analyses: (essentially) unbiased analyses in order to determine the exponents ζ and β_c and biased analyses in which ζ and β_c are fixed. In all cases we fix the value of Δ ($\Delta = 0.52$) and the exponent of the antiferromagnetic singularity. In the unbiased analyses, in order to have a linear problem, we consider $\ln c_n$ that behaves as

$$\ln c_n = -\ln(\beta_c) n + (\zeta-1)\ln n + b_0 + \frac{b_1}{n^\Delta} + \frac{b_2}{n} + \frac{b_3}{n^{1+\Delta}} + \frac{b_4}{n^2} + \cdots + (-1)^n n^{-(\zeta+\theta_{\rm af})}\left(d_0 + \frac{d_1}{n^\Delta} + \frac{d_2}{n} + \frac{d_3}{n^{1+\Delta}} + \frac{d_4}{n^2}\cdots\right). \tag{53}$$

As before, we have neglected terms that have exponents similar to those already present: for instance, terms $O(n^{-k\Delta-m})$ or $O(n^{-k\Delta_2-h\Delta-m})$. In the expansion of the antiferromagnetic part we have assumed $\Delta_{\rm af} = \Delta$, or $\Delta_{\rm af} = 1$. Note that if only analytic terms are present in Eq. (49), i.e., $B_{\Delta_{\rm af}} = 0$, then d_1 is proportional to A_1 and therefore it vanishes in improved models.

We first analyze improved models and we verify that $A_1 \approx 0$. For this purpose, we consider the susceptibility χ and, for each improved Hamiltonian, we generate two sequences of amplitudes in the following way:

(a) We choose two integers h,k and consider Eq. (53) keeping only $b_0, \ldots, b_{h-1}$ in the ferromagnetic part and $d_0, \ldots, d_{k-1}$ in the antiferromagnetic one. Then, we generate the sequences $\beta_c^{(n)}$, $\gamma^{(n)}$, $b_0^{(n)}, \ldots, b_{h-1}^{(n)}$, $d_0^{(n)}, \ldots, d_{k-1}^{(n)}$, by solving the $(h+k+2)$ equations $\ln c_{n-m} = R_{n-m}$, $m = 0, \ldots, h+k+1$, where R_n is the right-hand side of Eq. (53). We use $\Delta = 0.52$, $\gamma + \theta_{\rm af} = 2.1277$.

(b) We choose two integers h,k and consider Eq. (49) keeping only $A_0, \ldots, A_{h-1}$ in the ferromagnetic part and $B_0, \ldots, B_{k-1}$ in the antiferromagnetic one. We use $\Delta = 0.52$, $\gamma = 1.2373$, $\theta_{\rm af} = 0.8904$, the IA estimate of β_c, and $B_{\Delta_{\rm af}} = 0$. Then, we generate the sequences $A_0^{(n)}, \ldots, A_{h-1}^{(n)}$, $B_0^{(n)}, \ldots, B_{k-1}^{(n)}$, by solving the $(h+k)$ equations $c_{n-m} = R_{n-m}$, $m = 0, \ldots, h+k-1$, where R_n is the right-hand side of Eq. (49).

In both cases we vary h and k, trying to find the values that give the best stability of the exponents or of the leading amplitudes. In the unbiased analysis (a), the preferred choice is $(h,k) = (4,4)$, while for analysis (b) we use $(h,k) = (3,2)$. For these choices of the parameters, in Fig. 8 we report the corresponding sequence of $a_\chi^{(n)} \equiv a_1^{(n)}$, obtained using Eq. (52). In the unbiased analysis (a), $a_\chi^{(n)}$ clearly converges to zero for the improved Hamiltonians ϕ^4 and ϕ^6, as expected. For the spin-1 model, the situation is not that clear, and presumably more orders are needed to observe convincingly $a_\chi = 0$. In the case of the biased analysis, $a_\chi^{(n)}$ is very stable and small already for $n \gtrsim 15$. For all Hamiltonians we observe $|a_\chi| \lesssim 10^{-3}$.

As a second check of consistency we have verified that

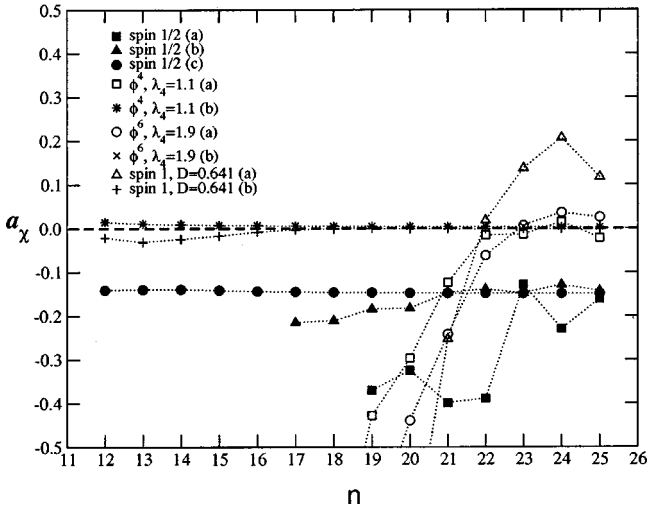

FIG. 8. Amplitude a_χ of the leading scaling correction as obtained from several different analyses of χ for the standard spin-1/2 Ising model and for the three improved models. Details are explained in the text.

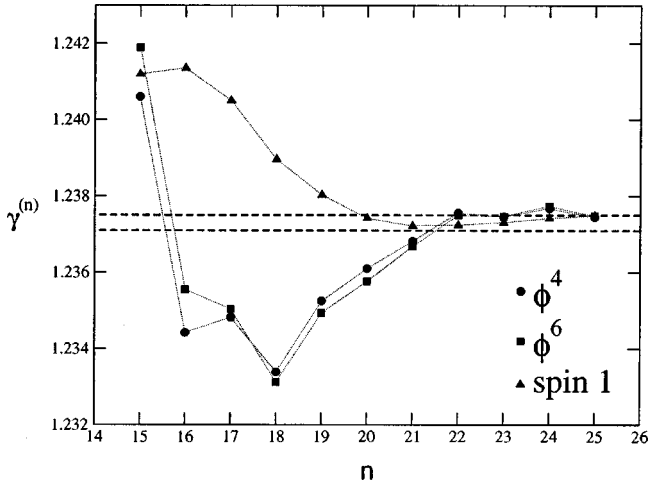

FIG. 9. Exponent γ as obtained from the analysis (a) of χ for the three improved models. Details are explained in the text.

our estimates of a_χ are compatible with the quoted error bars on λ_4^* and D^*. For this purpose, using the analysis of type (b) reported above, we have computed a_χ for $\lambda_4^* \pm \Delta\lambda_4$, where $\Delta\lambda_4$ is the quoted error bar—for the spin-1 model we are referring to $D^* \pm \Delta D$. In all cases, we find $|a_\chi(\lambda_4^* \pm \Delta\lambda_4)| > |a_\chi(\lambda_4^*)|$ and that $a_\chi(\lambda_4^* + \Delta\lambda_4)$ and $a_\chi(\lambda_4^* - \Delta\lambda_4)$ have opposite sign. This confirms the correctness of our estimates of λ_4^* and D^*. Of course, since we use β_c and γ obtained in the IA analysis, the above results represent only a check of consistency. Indeed: (i) we determine β_c and γ by performing a IA analysis whose results should be reliable only if the models are improved (in some sense we weakly assume here $a_\chi \approx 0$); (ii) using such values of β_c and γ, we estimate a_χ and find $a_\chi \approx 0$.

Once we have verified that A_1 is very small and compatible with zero within the precision of the analysis, we have performed several analyses fixing $A_1 = 0$ and $b_1 = 0$. At the same time, we have set $d_1 = 0$, which corresponds to assuming $\Delta_{\rm af} = 1$. We have determined the exponents by performing the analysis (a) reported above. In the case of the ϕ^4 model for $\lambda_4 = 1.10$, this analysis gives $\gamma \approx 1.2374$. Similarly, we obtain $\gamma \approx 1.2375$ for the ϕ^6 model at $\lambda_4 = 1.90$ and for the spin-1 model at $D = 0.641$. In Fig. 9 we show the sequence $\gamma^{(n)}$ for $(h,k) = (5,5)$ (since two coefficients vanish, we are considering four amplitudes in the ferromagnetic and antiferromagnetic expansion). We observe a very good agreement with the IA estimate $\gamma = 1.2373(2)$. It is difficult to estimate the uncertainty, since the results do not show a sufficiently robust stability with respect to the number (h,k) of coefficients used in the analysis.

Finally, we report the estimates of the amplitudes obtained in the analysis of type (b) for the magnetic susceptibility:

ϕ^4: $A_0^{(\chi)} \approx 0.5246$, $A_2^{(\chi)} \approx 0.13$, $B_0^{(\chi)} \approx -0.0351$;

ϕ^6: $A_0^{(\chi)} \approx 0.4601$, $A_2^{(\chi)} \approx 0.11$, $B_0^{(\chi)} \approx -0.0311$;

spin-1: $A_0^{(\chi)} \approx 0.5126$, $A_2^{(\chi)} \approx 0.12$, $B_0^{(\chi)} \approx -0.0359$.

Moreover, $|A_3^{(\chi)}| \lesssim 10^{-2}$ for the ϕ^4 and ϕ^6 models, while $A_3^{(\chi)} \approx -0.02$ for the spin-1 model. Errors should be ± 1 on the last reported digit and include the uncertainty on the $n \to \infty$ extrapolation of the sequences and the variation of the estimates for h and k in the range $h = 3$–5 and $k = 2$–4. Instead they do not take into account the variation of the estimates with γ and β_c. Note that the estimate of A_2 is purely phenomenological and in practice it should correspond to the sum of the amplitude of n^{-1} and of $n^{-\Delta_2}$ (note that in improved models the amplitude of $n^{-2\Delta}$ vanishes).

We have performed similar analyses for the spin-1/2 Ising model, in order to compute the nonuniversal amplitudes. We have performed: (a) an analysis of type (a) using $(h,k) = (4,4)$; (b) an analysis of type (a) in which we have fixed β_c to its MC value using $(h,k) = (4,3)$; (c) an analysis of type (b) using $(h,k) = (3,2)$. The results for $a_\chi^{(n)}$ are reported in Fig. 8. These analyses give perfectly consistent results and allow us to determine the amplitudes:

$$A_0^{(\chi)} = 1.233 - 10(\gamma - 1.2373) - 0.013(\Delta - 0.52), \quad (54)$$

$$A_1^{(\chi)} = -0.13 - 0.7(\Delta - 0.52) + 50(\gamma - 1.2373), \quad (55)$$

$$B_0^{(\chi)} = -0.073, \quad (56)$$

where we have explicitly written the dependence on the input parameters (when it turns out to be relevant). We have repeated the same analysis for the second moment m_2. We obtain

$$A_0^{(m_2)} = 1.301 - 10(\gamma + 2\nu - 2.497\,54) - 0.07(\Delta - 0.52), \quad (57)$$

$$A_1^{(m_2)} = -0.73 - 4(\Delta - 0.52) + 55(\gamma + 2\nu - 2.497\,54), \quad (58)$$

$$B_0^{(m_2)} = 0.06. \quad (59)$$

Using the above results and Eq. (52), one can determine the amplitudes a_χ and a_ξ, associated with the $O(t^\Delta)$ scaling corrections in the Wegner expansion of χ and ξ, respec-

tively, and evaluate their universal ratio. We obtain $a_\xi/a_\chi = 0.9(1)$, where the error takes also into account the uncertainty on the input parameters of the biased analysis. For comparison we mention the recent HT result $a_\xi/a_\chi = 0.76(6)$ [17], and the field theoretical estimate $a_\xi/a_\chi = 0.68(2)$ [35].

IV. THE CRITICAL EQUATION OF STATE

A. Definitions

The equation of state relates the magnetization M, the magnetic field H, and the reduced temperature $t \equiv (T - T_c)/T_c$. In the neighborhood of the critical point $t=0$, $H=0$, it can be written in the scaling form

$$H = B_c^{-\delta} M^\delta f(x), \tag{60}$$

$$x \equiv t(M/B)^{-1/\beta}, \tag{61}$$

where B_c and B are the amplitudes of the magnetization on the critical isotherm and on the coexistence curve,

$$M = B_c H^{1/\delta}, \quad t=0, \tag{62}$$

$$M = B(-t)^\beta, \quad H=0, t<0. \tag{63}$$

Using these normalizations the coexistence curve corresponds to $x=-1$, and the universal function $f(x)$ satisfies $f(-1)=0$, $f(0)=1$. Griffiths' analyticity implies that $f(x)$ is regular everywhere for $x>-1$. It has a regular expansion in powers of x,

$$f(x) = 1 + \sum_{n=1}^{\infty} f_n^0 x^n, \tag{64}$$

and a large-x expansion of the form

$$f(x) = x^\gamma \sum_{n=0}^{\infty} f_n^\infty x^{-2n\beta}. \tag{65}$$

At the coexistence curve, i.e., for $x \to -1$, $f(x)$ has at most an essential singularity [58]. It can be asymptotically expanded as

$$f(x) \approx \sum_{n=1}^{\infty} f_n^{\rm coex} (1+x)^n. \tag{66}$$

It is useful to rewrite the equation of state in terms of a variable proportional to $Mt^{-\beta}$, although in this case we must distinguish between $t>0$ and $t<0$. For $t>0$ we define

$$H = \left(\frac{C^+}{C_4^+}\right)^{1/2} t^{\beta\delta} F(z),$$

$$z \equiv \left[-\frac{C_4^+}{(C^+)^3}\right]^{1/2} M t^{-\beta}, \tag{67}$$

while for $t<0$ we set

$$H = \frac{B}{C^-}(-t)^{\beta\delta}\Phi(u),$$

$$u \equiv \frac{M}{B}(-t)^{-\beta}. \tag{68}$$

The constants $C^\pm$ and C_4^+ are the amplitudes appearing in the critical behavior of the zero-momentum connected n-point correlation functions χ_n:

$$\chi_n = C_n^\pm |t|^{-\gamma-(n-2)\beta\delta}. \tag{69}$$

The susceptibility χ corresponds to χ_2 and we simply write $C^\pm = C_2^\pm$.

With the chosen normalizations [41,46,49]

$$F(z) = z + \frac{1}{6}z^3 + \sum_{j=3} \frac{1}{(2j-1)!} r_{2j} z^{2j-1}, \tag{70}$$

$$\Phi(u) = (u-1) + \sum_{j=3}^{\infty} \frac{1}{(j-1)!} v_j (u-1)^{j-1}. \tag{71}$$

The functions $F(z)$ and $\Phi(u)$ are related to $f(x)$. Indeed,

$$z^{-\delta} F(z) = F_0^\infty f(x), \quad z = z_0 x^{-\beta}, \tag{72}$$

and

$$u^{-\delta}\Phi(u) = \frac{C^- B^{\delta-1}}{B_c^\delta} f(x), \quad u = (-x)^{-\beta}. \tag{73}$$

The constant F_0^∞ is defined by the large-z behavior of $F(z)$,

$$F(z) = z^\delta \sum_{k=0} F_k^\infty z^{-k/\beta}, \tag{74}$$

while

$$z_0 = \left[-\frac{C_4^+}{(C^+)^3}\right]^{1/2} B. \tag{75}$$

To compare with experimental data, it is useful to determine the magnetization as a function of $tH^{-1/\beta\delta}$. Therefore, we define

$$E(y) \equiv B_c^{-1} M H^{-1/\delta} = f(x)^{-1/\delta}, \tag{76}$$

$$y \equiv (B/B_c)^{1/\beta} t H^{-1/(\beta\delta)} = x f(x)^{-1/(\beta\delta)}. \tag{77}$$

Finally, we shall also determine the scaling behavior of the susceptibility, by defining

$$D(y) \equiv B_c^{-1} H^{1-1/\delta} \chi = \frac{f(x)^{1-1/\delta}}{\delta f(x) - \frac{1}{\beta} x f'(x)}. \tag{78}$$

B. Small-magnetization behavior

In this section we determine the first few coefficients r_{2j} appearing in the expansion of the scaling function $F(z)$, cf. Eq. (67). We shall also compute the four-point renormalized coupling constant g_4, which, although not related to the equation of state, is relevant for the field-theoretical approach and will be used to determine amplitude ratios involving the second-moment correlation length.

In order to estimate the critical limit of g_4 and of r_{2j} we first determine their HT expansions using the corresponding results for χ_{2j} and m_2,

$$g_4 = -\frac{\chi_4}{\chi^2 \xi^3}, \tag{79}$$

$$r_6 = 10 - \frac{\chi_6 \chi_2}{\chi_4^2}, \tag{80}$$

$$r_8 = 280 - 56\frac{\chi_6 \chi_2}{\chi_4^2} + \frac{\chi_8 \chi_2^2}{\chi_4^3}. \tag{81}$$

The corresponding series [59] have been analyzed by closely following the procedure presented in Appendix B.3 of Ref. [25]. We use biased IA1's with a singularity at β_c or a pair of singularities at $\pm\beta_c$, where β_c is obtained from the analysis of the susceptibility. Around β_c, IA1's behave like [60]

$$\mathrm{IA1} \approx f(\beta)(1-\beta/\beta_c)^{\zeta} + g(\beta), \tag{82}$$

where $f(\beta)$ and $g(\beta)$ are regular at β_c, provided ζ is not a negative integer. In particular

$$\zeta = \frac{P_0(\beta_c)}{P_1'(\beta_c)}, \quad g(\beta_c) = -\frac{R(\beta_c)}{P_0(\beta_c)} \tag{83}$$

[see Eq. (13) for the definition of the above quantities]. In the case we are considering, ζ is positive and, therefore, $g(\beta_c)$ provides the desired estimate.

In Table IV (first line) we report the estimates of g_4 obtained for the three improved Hamiltonians. The error in parentheses is related to the spread of the approximants and the second one in brackets to the uncertainty on λ_4^*, D^*. The error induced by the uncertainty on β_c is negligible. The results are perfectly consistent. Our final estimate is

$$g_4 = 23.56(2). \tag{84}$$

The result for the exponent ζ in Eq. (82) is $\zeta = 1.3(3)$, which is consistent with our expectation for improved models, i.e., $\zeta = \Delta_2 \approx 2\Delta$ and $\Delta \approx 0.5$. For comparison, the same analysis applied to the standard Ising model gives $g_4 = 23.5(5)$ and $\zeta = 0.6(3)$, in agreement with the fact that in this case $\zeta = \Delta$. Notice that the small difference with the estimate of g_4 reported in Ref. [19] is essentially due to the different analysis employed here, which is better justified due to the nonanalytic behavior at β_c predicted by renormalization group [61]. With respect to standard Padé approximants, biased IA1's require more terms of the series to give reasonable results, but they are less subject to systematic errors since they allow for confluent nonanalytic corrections at β_c. Biased IA1's give [1] $g_4 = 23.54(4)$ when applied to the 17th-order series of Ref. [19].

TABLE IV. Results for g_4, r_6, and r_8.

	ϕ^4	ϕ^6	Spin-1	Final estimates
g_4	23.559(8)[11]	23.554(8)[20]	23.560(20)[5]	23.56(2)
r_6	2.057(4)[1]	2.056(4)[2]	2.052(8)[2]	2.056(5)
r_8	2.29(9)[3]	2.31(5)[5]	2.37(7)[3]	2.3(1)

Results for r_6, r_8 are obtained using the same method and are reported in Table IV. We finally recall that a rough estimate of r_{10} was obtained in Ref. [19] from the analysis of its 15th-order series, obtaining $r_{10} = -13(4)$. A review of the available results for these quantities can be found in Ref. [1].

C. Parametric representations of the equation of state

In this section we shall determine the equation of state using parametric representations, improving the results of Refs. [19,41]. This method has also been applied in two dimensions [62], and to the three-dimensional XY [25,29] and Heisenberg [27] universality classes.

In order to obtain approximate expressions for the equation of state, we parametrize the thermodynamic variables in terms of two parameters R and θ, implementing all expected scaling and analytic properties. Explicitly, we write [63–65]

$$M = m_0 R^{\beta} \theta,$$

$$t = R(1-\theta^2),$$

$$H = h_0 R^{\beta\delta} h(\theta), \tag{85}$$

where h_0 and m_0 are normalization constants. The function $h(\theta)$ is odd and normalized so that $h(\theta) = \theta + O(\theta^3)$. The smallest positive zero of $h(\theta)$, which should satisfy $\theta_0 > 1$, corresponds to the coexistence curve, i.e., to $T < T_c$ and $H \to 0$. We mention that alternative versions of the parametric representations have been considered in Ref. [66].

It is easy to express the scaling functions introduced in Sec. IV A in terms of θ. The scaling function $f(x)$ is obtained from

$$x = \frac{1-\theta^2}{\theta_0^2 - 1}\left(\frac{\theta_0}{\theta}\right)^{1/\beta},$$

$$f(x) = \theta^{-\delta}\frac{h(\theta)}{h(1)}, \tag{86}$$

while $F(z)$ is obtained by

$$z = \rho\,\theta(1-\theta^2)^{-\beta},$$

$$F(z(\theta)) = \rho(1-\theta^2)^{-\beta\delta} h(\theta), \tag{87}$$

where ρ can be related to m_0, h_0, C^+, and C_4^+ using Eqs. (67) and (85).

TABLE V. Polynomial approximations of $h(\theta)$ using the global stationarity condition for various values of the parameter k. The reported expressions are obtained by using the central values of the input parameters. The last column shows the corrections to the simple linear model $h_{\rm lin}(\theta,\theta_0)\equiv\theta(1-\theta^2/\theta_0^2)$.

k	$h(\theta)/\theta$	θ_0^2	$h(\theta)/h_{\rm lin}(\theta,\theta_0)$
1	$1-0.734732\theta^2$	1.36104	1
2	$1-0.731630\theta^2+0.009090\theta^4$	1.39085	$1-0.0126429\theta^2$
3	$1-0.736743\theta^2+0.008904\theta^4-0.000472\theta^6$	1.37861	$1-0.0113775\theta^2+0.0006511\theta^4$

It is important to note that Eq. (85) and the normalization condition $h(\theta)\approx\theta$ for $\theta\to 0$ do not completely fix the function $h(\theta)$. Indeed, one can rewrite the relation between x and θ in the form

$$x^\gamma = h(1) f_0^\infty (1-\theta^2)^\gamma \theta^{1-\delta}. \tag{88}$$

Thus, given $f(x)$, the value of $h(1)$ can be arbitrarily chosen to completely fix $h(\theta)$. In the expression (87) we can fix this arbitrariness by choosing arbitrarily the parameter ρ [1,19,29,41].

As suggested by arguments based on the ϵ expansion [19,41], we approximate $h(\theta)$ with polynomials, i.e., we set

$$h(\theta) = \theta + \sum_{n=1}^{k} h_{2n+1}\theta^{2n+1}. \tag{89}$$

This choice is further supported by the effectiveness of its simplest version with $k=1$, which is the so-called linear model. If we require the approximate parametric representation to give the correct $(k-1)$ universal ratios r_6, $r_8,\ldots,r_{2k+2}$, we obtain

$$h_{2n+1} = \sum_{m=0}^{n} c_{nm} 6^m (h_3+\gamma)^m \frac{r_{2m+2}}{(2m+1)!}, \tag{90}$$

where

$$c_{nm} = \frac{1}{(n-m)!}\prod_{k=1}^{n-m}(2\beta m - \gamma + k - 1), \tag{91}$$

and we have set $r_2 = r_4 = 1$. Moreover, by requiring that $F(z) = z + \frac{1}{6}z^3 + \cdots$, we obtain the relation

$$\rho^2 = 6(h_3+\gamma). \tag{92}$$

In the exact parametric representation, the coefficient h_3 can be chosen arbitrarily. Of course, this is no longer true when we use our truncated function $h(\theta)$, and the related approximate function $f^{(k)}_{\rm approx}(x,h_3)$ depends on h_3. We must thus fix a particular value for this parameter. Here we use a variational approach, requiring the approximate function $f^{(k)}_{\rm approx}(x,h_3)$ to have the smallest possible dependence on h_3. Thus, we set $h_3 = h_{3,k}$, where $h_{3,k}$ is a solution of the global stationarity condition

$$\left.\frac{\partial f^{(k)}_{\rm approx}(x,h_3)}{\partial h_3}\right|_{h_3=h_{3,k}} = 0 \tag{93}$$

for all x. Equivalently one may require that, for *any* universal ratio R that can be obtained from the equation of state, its approximate expression $R^{(k)}_{\rm approx}$ obtained by using the parametric representation satisfies

$$\left.\frac{dR^{(k)}_{\rm approx}(h_3)}{dh_3}\right|_{h_3=h_{3,k}} = 0. \tag{94}$$

The existence of such a value of h_3 is a nontrivial mathematical fact. The stationary value of h_3 is the solution of the algebraic equation [19]

$$\left[2(2\beta-1)(h_3+\gamma)\frac{\partial}{\partial h_3} - 2\gamma + 2k\right] h_{2k+1} = 0. \tag{95}$$

For $k=1$, the so-called linear model, Eq. (95) gives

$$h_3 = \frac{\gamma(1-2\beta)}{\gamma-2\beta}, \tag{96}$$

which is the optimal value of h_3 considered in Ref. [64]. Thus, the optimal (sometimes called restricted) linear model represents the first approximation of our scheme.

D. Results

Following Ref. [19], we apply the variational method by using the HT results for $\gamma = 1.2373(2)$, $\nu = 0.630\,12(16)$, $r_6 = 2.056(5)$, $r_8 = 2.3(1)$, and $r_{10} = -13(4)$ as input parameters of the approximation scheme. This provides different approximations with $k=1,2,3,4$. In Table V we report the polynomials $h(\theta)$ for $k=1,2,3$, that are obtained in the variational approach for the central values of the input parameters. The fast decrease of the coefficients of the higher-order terms in $h(\theta)$ gives further support to the effectiveness of the approximation scheme. We do not report $h(\theta)$ for $k=4$, since it requires r_{10} and its available estimate is rather imprecise. Using the results reported in Table V and Eqs. (86), (87), and (73), one may easily compute the corresponding approximations for the scaling functions $f(x)$, $F(z)$, and $\Phi(u)$. The results show a good convergence with increasing k. Actually, the results for $k=2,3,4$ are already consistent within the errors induced by the uncertainty on the input parameters, indicating that the systematic error due to the truncation is at most of the same order of the error induced by the input data. In Figs. 10, 11, and 12 we show respectively the scaling functions as obtained from $h(\theta)$ for $k=1,2,3$.

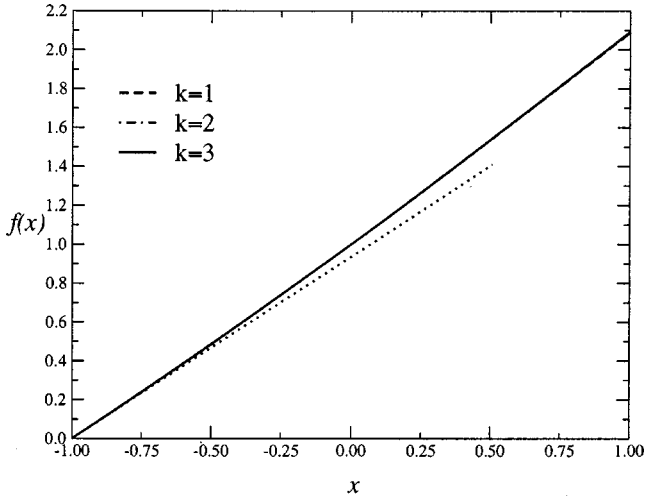

FIG. 10. The scaling function $f(x)$. We also plot the asymptotic behavior of $f(x)$ at the coexistence curve (dotted line), i.e., $f(x) \approx f_0^{\rm coex}(1+x)$ for $x \to -1$.

In Table VI we report results concerning the behavior of the scaling function $f(x)$, $F(z)$ and $\Phi(u)$ for $H=0$ and on the critical isotherm, cf. Eqs. (64), (65), (66), (70), (71), (74). Note that the results for $k=1,2,3$ oscillate and that the uncertainty due to the input parameters on the $k=3$ results is approximately the same as the difference between the estimates with $k=2$ and $k=3$. Therefore, it is reasonable to consider the $k=3$ truncation as the best approximation of the method using the available input parameters and to use the corresponding errors as final uncertainties.

In Fig. 13 we give the behavior of the magnetization as a function of t and H, reporting the scaling function $E(y)$. The behavior of the susceptibility can be obtained from the scaling function $D(y)$. The function $D(y)$ has a maximum for $y_{\rm max}=1.980(4)$, corresponding to the so-called crossover or pseudocritical line (see Sec. V). In order to simplify possible comparisons, it may be convenient to consider the rescaled function,

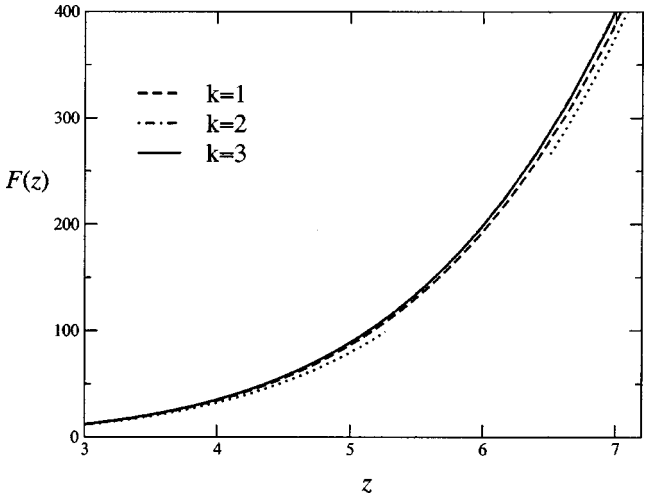

FIG. 11. The scaling function $F(z)$. We also show the small-z and large-z behaviors (dotted lines), i.e., $F(z) \approx z + \frac{1}{6}z^3 + \frac{1}{120}r_6 z^5$ for $z \to 0$ and $F(z) \approx F_0^\infty z^\delta$ for $z \to \infty$.

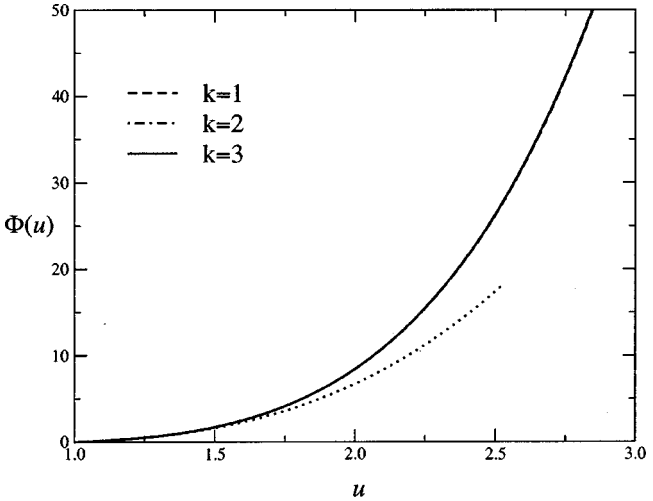

FIG. 12. The scaling function $\Phi(u)$. We plot also the asymptotic behavior of $\Phi(u)$ at the coexistence curve (dotted line), i.e., $\Phi(u) \approx (u-1) + \frac{1}{2}v_3(u-1)^2 + \frac{1}{6}v_4(u-1)^3$ for $u \to 1$.

$$C(y_R) = \frac{D(y)}{D(y_{\rm max})},$$

$$y_R = \frac{y}{y_{\rm max}}, \tag{97}$$

which is such that the maximum corresponds to $y_R=1$ and satisfies $C(1)=1$. In Fig. 14 we plot the scaling function $C(y_R)$, as obtained from the $k=1,2,3$ approximate parametric representations.

In experimental work on magnetic systems, it is customary to report [67,68] $h/m \equiv H|t|^{-\gamma}/M$ vs $m^2 = M^2|t|^{-2\beta}$. Such a function can be easily obtained from our approximations for $f(x)$, since $m^2 = B^2|x|^{-2\beta}$ and

$$\frac{h}{m} = k|x|^{-\gamma} f(x), \tag{98}$$

where the constant k can be written as

$$k = B_c^{-\delta} B^{\gamma/\beta} = \frac{R_\chi}{C^+}, \tag{99}$$

where $R_\chi \equiv C^+ B^{\delta-1}/B_c^\delta$ is a universal constant, see Sec. V. A plot of m^2/B^2 vs $C^+ h/m$ for the two cases $t>0$ and $t<0$ is reported in Fig. 15.

It is interesting to observe that in a neighborhood of the critical isotherm the equation of state can be written in the Arrott-Noakes form [69]

$$\left(\frac{H}{M}\right)^{1/\gamma} = at + bM^{1/\beta}, \tag{100}$$

where a and b are numerical constants. Indeed, using the results of Table VI for $k=3$, we obtain

TABLE VI. Expansion coefficients for the scaling equation of state obtained by the variational approach. See text for definitions. Numbers marked with an asterisk are inputs, not predictions.

	$k=1$	$k=2$	$k=3$	$k=4$
θ_0^2	1.3610(8)	1.390(2)	1.38(2)	1.34(6)
ρ	1.7365(8)	1.741(1)	1.733(10)	1.69(6)
r_6	1.938(3)	*2.056(5)	*2.056(5)	*2.056(5)
r_8	2.50(2)	2.39(3)	*2.3(1)	*2.3(1)
r_{10}	−12.59(2)	−12.08(5)	−10.6(1.8)	*−13(4)
F_0^∞	0.03277(8)	0.03388(11)	0.03382(15)	0.0338(2)
z_0	2.8254(7)	2.792(2)	2.794(3)	2.798(8)
f_1^0	1.05041(7)	1.0532(2)	1.0527(7)	1.051(2)
f_2^0	0.04298(6)	0.04494(13)	0.0446(4)	0.0439(13)
f_3^0	−0.02474(4)	−0.02595(8)	−0.0254(7)	−0.023(4)
f_0^∞	0.5960(4)	0.6031(7)	0.6024(15)	0.601(4)
$f_1^{\rm coex}$	0.93912(9)	0.9347(3)	0.9357(11)	0.938(4)
v_3	6.013(4)	6.062(4)	6.050(13)	6.02(5)
v_4	16.32(3)	16.10(4)	16.17(10)	16.4(3)

$$\left(\frac{H}{M}\right)^{1/\gamma} k^{-1/\gamma} = \left(\frac{M}{B}\right)^{1/\beta} + 0.851\, t - 0.050\, t^2 \left(\frac{M}{B}\right)^{-1/\beta} - 0.008\, t^3 \left(\frac{M}{B}\right)^{-2\beta} \cdots . \tag{101}$$

Thus, corrections to Eq. (100) are small and thus this expression has a quite wide range of validity.

V. UNIVERSAL AMPLITUDE RATIOS

From the critical equation of state one may derive estimates of several universal amplitude ratios. They are expressed in terms of the amplitudes of the magnetization, cf. Eqs. (62) and (63), of the magnetic susceptibility and n-point correlation functions, cf. Eq. (69), of the specific heat,

$$C_H = A^{\pm} |t|^{-\alpha}, \tag{102}$$

of the second-moment correlation length,

$$\xi = f^{\pm} |t|^{-\nu}, \tag{103}$$

and of the true (on-shell) correlation length, describing the large distance behavior of the two-point function,

$$\xi_{\rm gap} = f^{\pm}_{\rm gap} |t|^{-\nu}. \tag{104}$$

One can also define amplitudes along the critical isotherm, e.g.,

$$\chi = C^c |H|^{-\gamma/\beta\delta}, \tag{105}$$

$$\xi = f^c |H|^{-\nu/\beta\delta}, \tag{106}$$

$$\xi_{\rm gap} = f^c_{\rm gap} |H|^{-\nu/\beta\delta}. \tag{107}$$

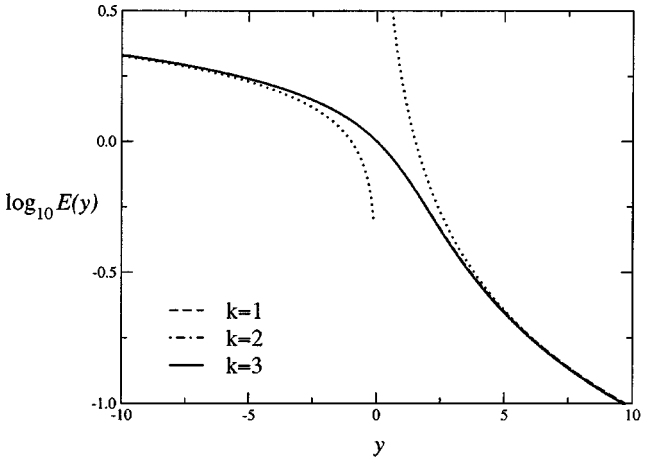

FIG. 13. The scaling function $E(y)$. We also report its asymptotic behaviors (dotted lines): $E(y) \approx R_\chi y^{-\gamma}$ for $y \to +\infty$, and $E(y) \approx (-y)^\beta$ for $y \to -\infty$.

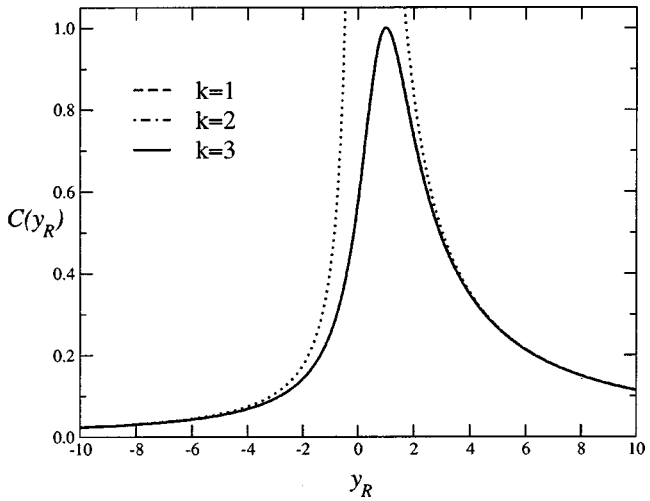

FIG. 14. The scaling function $C(y_R)$. We also report its asymptotic behaviors (dotted lines): $C(y_R) \approx R_\chi y_{\rm max}^{-\gamma} D(y_{\rm max})^{-1} y_R^{-\gamma} \approx 1.97 y_R^{-\gamma}$ for $y_R \to +\infty$, and $C(y_R) \approx \beta (f_1^{\rm coex})^{-1} y_{\rm max}^{-\gamma} D(y_{\rm max})^{-1} \times (-y_R)^{-\gamma} \approx 0.413 (-y_R)^{-\gamma}$ for $y_R \to -\infty$.

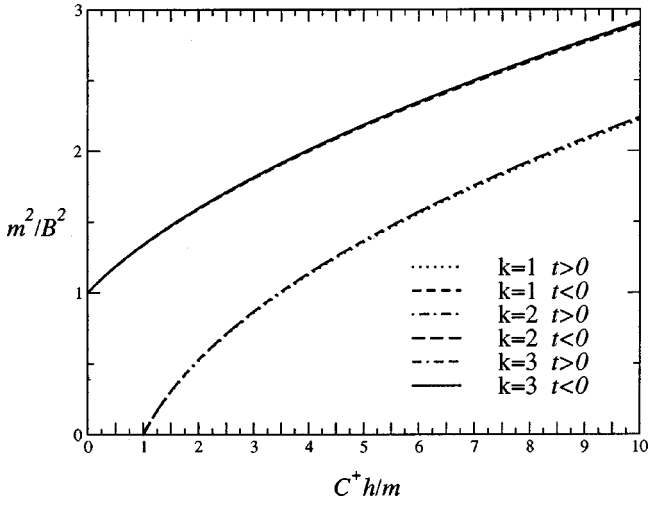

FIG. 15. Plot of m^2/B^2 vs C^+h/m.

We also consider the crossover (or pseudocritical) line $t_{\max}(H)$, that is defined as the reduced temperature for which the magnetic susceptibility has a maximum at H fixed. Renormalization group predicts

$$t_{\max}(H) = T_p H^{1/(\gamma+\beta)}, \tag{108}$$

$$\chi(t_{\max}, H) = C_p t_{\max}^{-\gamma}. \tag{109}$$

We consider several universal amplitude ratios. They are defined in Table VII.

In Table VIII we report the universal amplitude ratios, as derived by the approximate polynomial representations of the equation of state for $k=1,2,3,4$. The reported errors are only due to the uncertainty of the input parameters and do not include the systematic error of the procedure, which may be determined by comparing the results of the various approximations. In Table VIII we also show results for $z_{\max}$, $x_{\max}$, and $y_{\max}$, which are the values of the scaling variable z, x, and y [y was defined in Eq. (77)] associated with the crossover line. As already mentioned in Sec. IV D, we consider the $k=3$ results as our best estimates, and report them in Table I.

Estimates of universal ratios of amplitudes involving correlation-length amplitudes, such as Q^+, R_ξ^+, and Q_c, can be obtained using the HT estimate of g_4. For instance, $Q^+ = R_4^+ R_c^+ / g_4$. Other universal ratios can be obtained by supplementing the above results with the estimates of w^2 and Q_ξ^- (see Table VII), obtained by an analysis of the corresponding low-temperature expansions [14,34], and the HT estimate of Q_ξ^+ (see Sec. VI). Moreover, using approximate parametric representations of the correlation length, see Refs.

TABLE VII. Amplitude-ratio definitions.

$U_0 \equiv A^+/A^-$	$U_2 \equiv C^+/C^-$
$U_4 \equiv C_4^+/C_4^-$	$R_4^+ \equiv -C_4^+ B^2/(C^+)^3$
$R_c^+ \equiv \alpha A^+ C^+/B^2$	$R_c^- \equiv \alpha A^- C^-/B^2$
$R_4^- \equiv C_4^- B^2/(C^-)^3$	$R_\chi \equiv C^+ B^{\delta-1}/(B_c)^\delta$
$v_3 \equiv -C_3^- B/(C^-)^2$	$v_4 \equiv -C_4^- B^2/(C^-)^3 + 3v_3^2$
$g_4^+ \equiv -C_4^+/[(C^+)^2(f^+)^3]$	$w^2 \equiv C^-/[B^2(f^-)^d]$
$U_\xi \equiv f^+/f^-$	$U_{\xi_{\rm gap}} \equiv f_{\rm gap}^+/f_{\rm gap}^-$
$Q^+ \equiv \alpha A^+ (f^+)^3$	$Q^- \equiv \alpha A^- (f^-)^d$
$R_\xi^+ \equiv (Q^+)^{1/3}$	$Q_\xi^+ \equiv f_{\rm gap}^+/f^+$
$Q_\xi^- \equiv f_{\rm gap}^-/f^-$	$Q_\xi^c \equiv f_{\rm gap}^c/f^c$
$Q_c \equiv B^2(f^+)^3/C^+$	$Q_2 \equiv (f^c/f^+)^{2-\eta} C^+/C_c$
$P_m \equiv T_p^\beta B/B_c$	$P_c \equiv -T_p^{2\beta\delta} C^+/C_4^+$
$R_p \equiv C^+/C_p$	

TABLE VIII. Universal amplitude ratios obtained by taking different approximations of the parametric function $h(\theta)$.

	$k=1$	$k=2$	$k=3$	$k=4$
U_0	0.5231(11)	0.533(2)	0.5319(25)	0.529(6)
U_2	4.826(7)	4.745(10)	4.758(19)	4.78(5)
U_4	−9.73(3)	−8.85(6)	−9.0(2)	−9.3(5)
R_c^+	0.05545(7)	0.0570(1)	0.0567(3)	0.0562(11)
R_c^-	0.021967(11)	0.02253(3)	0.02242(12)	0.0222(4)
R_4^+	7.983(4)	7.794(8)	7.81(2)	7.83(4)
R_4^-	92.15(13)	94.13(13)	93.6(6)	92(2)
R_χ	1.6779(11)	1.658(2)	1.660(4)	1.665(10)
$U_2 R_4^+$	38.52(5)	36.98(10)	37.1(2)	37.4(6)
$R_4^+ R_c^+$	0.4427(7)	0.4444(7)	0.443(2)	0.440(6)
P_m	1.25203(6)	1.2493(2)	1.2498(6)	1.251(2)
P_c	0.3831(3)	0.3938(5)	0.3933(7)	0.3930(11)
R_p	1.9789(3)	1.9658(6)	1.9665(10)	1.9671(16)
$z_{\max}$	1.2443(4)	1.2317(5)	1.2322(8)	1.2326(12)
$x_{\max}$	12.32(3)	12.26(3)	12.27(4)	12.31(8)
$w_{\max}$	1.990(1)	1.977(2)	1.980(4)	1.984(9)
$D(w_{\max})$	0.36179(4)	0.36277(7)	0.36268(14)	0.3626(3)

TABLE IX. Estimates of the coefficients c_i, $i=2,3,4$, of the low-momentum expansion of the structure factor.

	ϕ^4	ϕ^6	Spin-1	Final estimates
c_2	$-0.390(7)\times10^{-3}$	$-0.390(6)\times10^{-3}$	$-0.389(12)\times10^{-3}$	$-0.390(6)\times10^{-3}$
c_3	$0.882(8)\times10^{-5}$	$0.882(6)\times10^{-5}$	$0.88(4)\times10^{-5}$	$0.882(6)\times10^{-5}$
c_4	$-0.4(1)\times10^{-6}$	$-0.4(1)\times10^{-6}$	$-0.4(1)\times10^{-6}$	$-0.4(1)\times10^{-6}$

[1,19] for details, one may also estimate the universal ratios Q_ξ^c and Q_2 defined in Table VII.

VI. LOW-MOMENTUM BEHAVIOR OF THE STRUCTURE FACTOR

In this section we update the determination of the first few coefficients that parametrize the low-momentum expansion of the scaling two-point function in the HT phase [19,34,70]

$$g(y)\equiv\frac{\chi}{\tilde{G}(k)}=1+y+\sum_{i=2}^{\infty}c_i y^i, \tag{110}$$

where $y=k^2\xi^2$.

The coefficients c_i can be related to the critical limit of appropriate dimensionless ratios of spherical moments m_{2j}. See Ref. [34] for details. We have estimated the first few coefficients c_i from the corresponding series derived from the 25th-order expansions of m_{2j}, using the analysis described in Sec. IV B. The results for the three improved models and our final estimates are reported in Table IX. Other interesting quantities are

$$S_M\equiv M_{\rm gap}^2/M^2, \tag{111}$$

$$S_Z\equiv\chi M^2/Z_{\rm gap}, \tag{112}$$

where $M_{\rm gap}$ (the mass gap of the theory) and $Z_{\rm gap}$ determine the long-distance behavior of the two-point function:

$$G(x)\approx\frac{Z_{\rm gap}}{4\pi|x|}e^{-M_{\rm gap}|x|}. \tag{113}$$

As discussed in Refs. [19,34], one may estimate S_M and S_Z from c_2, c_3, and c_4. Indeed, we have

$$S_M=1+c_2-c_3+c_4+2c_2^2+\cdots, \tag{114}$$

$$S_Z=1-2c_2+3c_3-4c_4-2c_2^2+\cdots, \tag{115}$$

where the ellipses indicate contributions that are negligible with respect to c_4. Therefore, one finds $S_M=0.999\,601(6)$ and $S_Z=1.000\,810(13)$. From the result for S_M, one obtains $Q_\xi^+\equiv f_{\rm gap}^+/f^+=1.000\,200(3)$.

A more detailed analysis of the behavior of the structure factor for all momenta can be found in Ref. [71].

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