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The $1/N$ Expansion of Two-Dimensional Spin Models.

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... ἱστορίης ἀπόδεξις ἥδε, ὡς μήτε τὰ γενόμενα ἐξ ἀνθρώπων τῷ χρόνῳ ἐξίτηλα γένηται, μήτε ἔργα μεγάλα τε καὶ θωμαστά ... ἀκλέα γένηται ...

(Herodotus, Histories, 1, 1)

... this (is) the statement of the research, lest human deeds should in time fade, and great and wonderful works ... become unknown ...

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1. – Introduction.

A better qualitative and quantitative understanding of quantum field theories requires an improvement of the analytical and numerical methods of approximation. Lattice field theories are a natural ground of application of large-scale numerical techniques. More efficient algorithms and more powerful computing machines lead to an ever increasing amount of numerical results. These results are, however, affected by two major limitations:

1) the quality of the information grows as the logarithm (or, at best, as a small power) of the numerical effort;

2) the lack of control on the systematic errors possibly induced by some of the numerical techniques is often tantalizing.

Both limitations are intrinsic to the field-theoretical nature of the problem addressed: off-critical systems, and even some bulk properties of critical systems, can usually be studied with great numerical precision as well as with sensible analytical methods. Systems at criticality, and the extraction of their scaling properties, however, constitute a much more difficult challenge.

In view of the above-mentioned limitations, the parallel development of more powerful analytical techniques, even with a limited domain of applicability, is certainly very welcome. On one side it allows the comparison between numerical and analytical results in a controlled environment, and therefore it leads to a check of applicability for those techniques and methods whose reliability cannot be taken for granted *a priori*. On the other side it may improve our understanding of those properties that cannot be directly tested with present-day numerical methods and may strengthen some of the theoretical hypotheses that must unavoidably be used in the field-theoretical interpretation of numerical data. One can easily understand the need for such theoretical pieces of evidence when one realizes that, notwithstanding all perturbative results and the substantial agreement existing in the theoretical physics community, there is at present no independent non-perturbative proof of existence of any asymptotically-free quantum field theory, whose relevance in the construction of models of the fundamental interactions cannot be overstressed.

In this perspective, we think that the $1/N$ expansion [1] (expansion in the number of field components) may be a rather rewarding instrument of analysis not only in the context of continuum field theories, where it has long been known as a major source of non-perturbative information, but also in the case of lattice field theories, where, to the best of our knowledge, the $1/N$ expansion is the only approach leading to some theoretical evidence for the existence of a continuum limit and of a non-vanishing scaling region, where the field-theoretical properties of the models can in principle be explicitly tested with predictable precision.

The conceptual foundations of the power of the $1/N$ expansion are essentially the following:

1) N is an intrinsically adimensional parameter, representing a dependence whose origin is basically group-theoretical, and leading to well-defined field representation for all integer values, hence it is not subject to any kind of renormalization.

2) N does not depend on any physical scale of the theory, therefore we may expect it to play no rôle in the parametrization of criticality. As a consequence there is no physical reason not to expect reasonable convergence properties from an expansion in $1/N$, at least in well-defined regions of the other physical parameters.

Evidence for a finite radius of convergence of the $1/N$ expansion has been produced in a number of instances, notably in the proposed exact S-matrices for a number of two-dimensional bosonic and fermionic models. More generally, the large-order behaviour of the coefficients of the $1/N$ expansion can be studied by applying inverse-scattering techniques to the problem of finding instanton solutions of the effective actions [2-4]. In the case of $O(N)$-symmetric $(\boldsymbol{\phi}^2)^2$ theories in less than four dimensions, the $1/N$ perturbation series can be resummed by a Borel transformation, and in the two-dimensional non-linear σ-model one is led to conjecture the convergence of the series also for the Green's functions.

Till now the major domain of application of the $1/N$ expansion has been in the evaluation of critical exponents for many different classes of models in the range of dimensions comprised between 2 and 4 (lower and upper critical dimensions). At the critical dimension the critical exponents are trivial, but the logarithmic deviations from scaling and the dynamical mass generation lead to the rich phenomenology characteristic of asymptotic freedom. This phenomenology can be studied, on the lattice as well as in the continuum, by applying the $1/N$ expansion in conjunction with proper modifications of the methods usually adopted in standard renormalizable quantum field theories, notably the techniques of regularization and renormalization of the physical parameters. Due to the non-renormalized character of N, the $1/N$ expansion leads to results whose renormalization group invariance properties are much more transparent than those of standard perturbation theory. However, as far as we could check, the $1/N$ expansion commutes with perturbation theory, and therefore it provides a direct reinterpretation and an unambiguous resummation of the perturbative results. Such phenomena as the effects of a change of regularization scheme, the rôle of dimensional transmutation in the parametrization of renormalization group invariance, the relationship between dynamical mass generation and Borel ambiguity in resummations, the interplay between dimensional regularization, minimal subtraction, and ε-expansion, the mixing effects and the subtraction of perturbative tails in the evaluation of quantum expectation values of composite operators, all find a specific and transparent illustration in the context of the $1/N$ expansion already at the lowest non-trivial order of computations [5, 6].

The main purpose of the present review is to describe the results that may be obtained by applying the $1/N$ expansion to the lattice versions of two-dimensional spin models. Therefore, we shall only briefly sketch the main results presented in the (wide) literature on all the above-mentioned topics, focusing only on those continuum results that are essential in order to introduce their lattice counterparts.

The subject of lattice $1/N$ expansion has not till now received a systematic treatment: as a consequence many sections of the present paper (especially sect. **5** and sects. **7-14**) are essentially original work by the authors.

In order to present a few peculiar techniques and results of the application of the $1/N$ expansion to renormalizable lattice field theories, we found it proper to focus on a specific class of models. These models should be simple enough to make all calculations as short and understandable as possible, as well as to make accurate numerical simulations feasible now or in the near future. However, it was necessary for the completeness of the presentation to deal with a sufficiently rich phenomenology, notably non-trivial mass spectra, gauge and topological properties, besides the obvious request of perturbative asymptotic freedom. After much thinking, we decided to study a two-parameter model of two-dimensional spin fields with $U(N)$ global symmetry [7]. This model interpolates between the standard $O(2N)$ [8-10] and CP^{N-1} models [11, 12], and for special values of the parameters it represents the gauge-fixed bosonized version of the minimal coupling of massless fermions to CP^{N-1} fields [13].

The present paper is organized as follows:

In sect. **2** we briefly discuss the general results that have been obtained in the study of d-dimensional spin models, with special emphasis on the topics related to perturbative and non-perturbative renormalizability.

In sect. **3** we introduce a class of $1/N$-expandable two-dimensional spin models, and discuss the qualitative picture of their properties that one may draw from a large-N analysis.

In sect. **4** we review a number of exact results, especially factorized S-matrices, that apply to the models under consideration for peculiar values of the parameters.

In sect. **5** we discuss the $1/N$ expansion in the continuum version of the models, introducing our regularization and renormalization procedure, defining observables, and extracting some quantitative $O(1/N)$ physical predictions.

Section **6** is dedicated to the presentation of a number of alternative lattice formulations and motivates our choice of a lattice action, which will turn out to depend explicitly on an Abelian vector field and on an extra parameter eventually allowing for a Symanzik tree-level improvement of the action.

In sect. **7** the basic ingredients of the $1/N$ expansion on the lattice, effective action, propagators, and vertices, are introduced.

Section **8** is devoted to a specific technical problem, the search for integral representations of the effective lattice propagators that, in the case of nearest-neighbour interactions, allow a substantial simplification of the numerical tasks in the evaluation of the effective Feynman diagrams.

In sect. **9** we introduce the basic ingredient of all lattice computations in the scaling (field-theoretical) regime, the asymptotic expansion of the lattice propagators for small values of the (dynamically generated) mass gap, *i.e.* for values of the correlation length much bigger than the lattice spacing (we are always working in the infinite-volume limit).

In sect. **10** special techniques for the asymptotic expansion in the case of nearest-neighbour interactions discussed in sect. **8** are presented.

Section **11** is dedicated to applying the above-mentioned results to the actual $O(1/N)$ evaluation of physical quantities in the scaling region of our class of models. We discuss the possible definitions of correlation length, show the universality of the lattice results, and give a full analysis of the simplest correlation function, the two-point correlator of the fundamental fields, including the evaluation of the lattice wave-function renormalization factor.

In sect. **12** the above-mentioned lattice contributions to physical quantities are explicitly evaluated in the case of nearest-neighbour interactions.

Section **13** is devoted to the issue of topological operators on the lattice; different definitions are analysed and compared in the context of the $1/N$ expansion of CP^{N-1} models.

In sect. **14** we rephrase our results in the language of standard perturbation theory and perturbative renormalization group. We discuss the evaluation of the ratio of Λ parameters in the context of the $1/N$ expansion and extract an explicit representation of the $O(1/N)$ contributions to the lattice renormalization group β-function, clarifying some subtleties concerning the non-commutativity of some limits at the border of the space of parameters, which however does not affect the physical predictivity of the model.

In sect. **15** we review some results concerning the possibility of performing a finite-size scaling analysis of spin models by the help of $1/N$-expansion techniques.

In sect. **16** we analyse the attempts at extracting physical predictions for models at low N by computing higher orders of the $1/N$ expansion on finite lattices.

In the same perspective, we discuss in sect. **17** an alternative approach to $1/N$-expandable spin models based on truncated Schwinger-Dyson equations.

Section **18** is devoted to a review of the results that can be obtained by applying the methods discussed here to a wide class of $1/N$-expandable fermionic lattice models.

Finally in sect. **19** we briefly discuss the relevance of our results and draw our conclusions.

2. – Spin models in d dimensions and renormalizability.

The renormalization group properties of two-dimensional spin models, notably asymptotic freedom, are the foundations of our belief in the existence of non-trivial renormalized quantum field theories describing the critical behaviour of these models in the coupling domain lying in the neighbourhood of the (trivial) critical coupling $g_c = 0$ ($\beta_c = \infty$). In the case of $1/N$-expandable spin models, the corresponding field theories are non-linear σ-models defined on symmetric spaces. These in turn may also be thought of as special limits of linear σ-models (sometimes coupled to gauge fields) enjoying the proper group symmetries.

In order to understand properly the renormalization group properties of these models, it is certainly convenient to extend the analysis by treating the physical dimension d as a continuous parameter in the range $2 \leq d \leq 4$. It is now possible to compare the $\varepsilon = d - 2$ expansion of the non-linear models with the $\varepsilon' = 4 - d$

expansion of the linear models. These latter theories are known to be superrenormalizable, with an ultraviolet-stable fixed point at the origin and an infrared-stable fixed point at «strong» coupling when $\varepsilon' > 0$. Since the infrared limit of linear models has the same relevant operator content as the ultraviolet limit of the non-linear models, the latter must also be renormalizable when $\varepsilon < 2$ around their non-trivial ultraviolet fixed point [14, 15]. As a consequence, the critical exponents should take the same value when computed in the linear and non-linear models at the same dimensionality, and in particular the critical exponents of the non-linear σ-models should become trivial at $d = 4$. As we shall see, this phenomenon is beautifully illustrated in the $1/N$ expansion.

In the two-dimensional limit $\varepsilon \to 0$, however, general theorems [16, 17] insure the impossibility of spontaneous breaking of continuous symmetries. Therefore there is no weak-coupling, broken-symmetry phase, and $g_c = 0$ is an ultraviolet fixed point, around which logarithmic deviations from scaling are allowed (asymptotic freedom). Since no massless modes can be present, non-perturbative mass generation must occur.

This by now standard scenario has been the starting point for most perturbative studies of non-linear σ-models. After Polyakov's pioneering paper [18], perturbative ultraviolet renormalizability was discussed by Brezin, Zinn-Justin, and Le Guillou [14, 19, 20] and by Bardeen, Lee, and Shrock [21] for $O(N)$ models, and by Valent [22] for CP^{N-1} models. The extension to more general symmetric spaces was suggested by Eichenherr and Forger [23, 24] and discussed by Pisarski [25], by Duane [26], and by Brezin and coworkers [27]. Quite naturally, one adopts dimensional regularization and evaluates the renormalization group β- and γ-functions in the minimal subtraction ($\overline{\text{MS}}$) scheme. However, a rigorous treatment of these models shows that generic Green's functions are plagued in two dimensions by severe infrared divergences. This problem was first tackled by Jevicki [28] and by Elitzur [29], who showed that two-dimensional Green's functions that are fully invariant under the symmetry group of the model could be computed (in low orders of perturbation theory) and found to be infrared-finite. This property was exploited in refs. [30-32] in the context of dimensional regularization, and was given a rigorous proof to all orders of perturbation theory by David [33-35]. Three-loop calculations for the renormalization group functions were first presented by Hikami and Brezin [36] for $O(N)$ models, and by Hikami [37] for CP^{N-1} models. Extensions to more general symmetric spaces was given in refs. [38, 39]. Anomalous dimensions were computed by Wegner and collaborators to three-loop order [40], and later to four-loop order [41, 42]. The four-loop order β-function computation was completed in refs. [43, 44]. It would be beyond the purposes of the present review to give any detail of the above-mentioned computations. A collection of results is presented in Appendix A for easy reference.

In the context of the perturbative approach, another thoroughly investigated issue is the classical thermodynamics of the models at non-zero external magnetic field, which ensures the absence of the infrared divergences discussed above. Renormalization and scaling behaviour were discussed in ref. [14] and reconsidered by Jolicoeur and Niel [45, 46], who exploited the scaling properties to devise an extrapolation method allowing for non-perturbative predictions in the limit of vanishing magnetic field.

When we turn to the approach based on the $1/N$ expansion, willing to investigate the renormalizability properties of the models for $2 \leq d \leq 4$, we must dramatically change our focus from a situation where the parameter ε can be considered infinitesimal and employed as an ultraviolet regulator to the case where the physical dimensionality is a fixed finite parameter. Nothing prevents us in principle from using some of the dimensional regularization techniques, and this is one of the basic ingredients in the study of d-dimensional models around criticality and in the evaluation of critical exponents in the $1/N$ expansion. An impressive series of results were obtained by Abe [47-49], Brezin and Wallace [50], and Ma [51, 52], and more recently with improved techniques by Vasilev and coworkers [53-56]. In $O(N)$ models, the critical exponent η is by now known to $O(1/N^3)$, while the critical exponent ν is known to $O(1/N^2)$; $O(1/N)$ results are available for CP^{N-1} models. Following the procedure indicated in ref. [36], these results are also the starting point for an evaluation of the renormalization group β- and γ-functions (in dimensional regularization and minimal subtraction) at the same orders of the $1/N$ expansion. All the results for critical exponents confirm the above-mentioned observations about universality between non-linear and linear models and triviality in $d = 4$.

We want to stress that, as far as the present evidence goes, the ε and $1/N$ expansions appear to be strictly commuting when applied to the evaluation of physical quantities, such as the critical exponents.

The problem of renormalizability in the framework of the $1/N$ expansion was studied by Symanzik [57, 58] and by Arefeva and collaborators. In refs. [59-63] the ultraviolet renormalizability of the three-dimensional $O(N)$ models in both the symmetric and the broken-symmetry phase was shown to all orders of $1/N$ by applying dimensional regularization. The ultraviolet renormalizability of CP^{N-1} models when $d = 2$, 3 was shown in ref. [64] by similar methods. However, in order to prove the existence of a renormalized critical theory free of infrared divergences, it was originally necessary to give up dimensional regularization and attack the problem from the point of view of BPHZL renormalization, which was done for three-dimensional $O(N)$ models in ref. [65]. Subsequently the results was generalized to all $2 < d < 4$ by the introduction of analytic regularization [66].

A different view of renormalizability for asymptotically-free $1/N$-expandable field theories has been put forward by Rim and Weisberger [67]. The essential, if subtle, equivalence of this point of view with more standard dimensional regularization approaches has, however, been exposed in ref. [5].

A very important issue in the context of the $1/N$ expansion of CP^{N-1} models is the relevance of classical instanton configurations, that appear to be non-perturbative in the expansion parameter $1/N$, and therefore might in principle invalidate conclusions obtained in a purely perturbative context. The problem was, however, solved by Jevicki [68], who showed that, at the quantum effective-action level, instantons, instead of being stationary points, appear in the form of poles. One may then demonstrate that the $1/N$ expansion and the semiclassical method correspond to two alternative contour integrations of the functional integral. Further insight on the rôs, it was originally necessaryby David [69], who discussed also the problem of summability of the instanton contributions, computed in refs. [70, 71]. The quantum statistics of CP^{N-1} models was studied by Affleck [72, 73] also in connection with the topological properties (θ-dependence) of the models, and later analysed and reviewed by Actor [74]. An extension to CP^{N-1} models coupled to fermions was discussed in ref. [75]. For the sake of completeness, we also mention that a different non-perturbative approach to

CP^{N-1} models, based on the rôle of «torons» (classical solutions with fractional topological charge) has been put forward in recent years by Zhitnitsky [76] and found to agree with large-N predictions.

A more general non-perturbative issue that may be addressed in the context of the $1/N$ expansion is the existence and the rôle of infrared renormalons [77], appearing as singularities on the positive real axis of the Borel transform in massless ultraviolet-free theories, and related to the appearance of non-perturbative expectation values. These in turn are the basic ingredients in the operator product expansion approach advocated by Shifman, Vainshtein and Zakharov in order to describe large-distance effects in asymptotically free theories [78].

David showed that, in the context of the $1/N$ expansion of $O(N)$ models, non-perturbative terms can be organized in an operator expansion, but they have infrared renormalons [79]; these renormalons cancel against the corresponding renormalons appearing in the coefficients of the operator product expansion when Green's functions (involving only zero-dimension operators) are computed. According to the same author [80, 81], only in well-definite instances (*e.g.*, the topological charge density, and other quantities with a direct physical meaning) non-perturbative expectation values can be defined unambiguously. In any case, it is possible to show that, in each order in $1/N$, the $O(N)$ two-dimensional S-matrix amplitudes can be written as series in powers of the dynamically generated mass times a convergent perturbative series [82]. For the partially different point of view supported by the ITEP group, one should see refs. [83, 84], where the issue of the operator product expansion in the context of the $1/N$ expansion of $O(N)$ models is also discussed.

The subject of the operator product expansion and renormalizability for critical $O(N)$ models in dimension $2 < d < 4$ (where non-trivial criticality exists) has been thoroughly investigated in recent years by Lang and Ruhl [85-90].

Finally we should mention that the possibility of including non-perturbative effects directly into the perturbative expansion has been explored by Davis and Nahm, who discussed both $O(N)$ models [91] and CP^{N-1} models [92], showing that proper normal-ordering may lead automatically to the inclusion of a non-perturbative mass gap in the perturbative series (and confinement in CP^{N-1} models), and the result of this procedure commutes with the $1/N$ expansion [93]. In a related development [94] the vacuum structure of the $O(N)$ model is studied by a variational technique and agreement with conventional large-N results is found.

3. – $1/N$-expandable two-dimensional spin models.

In order to achieve some generality, we shall investigate the properties of a two-parameter class of $1/N$-expandable spin models, described by the continuum action

$$S = N\int \mathrm{d}^2x\,\{\beta_v\,\partial_\mu\bar{z}\,\partial_\mu z + \beta_g\,\overline{D_\mu z}\,D_\mu z\}\,, \tag{3.1}$$

where z is an N-component complex field subject to the constraint

$$\bar{z}z = 1 \tag{3.2a}$$

and a covariant derivative $D_\mu = \partial_\mu + iA_\mu$ has been defined in terms of the composite gauge fields

$$A_\mu \equiv \frac{1}{2} i \{\bar{z}\,\partial_\mu z - \partial_\mu \bar{z}\,z\} = i\bar{z}\,\partial_\mu z. \tag{3.2b}$$

This action was introduced first by Samuel [7] and it is an interpolating action between pure CP^{N-1} models ($\beta_v = 0$) and $U(N)$ vector models ($\beta_g = 0$), which in turn are nothing but $O(2N)$ vector models.

We notice that pure CP^{N-1} models enjoy a $U(1)$ gauge invariance related to the local transformations

$$z(x) \rightarrow \exp[i\lambda(x)]\,z(x), \tag{3.3a}$$

$$\bar{z}(x) \rightarrow \exp[-i\lambda(x)]\,\bar{z}(x), \tag{3.3b}$$

$$A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \lambda(x). \tag{3.3c}$$

This invariance will play an important rôle in determining the structure of the effective vertices.

We can introduce the (rescaled) weak-coupling parameter f by the following change of variables:

$$\frac{1}{2f} = \beta_v + \beta_g, \tag{3.4a}$$

$$\varkappa f = \frac{\beta_v}{\beta_g}. \tag{3.4b}$$

The coupling constant $\varkappa$, as defined by eqs. (3.4), enjoys the property of non-renormalization, *i.e.* the renormalization group trajectories in the (β_g, β_v)-plane (in the continuum version of the model) are just the curves of constant $\varkappa$ and correspond to physically different theories. This property will emerge rather clearly from the discussion of the $1/N$ expansion. The renormalization group trajectories are plotted in fig. 1 for several values of $\varkappa$.

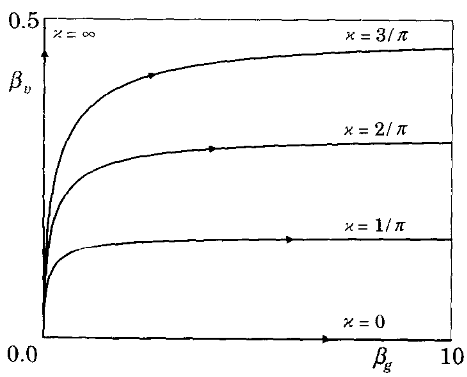


Fig. 1. – Renormalization group trajectories in the $\{\beta_g, \beta_v\}$-plane.

The $1/N$ expansion is achieved as usual by implementing the constraints (3.2) by Lagrange-multiplier fields α and θ'_μ. The manipulations are quite standard and we obtain

$$(3.5)\quad S=\frac{N}{2f}\int d^2x\left\{\partial_\mu\bar z\partial_\mu z+\frac{1}{1+\varkappa f}(\bar z\partial_\mu z)^2+i\alpha(\bar z z-1)+\frac{1}{1+\varkappa f}(\theta'_\mu-i\bar z\partial_\mu z)^2\right\}=$$
$$=\frac{N}{2f}\int d^2x\left\{\frac{\varkappa f}{1+\varkappa f}\partial_\mu\bar z\partial_\mu z+\frac{1}{1+\varkappa f}|(\partial_\mu+i\theta'_\mu)z|^2+i\alpha(\bar z z-1)\right\}.$$

We can now perform the Gaussian integration over the z-fields and obtain the effective action

$$(3.6)\quad S_{\text{eff}}=N\,\mathrm{Tr}\ln\left\{\frac{\varkappa f}{1+\varkappa f}(-\partial_\mu\partial_\mu)+\frac{1}{1+\varkappa f}(-\bar D_\mu D_\mu)+i\alpha\right\}+\frac{N}{2f}(-i\alpha),$$

where now $D_\mu=\partial_\mu+i\theta'_\mu$. Finally, rescaling the multiplier field θ'_μ to $\theta_\mu=\theta'_\mu/(1+\varkappa f)$ and introducing the vacuum expectation value of the α-field, $\alpha(x)=\langle\alpha\rangle+\alpha_q(x)$, $\langle\alpha\rangle=-im_0^2$, we obtain the following form of the effective action:

$$(3.7)\quad S_{\text{eff}}=N\,\mathrm{Tr}\ln\{-\partial_\mu\partial_\mu-i\{\partial_\mu,\theta_\mu\}+m_0^2+i\alpha_q\}+$$
$$+\frac{N}{2f}\{-m_0^2-i\alpha_q+(1+\varkappa f)\theta_\mu\theta_\mu\}.$$

In the large-N limit, the value of m_0^2 is determined, as a function of f only, by the saddle-point condition (gap equation)

$$(3.8)\quad \frac{1}{2f}=\int\frac{d^2p}{(2\pi)^2}\frac{1}{p^2+m_0^2}.$$

Equation (3.8) is in need of ultraviolet regularization; we shall come to this point in sect. **5**.

By taking the second functional derivative of the effective action around the saddle point, we may now obtain the propagators of the quantum fluctuations associated with the fields α_q and θ_μ; both are $O(1/N)$ quantities that can be expressed by the functions

$$(3.9a)\quad \Delta^{-1}_{(\alpha)}(p)=\int\frac{d^2q}{(2\pi)^2}\frac{1}{q^2+m_0^2}\frac{1}{(p+q)^2+m_0^2}=\frac{1}{2\pi p^2\xi}\ln\frac{\xi+1}{\xi-1},$$

$$(3.9b)\quad \Delta^{-1}_{(\theta)\mu\nu}(p)=\left(\varkappa+\frac{1}{f}\right)\delta_{\mu\nu}-\int\frac{d^2q}{(2\pi)^2}\frac{(p_\mu+2q_\mu)(p_\nu+2q_\nu)}{[q^2+m_0^2][(p+q)^2+m_0^2]}=$$
$$=\varkappa\delta_{\mu\nu}+\frac{1}{2\pi}\left(\xi\ln\frac{\xi+1}{\xi-1}-2\right)\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right),$$

where $\xi=\sqrt{1+4m_0^2/p^2}$. The effective propagators in d dimensions are presented for reference in Appendix B.

It is now crucial to observe that all the high-order effective vertices resulting from eq. (3.7) by taking higher functional derivatives are completely unaffected by the value of $\varkappa$. As a consequence, the vertices share the gauge properties enjoyed by the CP^{N-1} model, and in particular transversality. This fact in turn implies the possibility of replacing the propagator (3.9*b*), *i.e.*,

$$\Delta_{(\theta)\mu\nu}(p) = \Delta_{(\theta)}(p)\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \frac{1}{\varkappa}\frac{p_\mu p_\nu}{p^2}, \tag{3.10a}$$

where

$$\Delta_{(\theta)}^{-1}(p) = \frac{1}{2\pi}\left(\xi \ln\frac{\xi+1}{\xi-1} - 2 + 2\pi\varkappa\right), \tag{3.10b}$$

with its transverse part, when computing expectations of gauge-invariant operators. Equation (3.10) shows that $\varkappa$ is a «physical» parameter, related to the ratio of the mass of the Lagrangian field z (m_0 in the large-N limit) to the mass m_θ of the propagating field θ_μ. The exact relationship between the two masses is expressed in the large-N limit by the equation

$$\sqrt{\frac{4m_0^2}{m_\theta^2} - 1}\ \operatorname{arcctg}\sqrt{\frac{4m_0^2}{m_\theta^2} - 1} = 1 - \pi\varkappa, \qquad 0 < \varkappa < \frac{1}{\pi}. \tag{3.11}$$

This is the basic physical reason why $\varkappa$ is not subject to renormalization. $m_\theta(\varkappa)$ is plotted in fig. 2.

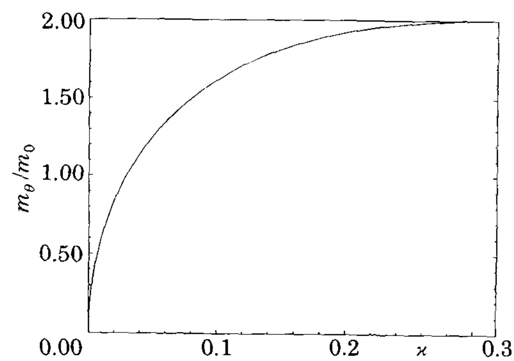


Fig. 2. – The large-N mass ratio m_θ/m_0, plotted as a function of $\varkappa$.

Let us now briefly describe the classes of different physical theories parametrized by $\varkappa$.

1) When $\varkappa = 0$, we get $m_\theta = 0$: θ_μ becomes a dynamical gauge field giving rise to a linear confining potential between z and $\bar{z}$, and the physical states are the bound states that are singlets under gauge transformations. This is the well-known physical picture of the CP^{N-1} models [95-97].

2) For very small values of $\varkappa$ the above picture is substantially unchanged: in the absence of a gauge symmetry the z-fields are not automatically confined by Elitzur's theorem; however, their mass is so much higher than that of their bound states to make them effectively disappear from the physical spectrum.

3) With growing $\varkappa$ an inversion occurs and the z-fields become the fundamental states of the model, while the mass of the bound states becomes bigger and bigger, when measured in units of m_0.

4) At $\varkappa = 1/\pi$ we meet a threshold: $m_\theta = 2m_0$ and the Yukawa potential that was the remnant of the linear confining potential completely disappears. This is a quite interesting model: it is easy to get convinced that the corresponding action is nothing but the effective action resulting from the functional integration over a set of N-component massless fermion fields minimally coupled to the «gauge» field A_μ (*cf.*, *e.g.*, refs. [13, 98]). This model has no quantum anomaly, and its factorized S-matrix is therefore known: the physical states are in the fundamental representation of $U(N)$ and the bound states have disappeared, as expected.

5) When $\varkappa > 1/\pi$, there are no bound states, and the models interpolate smoothly from $U(N)$ to $O(2N)$ symmetry. In particular, for integer values of $n = \pi\varkappa$ the models describe the minimal gauge-invariant coupling of n «flavours» of massless fermions [99]. This picture gives further support to the notion that $\varkappa$ is a physical parameter not subject to renormalization.

6) Finally, when $\varkappa \to \infty$ the effective field θ_μ completely decouples and we are left, as expected, with the well-known $O(2N)$ non-linear sigma-model, possessing a factorized S-matrix for the fundamental $2N$ real fields and showing absence of bound states.

4. – Review of exact results.

In sect. **3** we mentioned that, for special values of the parameter $\varkappa$, a number of exact results are available, especially concerning exact factorized S-matrices and bound-state spectra. We must, however, keep in mind that these results have been obtained in a rather indirect way, by applying such methods as analytic S-matrix theory or Bethe Ansatz. As a consequence, the $1/N$ expansion offers the possibility of verifying the applicability of the above-mentioned methods to the models at hand and therefore the correctness of the physical interpretation. For future reference and comparison, we would like to present here a short review of these exact results.

We start from the observation that the key ingredient for the possibility of constructing an exact S-matrix is the assumption of factorization of multiparticle amplitudes into two-particle amplitudes, *i.e.* the absence of particle production [100]. This property is in turn related to the existence of higher-order conservation laws. Consider the standard Noether current j_μ associated with the global symmetry, and notice that the non-local charge [101]

$$Q^{(\mathrm{nl})} = \frac{1}{2}\int \mathrm{d}x_1\,\mathrm{d}x_2\,\varepsilon(x_1 - x_2)\,[j_0(t,\,x_1)j_0(t,\,x_2)] + \int \mathrm{d}x\,j_1(t,\,x)\,, \tag{4.1}$$

where $\varepsilon(x)$ is the sign function, is classically conserved owing to the equations of motion. At the quantum level one may show that this conservation law is in general spoiled by anomalies generated by the renormalization process. These anomalies are non-perturbative, but their coefficients can be calculated in perturbation theory. A detailed analysis shows that in $O(2N)$ models the anomaly is actually absent [102], while in the pure CP^{N-1} case one finds [103, 104]

$$\frac{\mathrm{d}Q^{(\mathrm{nl})}}{\mathrm{d}t}=\frac{N}{\pi}\int\varepsilon^{\mu\nu}\partial_\mu\theta_\nu(t,\ x)\,\mathrm{d}x. \tag{4.2}$$

However, when we consider the inclusion of minimally-coupled massless fermions, we realize that the non-local charge is classically conserved only if we include a contribution from the fermionic axial vector current. The axial current in turn is known to possess a quantum anomaly, opposite in sign to the r.h.s. of eq. (4.2). As a consequence, the modified non-local charge has no net quantum anomaly and the corresponding model turns out to possess a factorized S-matrix [104, 105]. In our language, these results imply the possibility of finding explicit S-matrices in the cases $\varkappa=\infty$ and $\varkappa=1/\pi$, respectively.

Without belaboring on the techniques used in order to solve the factorization equations, we only recall that the Fock space is decomposed irreducibly into subspaces labelled by a definite particle number n. States are identified by sets of particle momenta $\{P_r\}$, and factorization is expressed by

$$\langle P'(\mathrm{out})\,|\,P(\mathrm{in})\rangle=\left\langle P'(\mathrm{in})\left|\prod_{1\le r\le s\le n}S(P'_r,\ P_s)\right|P(\mathrm{in})\right\rangle, \tag{4.3}$$

where $S(P'_r,\ P_s)$ is the two-particle S-matrix. S is best expressed in terms of rapidity variables $\theta_r=\mathrm{arctgh}(P_{r,1}/P_{r,0})$. Imposing $O(2N)$ symmetry [106] one may assume

$$\begin{aligned}\langle\theta'_1,\ i;\ \theta'_2,\ j(\mathrm{out})\,|\,\theta_1,\ k;\ \theta_2,\ l(\mathrm{in})\rangle&=\\=\delta(\theta_1-\theta'_1)\,\delta(\theta_2-\theta'_2)\,&[\delta_{ij}\delta_{kl}S_1(\theta)+\delta_{ik}\delta_{jl}S_2(\theta)+\delta_{il}\delta_{jk}S_3(\theta)]+\\+\delta(\theta_1-\theta'_2)\,\delta(\theta_2-\theta'_1)\,&[\delta_{ij}\delta_{kl}S_1(-\theta)+\delta_{il}\delta_{jk}S_2(-\theta)+\delta_{ik}\delta_{jl}S_3(-\theta)],\end{aligned} \tag{4.4}$$

where $\theta=\theta_1-\theta_2$. Solving the constraints one finds

$$S_3(\theta)=-\frac{i\pi}{(N-1)\,\theta}\,S_2(\theta), \tag{4.5a}$$

$$S_1(\theta)=-\frac{i\pi}{(N-1)\,(i\pi-\theta)}\,S_2(\theta), \tag{4.5b}$$

and, assuming minimality in the number of poles and zeros in the physical sheet,

$$S_2(\theta)=R(\theta)\,R(i\pi-\theta), \tag{4.6}$$

where

$$(4.7) \qquad R(\theta) = \frac{\Gamma\left(\frac{1}{2(N-1)} - \frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} - \frac{i\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} - \frac{1}{2(N-1)} - \frac{i\theta}{2\pi}\right)\Gamma\left(-\frac{i\theta}{2\pi}\right)}.$$

The result shows no bound-state poles [107], and was checked in a $1/N$ expansion up to second order [108].

CP^{N-1} models with minimally-coupled fermions in turn correspond to $SU(N)$ symmetry, and the form of the factorized S-matrix is [98, 109, 110]

$$(4.8a) \quad \langle \theta'_1, i; \theta'_2, j(\text{out}) | \theta_1, k; \theta_2, l(\text{in}) \rangle = \\ = \delta(\theta'_1 - \theta_1)\,\delta(\theta'_2 - \theta_2)\,[\delta_{ik}\delta_{jl}u_1(\theta) + \delta_{jk}\delta_{il}u_2(\theta)] - \\ - \delta(\theta'_1 - \theta_2)\,\delta(\theta'_2 - \theta_1)\,[\delta_{jk}\delta_{il}u_1(\theta) + \delta_{ik}\delta_{jl}u_2(\theta)],$$

$$(4.8b) \quad \langle \theta'_1, i; \bar\theta'_2, j(\text{out}) | \theta_1, k; \bar\theta_2, l(\text{in}) \rangle = \\ = \delta(\theta'_1 - \theta_1)\,\delta(\theta'_2 - \theta_2)\,[\delta_{ik}\delta_{jl}t_1(\theta) + \delta_{kl}\delta_{ij}t_2(\theta)] + \\ + \delta(\theta'_1 - \theta_2)\,\delta(\theta'_2 - \theta_1)\,[\delta_{ik}\delta_{jl}r_1(\theta) + \delta_{kl}\delta_{ij}r_2(\theta)],$$

where the bar indicates antiparticles. The constraints imply

$$(4.9a) \qquad u_2(\theta) = -\frac{2i\pi}{N\theta}\,u_1(\theta),$$

$$(4.9b) \qquad r_1(\theta) = r_2(\theta) = 0,$$

$$(4.9c) \qquad t_2(\theta) = -\frac{2i\pi}{N(i\pi - \theta)}\,t_1(\theta).$$

Moreover, the crossing symmetry requires

$$(4.9d) \qquad t_1(\theta) = u_1(i\pi - \theta),$$

$$(4.9e) \qquad t_2(\theta) = u_2(i\pi - \theta).$$

Finally, from minimality one obtains

$$(4.10) \qquad t_1(\theta) = \frac{\Gamma\left(\frac{1}{2} - \frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} + \frac{1}{N} + \frac{i\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} + \frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} + \frac{1}{N} - \frac{i\theta}{2\pi}\right)}.$$

There are no bound-state poles: the bosons interact repulsively and the fermions are screened by a «secret» long-range force, while the gauge field loses the zero mass pole. This result was again checked in the $1/N$ expansion.

We must notice that these integrable models can sometimes be solved by the Bethe Ansatz and quantum inverse-scattering methods, always reproducing the above-mentioned results.

For completeness we mention that the factorized S-matrix approach allows in some special cases the determination of exact form factors. In particular, in $O(2N)$ models it is possible to evaluate the matrix elements of the Noether current between the two-particle state and the vacuum. The result was checked in the $1/N$ expansion to $O(1/N)$ by the use of the (explicitly known) spectral representation of the propagator $\Delta_{(\alpha)}(p^2)$ [111]. One can write

$$\Delta_{(\alpha)}(p^2) = 4\pi m_0^2 \frac{\sinh\psi}{\psi}, \qquad \sinh^2 \frac{1}{2}\psi = \frac{p^2}{4m_0^2}. \tag{4.11}$$

Therefore

$$\begin{cases} \Delta_{(\alpha)}(p^2) = 4\pi m_0^2 \left[1 + \dfrac{p^2}{2\pi} \displaystyle\int_{4m_0^2}^{\infty} \mathrm{d}\mu^2 \, \frac{\rho_{(\alpha)}(\mu^2)}{\mu^2 (p^2 + \mu^2)} \right], \\[2ex] \rho_{(\alpha)}(\mu^2) = 2\pi \dfrac{\sinh\phi}{\phi^2 + \pi^2}, \qquad \cosh^2 \dfrac{1}{2}\phi = \dfrac{\mu^2}{4m_0^2}. \end{cases} \tag{4.12}$$

Likewise, when $\varkappa = 1/\pi$, we have

$$\begin{cases} \Delta_{(\theta)}(p^2) = 2\pi \dfrac{\operatorname{tgh}\frac{1}{2}\psi}{\psi} = \displaystyle\int_{4m_0^2}^{\infty} \mathrm{d}\mu^2 \, \frac{\rho_{(\theta)}(\mu^2)}{p^2 + \mu^2}, \\[2ex] \rho_{(\theta)}(\mu^2) = 2\pi \dfrac{\operatorname{ctgh}\frac{1}{2}\phi}{\phi^2 + \pi^2}. \end{cases} \tag{4.13}$$

A last very important exact result that can be obtained from the analysis of the S-matrices for integrable models is the analytic determination of the so-called mass–Λ-parameter ratio, where the Λ-parameter is defined in standard perturbation theory ($\overline{\mathrm{MS}}$ scheme) in terms of the subtraction scale μ and the universal coefficients b_0 and b_1 of the renormalization group β-function:

$$\Lambda_{\overline{\mathrm{MS}}} = \mu \exp\left[- \int \frac{\mathrm{d}g'}{\beta(g')} \right] \cong \mu (b_0 g)^{-b_1/b_0^2} \exp\left[-\frac{1}{b_0 g} \right]. \tag{4.14}$$

The seminal result was obtained by Hasenfratz and collaborators in the case of $O(2N)$ models [112, 113]:

$$m = \left(\frac{8}{e} \right)^{1/(2(N-1))} \frac{1}{\Gamma\left(1 + \dfrac{1}{2(N-1)} \right)} \Lambda_{\overline{\mathrm{MS}}}. \tag{4.15}$$

The analysis can be repeated in the $\varkappa = 1/\pi$ model; the final result is

$$m = \left(\frac{2}{e}\right)^{1/N} \frac{1}{\Gamma\left(1 + \frac{1}{N}\right)} \Lambda_{\overline{\rm MS}} . \tag{4.16}$$

As we shall see in the next section, these results can also be explicitly verified in the context of the $1/N$ expansion.

5. – The $1/N$ expansion in the continuum: regularization and renormalization.

The Feynman rules for the $1/N$ expansion of the model can be easily derived by introducing external currents coupled to the z-fields before perfoming the functional integration. They are summarized in fig. 3. The $1/N$ expansion is an expansion in the loops of the effective fields α_q and θ_μ. In the graphical representation, a closed loop of the z-fields stands for an effective vertex of the expansion; effective vertices can be obtained by taking functional derivatives of the effective action and carry a factor of N.

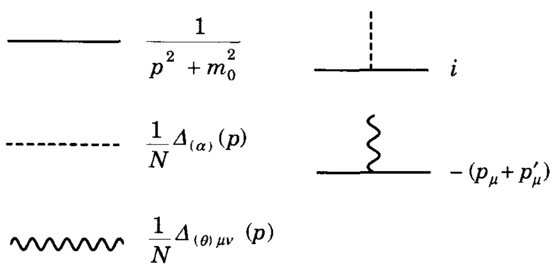


Fig. 3. – Feynman rules for the $1/N$ expansion in the continuum.

The effective vertices actually amount to one-loop integrals over the fundamental field propagators. In two dimensions, all the effective vertices may be in principle computed analytically, but the computation may become very cumbersome in the case of exceptional configurations of momenta, which are often those relevant to the actual computations one would like to perform. For a discussion of this technical problem, *cf.* ref. [114].

While no regularization is needed in the evaluation of the effective vertices, it becomes unavoidable when one wants to compute the Green's functions of the physical fields. In our choice of regularization we were not guided by the usual requirements of Poincaré invariance, gauge invariance, and consistency up to all perturbative orders that made dimensional regularization a favourite tool of quantum field theory. Having in mind our final purpose of performing explicit lattice computations, we rather focused on the requests of computational ease and transparency in the regularization mechanism, which is often obscured in dimensional regularization by the interplay of ultraviolet and infrared singularities.

Relaxing the consistency request down to one-loop consistency, we found that the simplest and most transparent scheme was a kind of sharp-momentum (SM) cut-off. The regularization procedure (roughly formulated for $O(N)$ models in ref. [115] and discussed more precisely in ref. [5]) starts from the observation that, since the Δ propagators are finite, the $O(1/N)$ (one-loop) Feynman integrals appearing in the computation of the n-point Green's functions are integrals of regular functions. They can therefore be regularized by subtracting explicitly the highest powers of the integration variable appearing in the Taylor expansion of the integrand. The lower limit for the integration of the subtraction terms is arbitrary, and we are therefore introducing a dependence on the cut-off value M^2. When the integrals are finite, this dependence disappears when taking the limit $M^2 \to \infty$.

Let us, however, consider the regularized gap equation

$$\frac{1}{2f} = \int \frac{d^2p}{(2\pi)^2} \frac{1}{p^2 + m_0^2} - \int_{M^2}^{\infty} \frac{dp^2}{4\pi} \frac{1}{p^2} = \frac{1}{4\pi} \ln \frac{M^2}{m_0^2}. \tag{5.1}$$

Equation (5.1) allow us to eliminate the dependence on M^2 of any superficially divergent diagram in favour of an explicit dependence on the coupling constant, which in turn will be readsorbed in the renormalization-group–invariant definition of the physical mass and in the wave-function renormalization.

A first obvious analogy with the lattice formulation lies in the fact that we are working with a «bare» coupling constant which can be varied together with the cut-off while keeping all physical quantities constant. As we shall show in sect. **9**, the analogy can be made much more stringent by finding the relationship between the SM cut-off and the lattice cut-off, which will also lead to a natural regularization of the infrared singularities appearing in massless lattice integrals.

It is important to observe that our regularization procedure differs by $O(1/M^2)$ terms from a naïve sharp-momentum cut-off (*i.e.* simply setting the upper integration limit to M^2). This difference is irrelevant in the continuum, but it will become crucial when discussing the relationship with the lattice scheme.

The connection between SM regularization and dimensional regularization (and renormalization) has been explored in some detail in the asymptotic regime of large Euclidean momenta, where perturbation theory holds [5]. We shall not belabor on this point in the present paper.

5'1. *The free energy.* – SM regularization leads to a simple parametrization of the perturbative tails that contribute to the non-scaling part of physical quantities. We can therefore evaluate the scaling contributions to the free energy to $O(1/N)$ in our models, thus also offering a first explicit example of our computational techniques. The free energy is the sum of the connected vacuum diagrams of the effective theory. In lowest orders only the trivial Gaussian integrations over the z, α and θ_μ fields do contribute. Subtracting the perturbative tail and keeping only the scaling part, according to the rules of SM regularization, leads to the following expression for the first two non-trivial contributions:

$$F = N \operatorname{Tr} \ln \frac{p^2 + m_0^2}{p^2} - \frac{N}{2f} m_0^2 + \frac{1}{2} \operatorname{Tr} \ln \frac{\Delta_{(\alpha)}^{(0)}(p)}{\Delta_{(\alpha)}(p)} + \frac{1}{2} \operatorname{Tr} \ln \frac{\Delta_{(\theta)}^{(0)}(p)}{\Delta_{(\theta)}(p)} + O\left(\frac{1}{N}\right), \tag{5.2}$$

where

$$\Delta_{(\alpha)}(p) \xrightarrow[p\to\infty]{} \frac{2\pi p^2}{\ln(p^2/m_0^2)} \equiv \Delta_{(\alpha)}^{(0)}(p), \tag{5.3a}$$

$$\Delta_{(\theta)}(p) \xrightarrow[p\to\infty]{} \frac{2\pi}{\ln(p^2/m_0^2) - 2 + 2\pi\varkappa} \equiv \Delta_{(\theta)}^{(0)}(p). \tag{5.3b}$$

According to our rules, the regularized expression of the free energy is therefore

$$\begin{aligned} F = {} & N\frac{m_0^2}{4\pi} + \frac{1}{2}\int \frac{d^2p}{(2\pi)^2}\left[\ln\ln\frac{\xi+1}{\xi-1} - \ln\ln\frac{p^2}{m_0^2}\right] - \frac{1}{2}\int_{M^2}^{\infty}\frac{dp^2}{4\pi}\frac{2m_0^2}{p^2}\frac{1}{\ln(p^2/m_0^2)} + \\ & + \frac{1}{2}\int\frac{d^2p}{(2\pi)^2}\left[\ln\left(\ln\frac{\xi+1}{\xi-1} + \frac{2\pi\varkappa-2}{\xi}\right) - \ln\left(\ln\frac{p^2}{m_0^2} + 2\pi\varkappa - 2\right)\right] - \\ & - \frac{1}{2}\int_{M^2}^{\infty}\frac{dp^2}{4\pi}\frac{2m_0^2}{p^2}\frac{3-2\pi\varkappa}{\ln(p^2/m_0^2) + 2\pi\varkappa - 2} + O\left(\frac{1}{N}\right) = \\ = {} & N\frac{m_0^2}{4\pi} + \frac{m_0^2}{4\pi}\left[\ln\ln\frac{M^2}{m_0^2} + (3-2\pi\varkappa)\ln\left(\ln\frac{M^2}{m_0^2} + 2\pi\varkappa - 2\right) + c_F(\varkappa)\right] + O\left(\frac{1}{N}\right), \end{aligned} \tag{5.4}$$

where $c_F(\varkappa)$ is a numerical constant which can be computed to all desired numerical accuracy; it is plotted in fig. 4. Notable special cases are

$$c_F(0) \cong 1.18887122, \tag{5.5a}$$

$$c_F\left(\frac{1}{\pi}\right) = 2\gamma_E. \tag{5.5b}$$

The issue of the evaluation of dimensionless SM-regulated one-loop continuum integrals is discussed in Appendix C.

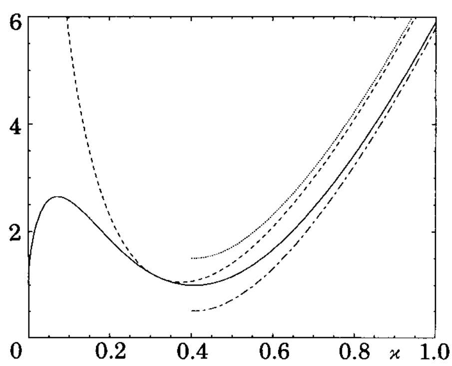


Fig. 4. – The $O(1)$ (subleading) finite scaling part of the free energy c_F, computed from eq. (5.4) (solid line) and from eq. (5.6) (dot-dashed line); the $O(1/N)$ (subleading) finite part of the mass gap c_m, computed from eq. (5.24) (dashed line) and from eq. (5.26) (dotted line).

$c_F(\varkappa)$ may also be evaluated in the context of a $1/\varkappa$ expansion of eq. (5.4). The result is

$$c_F(\varkappa) = (2\pi\varkappa - 3)\ln 2\pi\varkappa + \gamma_E + \ln 4 - 3 + \frac{5}{2\pi}\frac{1}{\varkappa} + O\left(\frac{1}{\varkappa^2}\right). \tag{5.6}$$

The large-$\varkappa$ limit of eq. (5.4) is [116]

$$F \cong N\frac{m_0^2}{4\pi} + \frac{m_0^2}{4\pi}\left[\ln\ln\frac{M^2}{m_0^2} + \gamma_E - \ln\frac{M^2}{4m_0^2} - 1\right]. \tag{5.7}$$

Substituting eq. (5.1) in (5.4), we obtain

$$F = N\frac{m_0^2}{4\pi} + \frac{m_0^2}{4\pi}\left[\ln\frac{2\pi}{f} + (3 - 2\pi\varkappa)\ln\left(\frac{2\pi}{f} + 2\pi\varkappa - 2\right) + c_F(\varkappa)\right] + O\left(\frac{1}{N}\right). \tag{5.8}$$

Now, exploiting the renormalization group invariance of the scaling part of the free energy, we are ready to obtain the renormalization group β-function:

$$\beta(f) = -2\left(\frac{\partial \ln F}{\partial f}\right)^{-1} = -\frac{f^2}{\pi}\left[1 + \frac{1}{N}\frac{f}{2\pi}\left(1 + \frac{3 - 2\pi\varkappa}{1 + f(\varkappa - 1/\pi)}\right) + O\left(\frac{1}{N^2}\right)\right]. \tag{5.9}$$

It is easy to check that the known universal coefficients are correctly reproduced. In particular, when $\varkappa = 1/\pi$

$$\beta(f) = -\frac{f^2}{\pi}\left[1 + \frac{1}{N}\frac{f}{\pi}\right] + O\left(\frac{1}{N^2}\right) \tag{5.10}$$

and when $\varkappa \to \infty$

$$\beta(f) = -\frac{f^2}{\pi}\left(1 - \frac{1}{N}\right)\left[1 + \frac{1}{N}\frac{f}{2\pi}\right] + O\left(\frac{1}{N^2}\right), \tag{5.11}$$

as expected for $O(2N)$ non-linear sigma-models.

5.2. *The two-point function: regularization.* – We now focus our attention on the properties of the invariant two-point correlation function of the fields in the fundamental representation. Let us define

$$G(p) = \frac{1}{2f}\int d^2x \exp[ipx]\langle \bar{z}(x) z(0)\rangle \equiv \tag{5.12}$$

$$\equiv \frac{1}{p^2 + m_0^2} - \frac{1}{N}\frac{1}{p^2 + m_0^2}\Sigma_1(p)\frac{1}{p^2 + m_0^2} + O\left(\frac{1}{N^2}\right).$$

The $O(1/N)$ contributions to the two-point function are drawn in fig. 5. We

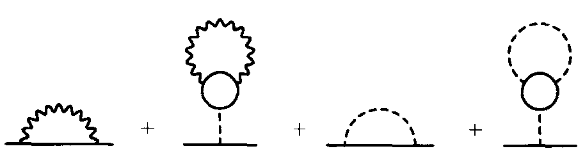


Fig. 5. – $O(1/N)$ contributions to the two-point function.

obtain

$$(5.13)\quad \Sigma_1(p) = \int \frac{d^2k}{(2\pi)^2}\frac{\Delta_{(\alpha)}(k)}{(p+k)^2+m_0^2} - \\ - \Delta_{(\alpha)}(0)\int \frac{d^2k}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\frac{1}{(q^2+m_0^2)^2}\frac{\Delta_{(\alpha)}(k)}{(q+k)^2+m_0^2} - \\ -\int \frac{d^2k}{(2\pi)^2}\frac{\Delta_{(\theta)\mu\nu}(k)}{(p+k)^2+m_0^2}(2p_\mu+k_\mu)(2p_\nu+k_\nu) + \\ + \Delta_{(\alpha)}(0)\int \frac{d^2k}{(2\pi)^2}\frac{d^2q}{(2\pi)^2}\frac{1}{(q^2+m_0^2)^2}\Delta_{(\theta)\mu\nu}(k)\frac{(2q_\mu+k_\mu)(2q_\nu+k_\nu)}{(q+k)^2+m_0^2}.$$

By straightforward manipulations (essentially replacing zero-momentum insertions of the α_q-field with derivatives with respect to m_0^2), eq. (5.13) can be recast in the form

$$(5.14)\quad \Sigma_1(p) = \int \frac{d^2k}{(2\pi)^2}\frac{\Delta_{(\alpha)}(k)}{(p+k)^2+m_0^2} + \frac{1}{2}\Delta_{(\alpha)}(0)\int\frac{d^2k}{(2\pi)^2}\Delta_{(\alpha)}(k)\frac{\partial}{\partial m_0^2}\Delta_{(\alpha)}^{-1}(k) + \\ + \int\frac{d^2k}{(2\pi)^2}\Delta_{(\theta)\mu\nu}(k)\left[\delta_{\mu\nu} - \frac{(2p_\mu+k_\mu)(2p_\nu+k_\nu)}{(p+k)^2+m_0^2}\right] + \\ + \frac{1}{2}\Delta_{(\alpha)}(0)\int\frac{d^2k}{(2\pi)^2}\Delta_{(\theta)\mu\nu}(k)\frac{\partial}{\partial m_0^2}\Delta_{(\theta)\mu\nu}^{-1}(k).$$

Furthermore, explicit knowledge of the propagators allows us to make use of the identities

$$(5.15a)\quad \frac{\partial}{\partial m_0^2}\Delta_{(\alpha)}^{-1}(p) = -\frac{2}{p^2\xi^2}\left[\Delta_{(\alpha)}^{-1}(p) + \frac{1}{4\pi m_0^2}\right],$$

$$(5.15b)\quad \frac{\partial}{\partial m_0^2}\Delta_{(\theta)}^{-1}(p) = \frac{2}{p^2\xi^2}\left[\Delta_{(\theta)}^{-1}(p) - \varkappa - \frac{p^2}{4\pi m_0^2}\right].$$

As a consequence, we obtain the representation

$$(5.16)\quad \Sigma_1(p) = \int \frac{d^2k}{(2\pi)^2}\frac{\Delta_{(\alpha)}(k)}{(p+k)^2+m_0^2} - \int\frac{d^2k}{(2\pi)^2}\frac{\Delta_{(\alpha)}(k)}{k^2+4m_0^2} + \\ + \int\frac{d^2k}{(2\pi)^2}\Delta_{(\theta)}(k)\left\{1 - \frac{4p^2k^2-4(p\cdot k)^2}{k^2[(p+k)^2+m_0^2]}\right\} - \int\frac{d^2k}{(2\pi)^2}\Delta_{(\theta)}(k)\frac{k^2+4\pi\varkappa m_0^2}{k^2+4m_0^2} + \\ + \frac{1}{\varkappa}\int\frac{d^2k}{(2\pi)^2}\frac{p^2+m_0^2}{k^2}\left[1 - \frac{p^2+m_0^2}{(p+k)^2+m_0^2}\right].$$

The last term in eq. (5.16) reflects the dependence on the longitudinal degrees of freedom of the field. Its contribution can be computed in closed form (after regularization) and it amounts to

$$\frac{1}{\varkappa}(p^2+m_0^2)\left[\frac{1}{4\pi}\ln\frac{M^2}{m_0^2}-\frac{1}{2\pi}\ln\frac{p^2+m_0^2}{m_0^2}\right]. \tag{5.17}$$

Equation (5.17) is singular in the $\varkappa \to 0$ limit, when the model becomes gauge-invariant and the contribution of the longitudinal degrees of freedom becomes gauge-dependent.

The regularized version of eq. (5.16) is obtained by applying the SM scheme prescriptions:

$$\Sigma_1^{\mathrm{reg}}(p) = \Sigma_1(p) - \int_{M^2}^{\infty}\frac{\mathrm{d}k^2}{4\pi}\left[\frac{2\pi}{\ln(k^2/m_0^2)}+\frac{2\pi(3-2\pi\varkappa)}{\ln(k^2/m_0^2)+2\pi\varkappa-2}\right]\frac{2m_0^2}{k^2} - \\ - \int_{M^2}^{\infty}\frac{\mathrm{d}k^2}{4\pi}\left[\frac{2\pi}{\ln(k^2/m_0^2)}-\frac{2\pi\cdot 2}{\ln(k^2/m_0^2)+2\pi\varkappa-2}+\frac{1}{\varkappa}\right]\frac{p^2+m_0^2}{k^2}. \tag{5.18}$$

Equation (5.18) implies the possibility of parametrizing $\Sigma_1^{\mathrm{reg}}(p)$ in the form

$$\Sigma_1^{\mathrm{reg}}(p) = \Sigma_1^{\mathrm{fin}}(p) + \\ + m_0^2\left[\ln\ln\frac{M^2}{m_0^2}+(3-2\pi\varkappa)\ln\left(\ln\frac{M^2}{m_0^2}+2\pi\varkappa-2\right)+c_m(\varkappa)\right] + \\ + (p^2+m_0^2)\left[\frac{1}{2}\ln\ln\frac{M^2}{m_0^2}-\ln\left(\ln\frac{M^2}{m_0^2}+2\pi\varkappa-2\right)+\frac{1}{4\pi\varkappa}\ln\frac{M^2}{m_0^2}\right], \tag{5.19}$$

where $\Sigma_1^{\mathrm{fin}}(p)$ is a regular, M-independent function of p^2 (and $\varkappa$) subject to the normalization condition

$$\Sigma_1^{\mathrm{fin}}(im_0) = 0, \tag{5.20}$$

and $c_m(\varkappa)$ is a numerically computable constant.

5.3. *Mass and wave-function renormalization.* – The interpretation of eqs. (5.19) and (5.20) becomes straightforward in the context of renormalization, when we identify $G(p)$ with the bare two-point function and write

$$\frac{1}{p^2+m_0^2+\frac{1}{N}\Sigma_1^{\mathrm{reg}}(p)} = \frac{Z}{p^2+m^2+\frac{1}{N}\Sigma_1^{\mathrm{fin}}(p)}. \tag{5.21}$$

Equation (5.20) is then the on-shell renormalization condition and

$$(5.22a)\qquad m^2 = m_0^2 + \frac{1}{N} m_0^2 \left[\ln \frac{2\pi}{f} + (3 - 2\pi\varkappa) \ln \left(\frac{2\pi}{f} + 2\pi\varkappa - 2 \right) + c_m(\varkappa) \right],$$

$$(5.22b)\qquad Z = 1 - \frac{1}{N} \left[\frac{1}{2} \ln \frac{2\pi}{f} - \ln \left(\frac{2\pi}{f} + 2\pi\varkappa - 2 \right) + \frac{1}{2\varkappa f} \right].$$

In particular eq. (5.20) ensures that the mass gap m^2 is identified as the pole of the two-point function and its $O(1/N)$ contribution m_1^2 satisfies the condition

$$(5.23)\qquad m_1^2 = \Sigma_1^{\text{reg}}(im_0).$$

Applying eq. (5.23) directly to eq. (5.18) we obtain the representation

$$(5.24)\qquad m_1^2 = \int \frac{d^2p}{(2\pi)^2} \left\{ \Delta_{(\alpha)}(p) \frac{\xi - 1}{p^2 \xi^2} + \Delta_{(\theta)}(p) \left[(\xi - 1) + (\pi\varkappa - 1)\left(\frac{1}{\xi^2} - 1 \right) \right] \right\} - \\ - \int_{M^2}^{\infty} \frac{dp^2}{4\pi} \left[\frac{2\pi}{\ln(p^2/m_0^2)} + \frac{2\pi(3 - 2\pi\varkappa)}{\ln(p^2/m_0^2) + 2\pi\varkappa - 2} \right] \frac{2m_0^2}{p^2},$$

allowing a direct determination of $c_m(\varkappa)$, which is plotted in fig. 4.

In the $\varkappa \to 0$ limit, $c_m(\varkappa)$ is an infrared-divergent quantity, which shows that in the CP^{N-1} models no single-particle mass for the fundamental fields can be consistently defined [114]. This is a reflection of the gauge properties of the model and is further evidence for confinement. More generally, it is pleasant to notice that m_1^2 does not depend on the longitudinal degrees of freedom of the vector field, as a consequence of gauge-invariance of the couplings and the on-shell condition. This allows a direct physical interpretation of eq. (5.24) also in those instances (*e.g.*, the $\varkappa = 1/\pi$ model) where we are dealing with a gauge-fixed version of a theory enjoying gauge symmetry properties. Actually we can compute exactly

$$(5.25)\qquad c_m(1/\pi) = 2\gamma_E.$$

Equation (5.25) can be shown to be consistent with the $1/N$ expansion of the exact result (4.16). The $1/\varkappa$ expansion of eq. (5.24) leads to

$$(5.26)\qquad c_m(\varkappa) = (2\pi\varkappa - 3) \ln 2\pi\varkappa + \gamma_E + \ln 4 - 2 + \frac{5}{2\pi} \frac{1}{\varkappa} + O\left(\frac{1}{\varkappa^2} \right).$$

The large-$\varkappa$ limit of eq. (5.24) is [116-118]

$$(5.27)\qquad m_1^2 \xrightarrow[\varkappa \to \infty]{} m_0^2 \left[\ln \frac{2\pi}{f} + \gamma_E - \frac{2\pi}{f} + \ln 4 \right],$$

consistently with the exact result (4.15). In the same limit one may also show that

$$(5.28)\qquad \lim_{p^2+m_0^2\to 0^+} \frac{\partial \Sigma_1(p)}{\partial p^2} \xrightarrow[\varkappa\to\infty]{} \frac{1}{2}\left[\ln\frac{2\pi}{f} + \gamma_E - \ln\frac{\pi}{2} - 1\right],$$

and the on-shell renormalization condition can be imposed on Z.

It is pleasant to notice that eq. (5.22a) allows an independent determination of the renormalization group β-function and the result is completely consistent with eq. (5.9). As a consequence the adimensional ratio

$$(5.29)\qquad \frac{F}{m^2} = \frac{1}{4\pi}[N + c_F(\varkappa) - c_m(\varkappa)] + O\left(\frac{1}{N}\right)$$

is universal and scheme-independent. It is interesting to notice that one obtains from eqs. (5.6) and (5.26) the relationship

$$(5.30)\qquad c_F(\varkappa) - c_m(\varkappa) = -1 + O\left(\frac{1}{\varkappa^2}\right).$$

An alternative renormalization-group–invariant definition of the correlation length can be defined starting from the second moment of the two-point correlation function

$$(5.31)\qquad \langle x^2 \rangle = \frac{\int d^2x\, \frac{1}{4}x^2 \langle \bar z(x) z(0)\rangle}{\int d^2x \langle \bar z(x) z(0)\rangle}.$$

In momentum space this definition leads to the relationship

$$(5.32)\qquad m_R^2 \equiv \frac{1}{\langle x^2\rangle} \cong \frac{m_0^2 + \frac{1}{N}\Sigma_1(0)}{1 + \frac{1}{N}\Sigma_1'(0)} \cong m_0^2 + \frac{1}{N}(\Sigma_1(0) - m_0^2\Sigma_1'(0)),$$

where

$$(5.33)\qquad \Sigma_1'(p) \equiv \frac{\partial \Sigma_1(p)}{\partial p^2}.$$

Substituting eq. (5.19) in eq. (5.32) and comparing with eq. (5.22a), we can easily show that

$$(5.34)\qquad m_R^2 \equiv m^2 + \delta m_R^2 = m^2 + \frac{1}{N}(\Sigma_1^{\rm fin}(0) - m_0^2\Sigma_1'^{\rm fin}(0)) + O\left(\frac{1}{N^2}\right).$$

Keeping also in mind eq. (5.20), we come to the conclusion that δm_R^2 is amenable to a (typically small) calculable constant, which can be interpreted as a universal scheme-independent adimensional ratio. The $O(1/N)$ contribution to δm_R^2 is plotted as a function of $\varkappa$ in fig. 6.

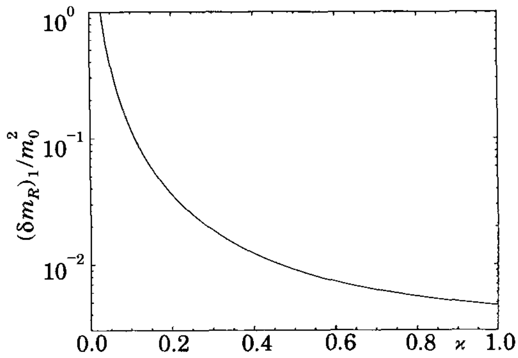


Fig. 6. – The $O(1/N)$ contribution to δm_R^2, as a function of $\varkappa$.

Equation (5.32) becomes singular in the $\varkappa \to 0$ limit, when only gauge-invariant correlations can be sensibly defined. The strategy for such computations is discussed in ref. [114] and is not especially relevant to our present analysis. We only mention that gauge invariance is obtained at the price of introducing «strings» connecting the z-fields, and defining

$$G_{\mathscr{C}}(x,\, y) = \frac{1}{2f}\left\langle \bar{z}(x) \exp\left[i \int_x^0 \mathrm{d}t_\mu \theta_\mu(t)\right] z(0) \right\rangle . \tag{5.35}$$

String renormalization is required. It is then possible to define $\langle x^2 \rangle_{\mathscr{C}}$ in analogy with eq. (5.31).

We stress that the gauge dependence of the definition (5.31) makes the observable m_R^2 dependent on the longitudinal degrees of freedom of the vector field. Therefore it has no direct physical interpretation in those special cases when we want to recover the gauge-invariant properties of a gauge-fixed model. Similar considerations hold for the so-called magnetic susceptibility

$$\chi = \int \mathrm{d}^2x \langle \bar{z}(x) z(0) \rangle = 2fG(0) \cong 2f\left(\frac{1}{m_0^2} - \frac{1}{N}\frac{\Sigma_1(0)}{m_0^4}\right). \tag{5.36}$$

The value of $\Sigma_1^{\mathrm{reg}}(0)$ can be computed analytically for special values of $\varkappa$; ignoring the contributions of the longitudinal degrees of freedom we have (*cf.* ref. [116])

$$\left.\frac{\Sigma_1^{\mathrm{reg}}(0)}{m_0^2}\right|_{\varkappa = 1/\pi} = \frac{3}{2}\left[\ln\frac{2\pi}{f} + \gamma_E - c_1\right], \tag{5.37a}$$

$$\left.\frac{\Sigma_1^{\mathrm{reg}}(0)}{m_0^2}\right|_{\varkappa = \infty} = \frac{3}{2}\left[\ln\frac{2\pi}{f} + \gamma_E - c_1\right] - \frac{2\pi}{f} + \ln 4\,, \tag{5.37b}$$

where

$$c_1 = \ln\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{7}{6}\right)}{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{5}{6}\right)} \cong 0.4861007\,. \tag{5.38}$$

We can adopt a wave-function renormalization condition slightly different from eq. (5.22*b*), defining the renormalization constant $\tilde{Z}$ by

$$\tilde{Z} = \frac{\chi m_R^2}{2f} = 1 - \frac{1}{N}\Sigma_1'(0) + O\left(\frac{1}{N^2}\right). \tag{5.39}$$

It is easy to check that the ratio of the two definitions is a β-independent constant:

$$\frac{\tilde{Z}}{Z} = 1 - \frac{1}{N}\Sigma_1'^{\,\mathrm{fin}}(0) + O\left(\frac{1}{N^2}\right). \tag{5.40}$$

Finally it is possible to compute the renormalization group function $\gamma(f)$, after its definition

$$\gamma(f) = -\beta(f)\frac{\partial}{\partial f}\ln\left[2fZ(f)\right]. \tag{5.41}$$

Ignoring the contributions from the longitudinal degrees of freedom of the vector field, we obtain

$$\gamma(f) = \frac{f}{\pi}\left(1 - \frac{1}{2N}\right)\left[1 + \frac{1}{N}\frac{f}{\pi}\left(1 + \frac{1}{1 + f(\varkappa - 1/\pi)}\right)\right] + O\left(\frac{1}{N^2}\right), \tag{5.42}$$

and, specifically for $O(2N)$ models,

$$\gamma(f) \approx \frac{f}{\pi}\left(1 - \frac{1}{2N}\right)\left[1 + \frac{1}{N}\frac{f}{2\pi}\right]. \tag{5.43}$$

Here and in the following, when checking agreement with the expressions presented in Appendix A, one must perform an appropriate change of variables, whose form may be extracted by comparing SM and $\overline{\mathrm{MS}}$ β-functions.

5·4. *Correlations of composite operators.* – Another very important class of correlation functions is obtained by considering the Green's function of the (gauge-invariant) composite operators

$$P_{ij}(x) = \bar{z}_i(x)\, z_j(x). \tag{5.44}$$

We have shown in ref. [114] that, for CP^{N-1} models ($\varkappa = 0$)

$$G_{ij,kl}(x-y) \equiv \langle P_{ij}(x)\, P_{kl}(y)\rangle_{\mathrm{conn}} = \frac{B(x-y)}{N(N+1)}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right), \tag{5.45}$$

where

$$B(x-y) = \frac{N\langle \bar{z}_i(x)\, z_j(x)\, z_i(y)\, \bar{z}_j(y)\rangle - 1}{N-1} = \frac{4f^2}{N}\left(1 + \frac{1}{N}\right)(\Delta_0^{-1} + \Delta_1^{-1}), \tag{5.46}$$

and in turn $\Delta_0^{-1} = N\Delta_{(\alpha)}^{-1}$ and Δ_1^{-1} is expressed to $O(1/N)$ by the sum of Feynman diagrams drawn in fig. 7. More generally, Δ_1^{-1} is the sum of all the diagrams such that the external α legs emerge from the same effective vertex (generalized tadpole contributions).

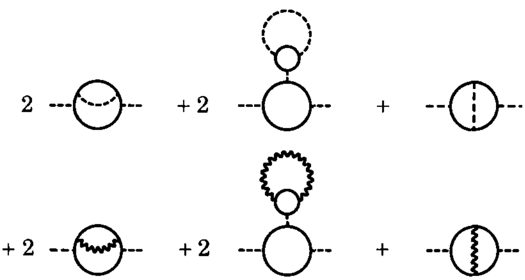


Fig. 7. – Contributions to Δ_1^{-1}, the correlation function of the composite operator $P_{ij}(x)$.

This analysis applies almost literally to the more general case presented here, and we can quote the final result which is just a very slight generalization of eq. (8.20) in ref. [114]:

$$(5.47)\qquad \Delta_1^{-1}(p) = -\int \frac{\mathrm{d}^2k}{(2\pi)^2}\Delta_{(\alpha)}(k)\Big[V_4^{(a)}(p,k) + V_4^{(a)}(p,-k) + V_4^{(b)}(p,k) + \frac{1}{k^2+4m_0^2}\frac{\partial}{\partial m_0^2}\Delta_{(\alpha)}^{-1}(p)\Big] - \int \frac{\mathrm{d}^2k}{(2\pi)^2}\Delta_{(\theta)}(k)\Big\{(k^2+4m_0^2)\cdot [V_4^{(a)}(p,k) + V_4^{(a)}(p,-k)] - 4[V_3(p,k)+V_3(p,-k)] + (k^2+4m_0^2+2p^2)V_4^{(b)}(p,k) + \frac{k^2+4\pi\varkappa m_0^2}{k^2+4m_0^2}\frac{\partial}{\partial m_0^2}\Delta_{(\alpha)}^{-1}(p)\Big\}.$$

The effective vertices entering eq. (5.47) are drawn in fig. 8. The formal definitions of V_3, $V_4^{(a)}$, and $V_4^{(b)}$, together with their explicit expressions in terms of

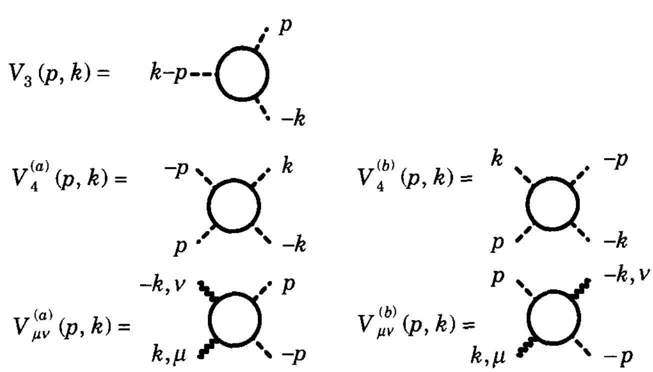


Fig. 8. – Effective vertices in the continuum. All momenta are entering in the diagrams.

elementary functions, can be found in Appendix D. We stress that algebraic manipulations, based on the gauge invariance of the effective vertices, lead to the possibility of replacing all explicit dependence on the mixed vector-scalar vertices appearing in fig. 7 with purely scalar vertices.

Regularization and renormalization are straightforward along the lines presented in ref. [114]. One may analyse the large-k behaviour of the effective vertices and find that the ultraviolet divergence of eq. (5.47) is regulated by the SM counterterm

$$\int_{M^2}^{\infty} \frac{\mathrm{d}k^2}{4\pi} \frac{\Delta_{(\alpha)}^{(0)}(k)}{k^4}\left[4\Delta_{(\alpha)}^{-1}(p) - 2m_0^2 \frac{\partial}{\partial m_0^2}\Delta_{(\alpha)}^{-1}(p)\right] - \int_{M^2}^{\infty} \frac{\mathrm{d}k^2}{4\pi} \frac{\Delta_{(\theta)}^{(0)}(k)}{k^2}\left[4\Delta_{(\alpha)}^{-1}(p) + 2(3-2\pi\varkappa)\, m_0^2 \frac{\partial}{\partial m_0^2}\Delta_{(\alpha)}^{-1}(p)\right]. \tag{5.48}$$

Equation (5.48) shows that renormalized Green's functions are obtained by mass and wave-function renormalization, and in particular mass renormalization is once more consistent with eq. (5.4) and eq. (5.22a), while wave-function renormalization is obtained by defining

$$Z_P \cong 1 - \frac{1}{N}\left[2\ln\frac{2\pi}{f} - 2\ln\left(\frac{2\pi}{f} + 2\pi\varkappa - 2\right) + c_Z(\varkappa)\right]. \tag{5.49}$$

We can extract from eq. (5.49) the anomalous dimension of the composite field P_{ij} in the SM regularization scheme:

$$\gamma_P(f) = -\beta(f)\frac{\partial}{\partial f}\ln\left[4f^2 Z_P\right] = \frac{2f}{\pi}\left[1 + \frac{1}{N}\frac{f}{2\pi}\left(1 + \frac{1}{1+f(\varkappa - 1/\pi)}\right)\right] + O\left(\frac{1}{N^2}\right). \tag{5.50}$$

We notice that, in contrast with eq. (5.22b) and consistently with the gauge properties of P_{ij}, Z_P is independent of the longitudinal degrees of freedom of θ_μ.

A magnetic susceptibility and a second moment of the correlation function can be defined for the field P_{ij}. These quantities can be shown to satisfy all the renormalization group requirements. In particular

$$\langle x^2\rangle_P = \frac{\int \mathrm{d}^2x\, \frac{1}{4}x^2 \langle \operatorname{tr}\{P(x)\,P(0)\}\rangle}{\int \mathrm{d}^2x \langle \operatorname{tr}\{P(x)\,P(0)\}\rangle} \tag{5.51}$$

can be used as an alternative definition of the correlation length. In the large-N limit,

$$\langle x^2\rangle_P = \frac{1}{6m_0^2} + O\left(\frac{1}{N}\right). \tag{5.52}$$

In analogy with eq. (5.39), we can obtain a computationally convenient definition of renormalization constant $\tilde{Z}_P$ by the prescription

$$(5.53)\quad \tilde{Z}_P = \frac{2\pi}{3(2f)^2}\frac{\chi_P}{\langle x^2\rangle_P} = \\ = \frac{2\pi}{3}\left[\frac{\partial}{\partial p^2}\left(\Delta_{(\alpha)}(p) - \frac{1}{N}\Delta^2_{(\alpha)}(p)\,\Delta_1^{-1}(p)\right)\right]^{-1}\Bigg|_{p^2=0} + O\left(\frac{1}{N^2}\right),$$

where

$$(5.54)\quad \chi_P = \int d^2x\,\langle \operatorname{tr}\{P(x)\,P(0)\}\rangle$$

and we fixed the normalization by noticing that

$$(5.55)\quad \left[\frac{\partial}{\partial p^2}\Delta_{(\alpha)}(p)\right]^{-1}\Bigg|_{p^2=0} = \frac{3}{2\pi}.$$

In order to regularize eq. (5.53), we apply eq. (5.48) and recognize that the structure of the counterterms of $\Delta^2_{(\alpha)}\Delta_1^{-1}$ is simply

$$(5.56)\quad \int_{M^2}^{\infty}\frac{dk^2}{4\pi}\left(4\,\frac{\Delta^{(0)}_{(\alpha)}(k)}{k^4} - 4\,\frac{\Delta^{(0)}_{(\theta)}(k)}{k^2}\right)\Delta_{(\alpha)}(p) + \\ + \int_{M^2}^{\infty}\frac{dk^2}{4\pi}\left(2\,\frac{\Delta^{(0)}_{(\alpha)}(k)}{k^4} + 2(3-2\pi\varkappa)\,\frac{\Delta^{(0)}_{(\theta)}(k)}{k^2}\right)m_0^2\,\frac{\partial}{\partial m_0^2}\Delta_{(\alpha)}(p).$$

Now, noticing that

$$(5.57)\quad \frac{\partial}{\partial p^2}\left[m_0^2\,\frac{\partial}{\partial m_0^2}\Delta_{(\alpha)}(p)\right]\Bigg|_{p^2=0} = 0,$$

we immediately check that eq. (5.53) is consistent with the parametrization (5.49); $c_Z(\varkappa)$ can be computed numerically and is plotted in fig. 9. It is worth noticing that $c_Z(\varkappa)$ is finite for $\varkappa \to 0$.

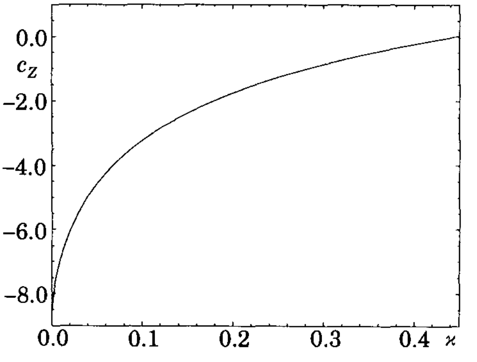


Fig. 9. – c_Z, the finite part of the renormalization constant $\tilde{Z}_P$.

5.5. *Wilson loops and static potential.* – Having exhausted the discussion of the correlations of fundamental fields that may be relevant to a $O(1/N)$ analysis, we would like to consider also the properties of (gauge-invariant) correlations of the vector field θ_μ. These correlations are most naturally expressed in terms of expectation values of the Wilson loops. The vector field being Abelian, no path ordering is required and the general definition, for arbitrary loops $\mathscr{C}$, is

$$L(\mathscr{C}) = \left\langle \exp\left\{ i \oint_{\mathscr{C}} dt_\mu \theta_\mu(t) \right\} \right\rangle = \tag{5.58}$$

$$= 1 - \frac{1}{2N} \oint_{\mathscr{C}} dt_\mu \oint_{\mathscr{C}} dt'_\nu \int \frac{d^2k}{(2\pi)^2} \exp[ik\cdot(t-t')]\, \Delta_{(\theta)\mu\nu}(k) + O\left(\frac{1}{N^2}\right).$$

We defined $L(\mathscr{C})$ having in mind the interpretation of our Lagrangian as an effective theory for an underlying gauge-invariant model where the gauge field is θ_μ, *not* the original vector field θ'_μ.

For our purposes we shall only consider long rectangular loops of size $R \times T$, and consider the limit $T \to \infty$. The quantity

$$V(R) = -\lim_{T\to\infty} \frac{1}{T} \ln L(R,\, T) \tag{5.59}$$

can be interpreted as the interaction potential generated by vector fields between two static sources. This quantity is relevant to the discussion of the non-relativistic bound-state spectrum, a reasonable approximation in the large-N limit. One can easily show that $V(R)$ does not depend on the longitudinal components of θ_μ, and, within the $1/N$ expansion, one may generalize the result of ref. [114] to

$$NV(R) \cong -2 \int_0^\infty \cos(kR)\, \Delta_{(\theta)}(k) \frac{dk}{2\pi} + 2 \int_0^\infty [\Delta_{(\theta)}(k) - \Delta^{(0)}_{(\theta)}(k)] \frac{dk}{2\pi}, \tag{5.60}$$

where $\Delta^{(0)}_{(\theta)}(k)$ is defined in eq. (5.3*b*); it has been introduced in the context of SM regularization of the loop ultraviolet singularities giving rise to the so-called «perimeter term». Rotating the integration contour in the complex k plane to $k = ix + \varepsilon$ we finally obtain the following representation of the static potential, for $0 \le \varkappa \le 1/\pi$:

$$N\frac{V(R)}{m_0} \cong 2 \int_0^\infty [\Delta_{(\theta)}(k) - \Delta^{(0)}_{(\theta)}(k)] \frac{dk}{2\pi m_0} - \tag{5.61}$$

$$-\pi \frac{m_\theta}{m_0} \frac{1 - m_\theta^2/(4m_0^2)}{m_\theta^2/(4m_0^2) - \pi\varkappa} \exp[-m_\theta R] -$$

$$- \int_{2m_0}^\infty \exp[-xR] \frac{2\pi\xi'}{\left[\xi' \ln \dfrac{1+\xi'}{1-\xi'} + 2\pi\varkappa - 2\right]^2 + \pi^2\xi'^2} \frac{dx}{m_0},$$

where $\xi' = \sqrt{1 - 4m_0^2/x^2}$ and we have defined the mass m_θ according to eq. (3.11). In fig. 10 we have drawn the function $NV(R)/m_0$ for a few different values of $\varkappa$.

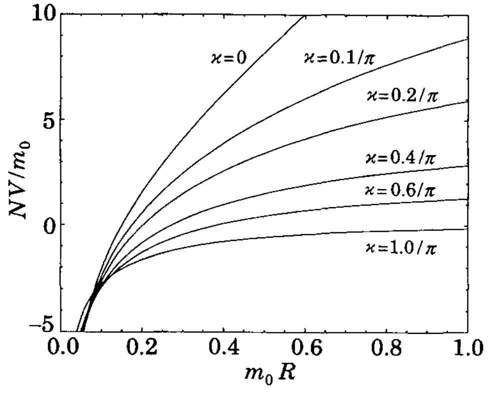


Fig. 10. – The static potential $V(R)$ for several values of $\varkappa$.

Equation (5.61), considered as a function of $\varkappa$, interpolates between the linear confining potential at $\varkappa = 0$ and the limiting case $\varkappa = 1/\pi$, when $\lim_{R\to\infty} V(R) = 0$ and the bound-state spectrum finally disappears, while $m_\theta = 2m_0$. The qualitative discussion of the properties of the models as a function of $\varkappa$, presented in sect. **3**, is essentially based upon an analysis of this static potential, plus the information coming from integrability at $\varkappa = 1/\pi$.

A more quantitative discussion of the bound-state spectrum can be obtained, for sufficiently large N, by considering the semiclassical approximation, *i.e.* by solving the Schrödinger equation in the asymptotic potential $V_{as}(R)$. $V_{as}(R)$ is obtained by removing from eq. (5.61) all the contributions not affecting leading order predictions, and it can be represented in the form

$$V_{as}(R, \varkappa) = \frac{6\pi m_0}{N} A\left(\frac{m_\theta}{m_0}\right)(1 - \exp[-m_\theta R]), \tag{5.62}$$

where

$$A(x) = \frac{x}{6} \frac{1 - \frac{1}{4}x^2}{\frac{1}{4}x^2 - 1 + \sqrt{4/x^2 - 1}\ \mathrm{arcctg}\sqrt{4/x^2 - 1}}, \tag{5.63}$$

enjoying the property

$$\lim_{x\to\infty} xA(x) = 1. \tag{5.64}$$

The analysis of the resulting Schrödinger equation is presented in some detail in Appendix E. In the case $\varkappa = 0$ (CP^{N-1} models), the analysis of the Schrödinger equation in the linear potential was presented in refs. [97, 119] and in more detail in ref. [93].

5.6. *Topological charge and susceptibility.* – Before concluding this section, we must mention that, in the special case $\varkappa = 0$, the $1/N$ expansion can also be applied to the problem of the so-called topological susceptibility

$$\chi_t = \int d^2x \langle q(x)\, q(0) \rangle, \tag{5.65}$$

where

$$q(x) = \frac{i}{2\pi} \varepsilon_{\mu\nu} \overline{D_\mu z}\, D_\nu z = \frac{1}{2\pi} \varepsilon_{\mu\nu} \partial_\mu \theta_\nu \tag{5.66}$$

is the topological charge density. One can easily show that

$$\chi_t = \lim_{p^2 \to 0} \frac{1}{(2\pi)^2} p^2 \tilde{\Delta}_{(\theta)}(p), \tag{5.67}$$

where $\tilde{\Delta}_{(\theta)}(p)$ is the full propagator of the quantum field θ_μ. χ_t is trivially zero when $\varkappa \neq 0$.

In the large-N limit, eq. (3.10b) implies the simple relationship [95]

$$\chi_t = \frac{3}{\pi N} m_0^2 + O\left(\frac{1}{N^2}\right). \tag{5.68}$$

The computation of the $1/N^2$ corrections to the inverse vector field propagator was performed in ref. [120]. They can be obtained from the diagrams of fig. 11. The vector and mixed scalar-vector vertices can ben replaced by combination of scalar vertices; the result is

$$\Delta^{-1}_{1(\theta)}(p) = \int \frac{d^2k}{(2\pi)^2} \Delta_{(\alpha)}(k)\, W_1(k, p) - \int \frac{d^2k}{(2\pi)^2} \Delta_{(\theta)}(k)\, W_2(k, p) - \int \frac{d^2k}{(2\pi)^2} \frac{\Delta_{(\alpha)}(k) + k^2 \Delta_{(\theta)}(k)}{k^2 + 4m_0^2} \frac{\partial}{\partial m_0^2} \Delta^{-1}_{(\theta)}(p), \tag{5.69}$$

where

$$W_1(k, p) = -(p^2 + 4m_0^2)\,[V_4^{(a)}(k, p) + V_4^{(a)}(k, -p)] - (p^2 + 2k^2 + 4m_0^2)\, V_4^{(b)}(k, p) + 4\,[V_3(k, p) + V_3(k, -p)], \tag{5.70a}$$

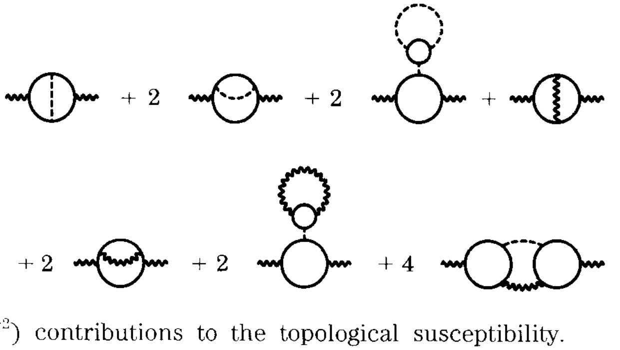


Fig. 11. – $O(1/N^2)$ contributions to the topological susceptibility.

(5.70b)
$$W_2(k,p) = (p^2+4m_0^2)(k^2+4m_0^2)[V_4^{(a)}(k,p)+V_4^{(a)}(k,-p)] + (p^2+2k^2+4m_0^2)(k^2+2p^2+4m_0^2)V_4^{(b)}(k,p) - 4(p^2+k^2+4m_0^2)[V_3(k,p)+V_3(k,-p)] - 2\frac{k^2p^2}{(k\cdot p)^2}[\Delta_{(\alpha)}(k+p)Z_+^2(k,p)+\Delta_{(\alpha)}(k-p)Z_-^2(k,p)],$$

and in turn

(5.71)
$$Z_\pm(k,p) = (p^2+k^2\pm 4m_0^2)V_3(\mp k,p) - \Delta_{(\alpha)}^{-1}(k) - \Delta_{(\alpha)}^{-1}(p).$$

A property of $W_2(k,p)$ relevant to the computation of higher-order corrections to the slope of the linear static potential is

(5.72)
$$\lim_{\substack{p\to 0\\ k\to 0}} \frac{W_2(k,p)}{k^2p^2} = \frac{4}{45\pi}\frac{1}{m_0^6}.$$

Ultraviolet regularization in the SM scheme is straightforward and can be proven to be consistent with the expected renormalizability properties of the model: only mass and z-field wave-function renormalization are required.

The $1/N$-expandable, dimensionless ratio $R=\chi_t\langle x^2\rangle_P$ is computed; it takes the scheme-independent value

(5.73)
$$R = \frac{1}{2\pi N}\left[1+\frac{c_R}{N}+O\left(\frac{1}{N^2}\right)\right],$$

where $c_R \cong -0.380088$.

6. – Lattice formulation of two-dimensional spin models.

It is well known that infinitely many lattice actions share the same naïve continuum limit. However, from the point of view of numerical simulations, the choice of a lattice action is a quite important topic, because one is trying to optimize the speed of the computation and the width of the scaling region. In the lattice $1/N$ expansion, which we are discussing, another relevant criterion of choice is the possibility of performing analytic calculations as far as possible, in order to keep under control and possibly minimize the number of the altogether unavoidable numerical integrations.

In a formal approach (and for one-coupling models), one might follow the original suggestion by Stone [121], and adopt a lattice action defined by the kernel of the heat equation on the manifold where the fundamental degrees of freedom are defined. This choice has the advantage of corresponding to the fixed-point (continuum limit) action in the exactly-solvable one-dimensional case [122]. However, for our purposes it is definitely more convenient to consider actions that are polynomial in the fields.

In passing we mention that a lattice Hamiltonian formulation was introduced by Hamer, Kogut and Susskind [123], and studied in the large-N limit in refs. [124-126]. $1/N$ corrections have not however been evaluated.

We shall discuss a number of possibilities, and finally focus on those formulations that meet the above-mentioned criterion.

6'1. *Nearest-neighbour quadratic actions.* – The lattice counterpart of eq. (3.1) satisfying the request that only nearest-neighbour interactions be involved and no term higher than quadratic in any given field be present is [7]

$$S^{(1)} = N \sum_{n,\mu} \{\beta_v [2 - \bar{z}_{n+\mu} z_n - \bar{z}_n z_{n+\mu}] + \beta_g [2 - \bar{z}_{n+\mu} \lambda_{n,\mu} z_n - \bar{z}_n \bar{\lambda}_{n,\mu} z_{n+\mu}]\}, \tag{6.1}$$

where the N-component complex field z_n satisfies the constraint

$$\bar{z}_n z_n = 1 \tag{6.2a}$$

and we have introduced explicitly in the action a $U(1)$ gauge field $\lambda_{n,\mu}$ satisfying

$$\bar{\lambda}_{n,\mu} \lambda_{n,\mu} = 1 . \tag{6.2b}$$

This form of the action was introduced in the literature and its large N features were discussed in detail in ref. [7]. Recently the case $\beta_v = 0$ (CP^{N-1} models) of eq. (6.1) has been the starting point of many numerical simulations at small and intermediate values of N [127-130].

6'2. *Nearest-neighbour quartic actions.* – It is possible to write down a gauge-invariant lattice action without explicitly introducing a $U(1)$ gauge field [131, 132]:

$$S_{g4}^{(1)} = N\beta_g \sum_{n,\mu} [1 - |\bar{z}_{n+\mu} z_n|^2] . \tag{6.3}$$

This used to be a favourite version of lattice CP^{N-1} models, especially because of the property that the $N = 2$ action is completely equivalent to the popular standard lattice action of the $SU(2) \approx O(3)$ non-linear σ-model. $S_{g4}^{(1)}$ is sometimes referred to as the «adjoint» form of lattice CP^{N-1} models, and is characterized by possessing a large-N first-order phase transition, that does not, however, show up for any finite value of N.

The mixed action obtained by combining $S^{(1)}$ (for $\beta_g = 0$) and $S_{g4}^{(1)}$ is also solvable in the large-N limit [7]. More generally, the qualitative features of the phase diagrams for mixed actions can be explored, for finite N and in the $1/N$ expansion, by the mean-field technique [133-135], which is an increasingly accurate description of the lattice models when one considers higher and higher space dimensionality.

However large-N studies and numerical simulations have shown that the approach to scaling of $S_{g4}^{(1)}$ is very slow even for quite large correlation lengths, and the situation is even worse for what concerns asymptotic scaling, since the large-N β-function gets contributions to all loops, in contrast with the continuum and the action (6.1) (*cf.* ref. [134]). We shall therefore avoid any further effort concerning eq. (6.3) and its $1/N$ expansion.

6.3. *Next-to-nearest-neighbour actions.* – It is obviously possible to formulate the models in terms of next-to-nearest-neighbour interactions:

$$(6.4)\qquad S^{(2)} = N \sum_{n,\mu} \frac{1}{4}\beta_v [2 - \bar{z}_{n+2\mu} z_n - \bar{z}_n z_{n+2\mu}] + \\ + \frac{1}{4}\beta_g [2 - \bar{z}_{n+2\mu} \lambda_{n+\mu,\mu} \lambda_{n,\mu} z_n - \bar{z}_n \bar{\lambda}_{n,\mu} \bar{\lambda}_{n+\mu,\mu} z_{n+2\mu}]$$

and similarly

$$(6.5)\qquad S^{(2)}_{g4} = \frac{1}{4} N\beta_g \sum_{n,\mu} [1 - |\bar{z}_{n+2\mu} z_n|^2].$$

More generally, any combination in the form

$$(6.6)\qquad c_1 S^{(1)} + c_2 S^{(2)}, \qquad c_1 + c_2 = 1, \qquad c_1 > 0$$

is a lattice action belonging to the same universality class. Amongst these combinations, a special rôle is played by the choice

$$(6.7)\qquad S^{\text{Sym}} = \frac{4}{3} S^{(1)} - \frac{1}{3} S^{(2)},$$

corresponding to the so-called Symanzik tree-improved version of the model [136, 137]. The choice S^{Sym} corresponds to forcing a short-distance (ultraviolet) behaviour markedly more similar to the behaviour of the continuum action. As a byproduct, the leading logarithmic dependence on the correlation length of the deviations from scaling is removed and the scaling region is therefore enlarged. This makes eq. (6.7) a natural candidate for numerical simulations of the models.

In our discussion of the $1/N$ expansion, we shall try to carry the analysis of the general case $c_1 S^{(1)} + c_2 S^{(2)}$ as far as possible, and we shall concentrate on $S^{(1)}$ when its peculiar analytic properties will become crucial to the development of our study.

7. – $1/N$ expansion on the lattice: effective action, propagators and vertices.

In the context of the $1/N$ expansion, the crucial property that selects quadratic actions is the possibility of performing the exact Gaussian integration over the z-fields at the only price of introducing a scalar Lagrange multiplier for the constraint (6.2a).

Let us introduce the matrices

$$(7.1a)\qquad \mathcal{M}^{(1)}_{mn} = \beta_v \sum_{\mu} (2\delta_{m,n} - \delta_{m+\mu,n} - \delta_{m,n+\mu}) + \\ + \beta_g \sum_{\mu} (2\delta_{m,n} - \lambda_{m,\mu}\delta_{m+\mu,n} - \bar{\lambda}_{m,\mu}\delta_{m,n+\mu}),$$

$$(7.1b)\qquad \mathcal{M}^{(2)}_{mn} = \frac{1}{4}\beta_v \sum_{\mu} (2\delta_{m,n} - \delta_{m+2\mu,n} - \delta_{m,n+2\mu}) + \\ + \frac{1}{4}\beta_g \sum_{\mu} (2\delta_{m,n} - \lambda_{m,\mu}\lambda_{m+\mu,\mu}\delta_{m+2\mu,n} - \bar{\lambda}_{m,\mu}\bar{\lambda}_{m-\mu,\mu}\delta_{m,n+2\mu}),$$

and obtain the effective action in the form

$$(7.2)\qquad S_{\rm eff}=N\,{\rm Tr}\ln\left[c_1\mathcal{M}^{(1)}_{mn}+c_2\mathcal{M}^{(2)}_{mn}+\frac{i}{2f}\alpha_n\delta_{m,n}\right]+\frac{N}{2f}\sum_n(-i\alpha_n)\,.$$

In order to perform the expansion, we must solve also the constraint involving the $\lambda_{n,\mu}$ fields, which can be done by setting

$$(7.3)\qquad \lambda_{n,\mu}=\exp\left[i\theta'_{n,\mu}\right],$$

where $\theta'_{n,\mu}$ is an (unconstrained) real field essentially playing the rôle of θ'_μ in the continuum version.

Assuming translation invariance of the classical fields α_n and $\theta'_{n,\mu}$, we can write down the effective action in momentum space:

$$(7.4)\qquad S_{\rm eff}=N\int\frac{{\rm d}^2p}{(2\pi)^2}\left\{\ln\left[\frac{i}{2f}\alpha+c_1\left(\beta_v\sum_\mu 4\sin^2\frac{p_\mu}{2}+\beta_g\sum_\mu 4\sin^2\frac{p_\mu+\theta'_\mu}{2}\right)+\right.\right.$$
$$\left.\left.+c_2\left(\beta_v\sum_\mu\sin^2p_\mu+\beta_g\sum_\mu\sin^2(p_\mu+\theta'_\mu)\right)\right]-\frac{i}{2f}\alpha\right\}.$$

A solution of the saddle-point equation is easily found to be

$$(7.5)\qquad \theta'_\mu=0\,,\qquad \alpha=-\,im_0^2\,,$$

where m_0^2 is defined by the mass gap equation

$$(7.6)\qquad \frac{1}{2f}=\int\frac{{\rm d}^2p}{(2\pi)^2}\,\frac{1}{c_1\sum_\mu 4\sin^2\frac{1}{2}p_\mu+c_2\sum_\mu\sin^2p_\mu+m_0^2}\equiv\int\frac{{\rm d}^2p}{(2\pi)^2}\,\frac{1}{\bar{p}^2+m_0^2};$$

we adopted the standard notations

$$(7.7a)\qquad \hat{p}_\mu\equiv 2\sin\frac{1}{2}p_\mu\,,\qquad \hat{p}^2\equiv\sum_\mu\hat{p}_\mu^2\,,\qquad \hat{p}^4\equiv\sum_\mu\hat{p}_\mu^4\,,$$

$$(7.7b)\qquad \bar{p}^2\equiv c_1\sum_\mu 4\sin^2\frac{1}{2}p_\mu+c_2\sum_\mu\sin^2p_\mu=$$
$$=\sum_\mu\hat{p}_\mu^2\left(c_1+c_2\cos^2\frac{1}{2}p_\mu\right)=\hat{p}^2-\frac{1}{4}c_2\hat{p}^4\,,$$

$$(7.7c)\qquad \int\frac{{\rm d}^2p}{(2\pi)^2}\equiv\int_{-\pi}^{\pi}\frac{{\rm d}p_1}{2\pi}\int_{-\pi}^{\pi}\frac{{\rm d}p_2}{2\pi}\,.$$

Going back to coordinate space, taking the second functional derivatives with respect to the fields α_n and $\theta'_{n,\mu}$, and evaluating them at the saddle point defined by eq. (7.5), we obtain a representation of the lattice propagators of the effective fields:

$$(7.8a) \qquad \Delta^{-1}_{(\alpha)}(k) = \int \frac{d^2p}{(2\pi)^2} \frac{1}{\overline{p+\frac{1}{2}k}^2 + m_0^2} \frac{1}{\overline{p-\frac{1}{2}k}^2 + m_0^2},$$

$$(7.8b) \qquad \Delta^{-1}_{(\theta)\mu\nu}(k) = (1+\varkappa f)\,\delta_{\mu\nu} \int \frac{d^2p}{(2\pi)^2} \frac{2c_1 \cos p_\mu + c_2 \cos 2p_\mu (1+\cos k_\mu)}{\overline{p}^2 + m_0^2} -$$

$$- \int \frac{d^2p}{(2\pi)^2} \exp\left[\frac{1}{2} i (k_\mu - k_\nu)\right] \cdot$$

$$\cdot \frac{(2c_1 \sin p_\mu + c_2 \sin 2p_\mu \cos \frac{1}{2}k_\mu)(2c_1 \sin p_\nu + c_2 \sin 2p_\nu \cos \frac{1}{2}k_\nu)}{\left(\overline{p+\frac{1}{2}k}^2 + m_0^2\right)\left(\overline{p-\frac{1}{2}k}^2 + m_0^2\right)},$$

where the θ'_μ-field has been rescaled to $\theta_\mu = \theta'_\mu/(1+\varkappa f)$.

The effective vertices are obtained by taking higher-order functional derivatives of the lattice effective action. They are in the form of one-loop integrals of the fundamental fields propagators. Computations can be cumbersome, but they are conceptually straightforward. The Feynman rules needed for the computations are summarized in fig. 12. We used the notation

$$(7.9a) \qquad V_{3\mu}(p, k) = 2c_1 \sin p_\mu + c_2 \sin 2p_\mu \cos \frac{1}{2} k_\mu,$$

$$(7.9b) \qquad V_{4\mu}(p, k) = c_1 \cos p_\mu + c_2 \cos 2p_\mu \cos^2 \frac{1}{2} k_\mu.$$

As discussed in sect. **3**, the vertices are gauge-invariant for all values of $\varkappa$. As a consequence, the following relationships hold:

$$(7.10a) \qquad \sum_\mu \hat{k}_\mu V_{3\mu}(p, k) = \overline{p+\frac{1}{2}k}^2 - \overline{p-\frac{1}{2}k}^2,$$

$$(7.10b) \qquad 2\hat{k}_\mu V_{4\mu}(p, k) = V_{3\mu}\left(p + \frac{1}{2}k, k\right) - V_{3\mu}\left(p - \frac{1}{2}k, k\right).$$

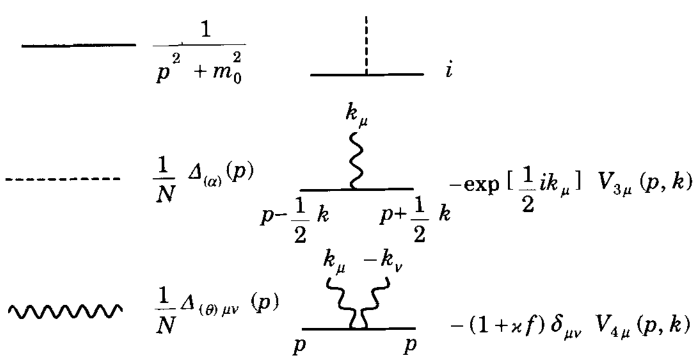


Fig. 12. – Feynman rules for the $1/N$ expansion on the lattice.

The notations we have introduced allow the following representation of the effective vector propagator:

$$(7.11)\quad \Delta^{-1}_{(\theta)\mu\nu}(k) = 2(1+\varkappa f)\,\delta_{\mu\nu}\int\frac{d^2p}{(2\pi)^2}\frac{V_{4\mu}(p,\,k)}{\bar{p}^2+m_0^2} - \\ - \exp\left[\frac{1}{2}i(k_\mu-k_\nu)\right]\int\frac{d^2p}{(2\pi)^2}\frac{V_{3\mu}(p+\frac{1}{2}k,\,k)\,V_{3\nu}(p+\frac{1}{2}k,\,k)}{(\bar{p}^2+m_0^2)\left(\overline{p+k}^2+m_0^2\right)}.$$

By applying eqs. (7.10), we may check that

$$(7.12)\quad \sum_\mu \exp\left[-\frac{1}{2}ik_\mu\right]\hat{k}_\mu\left\{2\delta_{\mu\nu}\int\frac{d^2p}{(2\pi)^2}\frac{V_{4\mu}(p,\,k)}{\bar{p}^2+m_0^2} - \right. \\ \left. - \exp\left[\frac{1}{2}i(k_\mu-k_\nu)\right]\int\frac{d^2p}{(2\pi)^2}\frac{V_{3\mu}(p+\frac{1}{2}k,\,k)\,V_{3\nu}(p+\frac{1}{2}k,\,k)}{(\bar{p}^2+m_0^2)\left(\overline{p+k}^2+m_0^2\right)}\right\} = 0\,,$$

as an effect of gauge invariance: when $\varkappa = 0$ the inverse propagator $\Delta^{-1}_{(\theta)\mu\nu}(k)$ is transverse on the lattice, *i.e.* it vanishes when contracted with $\exp[-ik_\mu]-1$ or $\exp[ik_\nu]-1$. It is, therefore, convenient to restate eq. (7.11) in a form reminiscent of eq. (3.9*b*):

$$(7.13)\qquad \Delta^{-1}_{(\theta)\mu\nu}(k) \equiv \Delta^{-1}_{(\theta)\mu}(k)\,\varkappa\delta_{\mu\nu} + \Delta^{-1}_{(\theta)}(k)\,\delta^t_{\mu\nu}(k)\,,$$

where

$$(7.14)\qquad \delta^t_{\mu\nu}(k) \equiv \delta_{\mu\nu} - \exp\left[\frac{1}{2}i(k_\mu-k_\nu)\right]\frac{\hat{k}_\mu\hat{k}_\nu}{\hat{k}^2},$$

$$(7.15a)\qquad \frac{1}{2f}\Delta^{-1}_{(\theta)\mu}(k) = \int\frac{d^2p}{(2\pi)^2}\frac{V_{4\mu}(p,\,k)}{\bar{p}^2+m_0^2},$$

$$(7.15b)\quad \Delta^{-1}_{(\theta)}(k) = 2\sum_\mu\int\frac{d^2p}{(2\pi)^2}\frac{V_{4\mu}(p,\,k)}{\bar{p}^2+m_0^2} - \sum_\mu\int\frac{d^2p}{(2\pi)^2}\frac{[V_{3\mu}(p+\frac{1}{2}k,\,k)]^2}{(\bar{p}^2+m_0^2)\left(\overline{p+k}^2+m_0^2\right)},$$

enjoying the important property

$$(7.16)\qquad \Delta^{-1}_{(\theta)}(0) = 0\,.$$

It is worth noticing that

$$(7.17)\qquad \Delta^{-1}_{(\theta)\mu}(k) \xrightarrow[c_2\to 0]{} 1 - \frac{f}{2}\int\frac{d^2p}{(2\pi)^2}\frac{\hat{p}^2}{\hat{p}^2+m_0^2} = 1 - \frac{f}{2} + \frac{m_0^2}{4},$$

independently of k and μ. However, even in this simplified case one cannot trivially identify $\varkappa$ with its continuum counterpart, even in the scaling regime, and, for finite values of f, $\varkappa_{\text{latt}}$ must be tuned in order to recover an assigned value of $\varkappa_{\text{cont}}$.

8. – Integral representations of lattice propagators.

We mentioned in sect. **6** that some powerful analytical techniques can be applied in some specific version of the model. In this section we introduce the first of these techniques, the use of integral representations of lattice propagators. We could only apply this technique to the case $c_2 = 0$ (nearest-neighbour interactions) which we shall discuss in great detail.

When $c_2 = 0$ dramatic simplifications occur already in eqs. (7.8). By simple manipulations and use of the gap equation

$$\frac{1}{2f} = \int \frac{d^2p}{(2\pi)^2} \frac{1}{\hat{p}^2 + m_0^2} \tag{8.1}$$

we obtain

$$\Delta_{(\alpha)}^{-1}(k) \xrightarrow[c_2 \to 0]{} \int \frac{d^2p}{(2\pi)^2} \frac{1}{\hat{p}^2 + m_0^2} \frac{1}{\widehat{p+k}^2 + m_0^2}, \tag{8.2a}$$

$$\Delta_{(\theta)}^{-1}(k) \xrightarrow[c_2 \to 0]{} \left[\frac{4 + m_0^2}{2f} - 1 - \int \frac{d^2p}{(2\pi)^2} \frac{2\widehat{p+k}^2}{(\hat{p}^2 + m_0^2)(\widehat{p+k}^2 + m_0^2)}\right]. \tag{8.2b}$$

The first analytical result concerns eq. (8.1), that can be cast into the form

$$\frac{1}{2f} = \frac{1}{2\pi} \frac{1}{1 + m_0^2/4} K\left(\frac{1}{1 + m_0^2/4}\right), \tag{8.3}$$

where K is the complete elliptic integral of the first kind. It is also possible to evaluate in closed form the inverse propagators along the principal diagonal of the momentum lattice, *i.e.* when $k_1 = k_2 \equiv l$. We change variables to $q_1 = \frac{1}{2}(p_1 + p_2)$ and $q_2 = \frac{1}{2}(p_1 - p_2)$, and notice that, for any periodic function $f(p_1, p_2)$ we have

$$\int_{-\pi}^{\pi} \frac{dp_1}{2\pi} \int_{-\pi}^{\pi} \frac{dp_2}{2\pi} f(p_1, p_2) = \int_{-\pi}^{\pi} \frac{dq_1}{2\pi} \int_{-\pi}^{\pi} \frac{dq_2}{2\pi} f(q_1 + q_2, q_1 - q_2). \tag{8.4}$$

In the case $k_1 = k_2 = l$ in particular

$$\hat{p}^2 = 4(1 - \cos q_1 \cos q_2), \tag{8.5}$$

$$\widehat{p+k}^2 = 4(1 - \cos(q_1 + l) \cos q_2). \tag{8.6}$$

Therefore the q-integrations of eqs. (8.2) can be performed in terms of the standard complete elliptic integrals K, E, and Π (we use the conventions of ref. [138]); the result is

$$\Delta_{(\alpha)}^{-1}(l, l) = \frac{1}{8\pi(1 + m_0^2/4)^2} \Pi\left(\frac{\cos^2 \frac{1}{2} l}{(1 + m_0^2/4)^2}, \frac{1}{1 + m_0^2/4}\right), \tag{8.7a}$$

$$(8.7b)\qquad \Delta_{(\theta)}^{-1}(l,\ l) = -\frac{1}{\pi}E\left(\frac{1}{1+m_0^2/4}\right) + \left(1-\frac{1}{\cos^2\frac{1}{2}l}\right)\frac{1}{\pi}K\left(\frac{1}{1+m_0^2/4}\right) + \\ + \left(\frac{1}{\cos^2\frac{1}{2}l} - \frac{1}{(1+m_0^2/4)^2}\right)\frac{1}{\pi}\Pi\left(\frac{\cos^2\frac{1}{2}l}{(1+m_0^2/4)^2},\ \frac{1}{1+m_0^2/4}\right).$$

In the general case, we had to resort to an integral representation of the inverse propagators. Let us make use of the standard Feynman parameters to write

$$(8.8)\qquad \frac{1}{\hat{p}^2+m_0^2}\,\frac{1}{\widehat{p+k}^2+m_0^2} = \int_0^1 \mathrm{d}x\,\frac{1}{[\hat{p}^2(1-x)+\widehat{p+k}^2x+m_0^2]^2}$$

and notice that, via trigonometric identities,

$$(8.9)\qquad (\widehat{p_\mu+k_\mu})^2 = \hat{p}_\mu^2(1-\tfrac{1}{2}\hat{k}_\mu^2)+\hat{k}_\mu^2+2\sin p_\mu \sin k_\mu\,.$$

Let us now change variables to $q_\mu = p_\mu + z_\mu$, where z_μ is defined by the relationships

$$(8.10)\qquad \sin z_\mu = \frac{x\sin k_\mu}{\sqrt{1-x(1-x)\hat{k}_\mu^2}},\qquad \cos z_\mu = \frac{1-\frac{1}{2}x\hat{k}_\mu^2}{\sqrt{1-x(1-x)\hat{k}_\mu^2}}\,.$$

Equation (8.10) implies the identities

$$(8.11a)\qquad \hat{p}_\mu^2(1-x)+(\widehat{p_\mu+k_\mu})^2x = \\ = \hat{q}_\mu^2\sqrt{1-x(1-x)\hat{k}_\mu^2}+2-2\sqrt{1-x(1-x)\hat{k}_\mu^2}\,,$$

$$(8.11b)\qquad \sin(p_\mu+\tfrac{1}{2}k_\mu) = \frac{\sin q_\mu\cos\frac{1}{2}k_\mu+(1-2x)\cos q_\mu\sin\frac{1}{2}k_\mu}{\sqrt{1-x(1-x)\hat{k}_\mu^2}}\,.$$

Substituting eqs. (8.8) and (8.11) in the relevant ome-loop integrals, we obtain

$$(8.12a)\qquad \Delta_{(\alpha)}^{-1} = \int_0^1 \mathrm{d}x\int\frac{\mathrm{d}^2q}{(2\pi)^2}\,\frac{1}{\left[4+m_0^2-\sum_\mu a_\mu\cos q_\mu\right]^2}\,,$$

$$(8.12b)\qquad \Delta_{(\theta)}^{-1} = \frac{4+m_0^2}{2f}-1-\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^2q}{(2\pi)^2}\,\frac{c-\sum_\mu b_\mu\sin^2q_\mu}{\left[4+m_0^2-\sum_\mu a_\mu\cos q_\mu\right]^2}\,,$$

where in order to simplify the notation, we have defined the auxiliary variables

$$a_\mu = 2\sqrt{1 - x(1-x)\,\hat{k}_\mu^2}, \tag{8.13a}$$

$$b_\mu = \frac{(1-2x)^2\,\hat{k}_\mu^2 + \hat{k}_\mu^2 - 4}{1 - x(1-x)\,\hat{k}_\mu^2}, \tag{8.13b}$$

$$c = \sum_\mu \frac{(1-2x)^2\,\hat{k}_\mu^2}{1 - x(1-x)\,\hat{k}_\mu^2}. \tag{8.13c}$$

We can now exploit the relationships

$$b_\mu = a_\mu^2 \frac{\mathrm{d}}{\mathrm{d}x} \frac{2(1-2x)}{a_\mu^2}, \tag{8.14a}$$

$$c = -2(1-2x) \sum_\mu \frac{1}{a_\mu} \frac{\mathrm{d}a_\mu}{\mathrm{d}x}, \tag{8.14b}$$

to perform an integration by parts in the variable x in eq. (8.12b) and obtain the more convenient representation

$$\Delta_{(\theta)}^{-1} = \int_0^1 \mathrm{d}x\, 2(1-2x) \sum_\mu \left\{ \frac{1}{a_\mu} \frac{\mathrm{d}a_\mu}{\mathrm{d}x} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{1}{\left[4 + m_0^2 - \sum_\mu a_\mu \cos q_\mu\right]^2} - \frac{1}{a_\mu^2} \frac{\mathrm{d}}{\mathrm{d}x} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{a_\mu^2 \sin^2 q_\mu}{\left[4 + m_0^2 - \sum_\mu a_\mu \cos q_\mu\right]^2} \right\}. \tag{8.15}$$

Trivial algebraic manipulations (involving repeated use of integration by parts in the momentum variables) lead finally to the form

$$\Delta_{(\theta)}^{-1} = \hat{k}^2 \int_0^1 \mathrm{d}x \frac{(1-2x)^2}{a_1 a_2} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{4 \cos q_1 \cos q_2}{\left[4 + m_0^2 - \sum_\mu a_\mu \cos q_\mu\right]^2}. \tag{8.16}$$

Starting from eqs. (8.12a) and (8.16) we can now perform the momentum integrations. One momentum component can be integrated easily, thanks to the relationship

$$\int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} \frac{1}{b - a\cos\theta} = \frac{1}{\sqrt{b^2 - a^2}} \tag{8.17}$$

and its parametric derivatives. As a consequence, we obtain

(8.18*a*) $$\Delta^{-1}_{(\alpha)} = \int_0^1 dx \int_{-\pi}^{\pi} \frac{dq_1}{2\pi} \frac{4 + m_0^2 - a_1 \cos q_1}{[(4 + m_0^2 - a_1 \cos q_1)^2 - a_2^2]^{3/2}},$$

(8.18*b*) $$\Delta^{-1}_{(\theta)} = \hat{k}^2 \int_0^1 dx \frac{(1-2x)^2}{a_1} \int_{-\pi}^{\pi} \frac{dq_1}{2\pi} \frac{4 \cos q_1}{[(4 + m_0^2 - a_1 \cos q_1)^2 - a_2^2]^{3/2}}.$$

The q_1 integrations in eqs. (8.18) are easily reducible to standard elliptic integrals [138]. Without belaboring on the straightforward algebraic tricks involved in the derivation, we present the final result in the following form:

(8.19*a*) $$\Delta^{-1}_{(\alpha)} = \frac{4 + m_0^2}{4\pi} \int_0^1 dx \frac{1}{(a_1 a_2)^{3/2}} \frac{\zeta^3 E(\zeta)}{1 - \zeta^2},$$

(8.19*b*) $$\Delta^{-1}_{(\theta)} = \hat{k}^2 \frac{4 + m_0^2}{\pi} \int_0^1 dx \frac{(1-2x)^2}{(a_1 a_2)^{5/2}} \left[\frac{\zeta^3 E(\zeta)}{1 - \zeta^2} + 2\zeta\,(E(\zeta) - K(\zeta)) \right],$$

where

(8.20) $$\zeta = \sqrt{\frac{4 a_1 a_2}{(4 + m_0^2)^2 - (a_1 - a_2)^2}}.$$

Equations (8.19) lead to an enormous computational gain: from the numerical point of view, the values of the elliptic integrals can be routinely generated with high accuracy, and therefore the number of numerical integrations is reduced from two to one; moreover, the x integration is much more regular than the original momentum integration: if we exploit the explicit expressions of $\Delta^{-1}_{(\alpha)}$ and $\Delta^{-1}_{(\theta)}$ on the principal diagonal (8.7) to further regularize the integrand, a moderate-size Gauss-Legendre integration suffices to produce very accurate results. From the point of view of analytic manipulations, we can exploit the knowledge of the asymptotic expansions of the elliptic integrals to simplify dramatically the otherwise quite complicated problem of generating an asymptotic expansion of the propagators in the small m_0 (scaling) region. This expansion in turn is the essential ingredient in quantitative computations for lattice models in the scaling (*i.e.* field-theoretical) regime. We shall discuss this point in the next section. Let us only recall here the relevant expansion formulae of the complete elliptic integrals, in the regime $k \approx 1$, $k' = \sqrt{1 - k^2} \ll 1$:

(8.21*a*) $$E(k) \approx 1 + \frac{k'^2}{2}\left(\ln \frac{4}{k'} - \frac{1}{2}\right),$$

(8.21*b*) $$K(k) \approx \ln \frac{4}{k'} + \frac{k'^2}{4}\left(\ln \frac{4}{k'} - 1\right),$$

$$(8.21c)\qquad \Pi(n,\ k) \approx \frac{1}{1-n}\left(\ln\frac{4}{k'} + \frac{\sqrt{n}}{2}\ln\frac{1-\sqrt{n}}{1+\sqrt{n}}\right) + \\ + \frac{k'^2}{4(1-n)^2}\left((1-n)\ln\frac{4}{k'} + \sqrt{n}\ln\frac{1-\sqrt{n}}{1+\sqrt{n}}\right).$$

9. – Asymptotic expansions in the scaling region.

We are interested in evaluating physical quantities in a lattice model possessing a non-trivial (continuum) field theory limit when the ultraviolet regulator ($1/a$, where a is the lattice spacing) is sent to infinity, while properly tuning the lattice coupling f according to the renormalization group equation of the model. In such a model, lattice expectation values for finite a and $f(a)$ will necessarily receive contributions from the irrelevant operators included in the lattice definition of a physical quantity. In order to isolate the contribution that will survive in the continuum limit, *i.e.* the scaling part of the expectation value, we must be able to perform an expansion in the powers of the lattice spacing. Within the $1/N$ expansion of lattice spin models, there is another dimensionful parameter that can be employed to classify the relevance of the contributions to any given expectation value: the bare large-N vacuum expectation value m_0^2 («large-N mass»). Since the continuum limit is the limit of infinite correlation length, it is also the limit of vanishing m_0^2: contributions carrying higher powers of m_0^2 with respect to the scaling contribution will be *irrelevant*. Actually, since the physical value of the mass in these asymptotically-free models is not strictly zero, we must be careful not to set $m_0^2 = 0$ and to get our results in the form of asymptotic expansions in which $1 \gg |1/\ln(m_0^2 a^2)| \gg m_0^2 a^2$.

The detailed analysis of the procedures for the evaluation of physical quantities in the scaling region, and the related problem of regularization of the infrared divergences generated by the naïve $m_0^2 \to 0$ limit, will be discussed in sect. **11**. Preliminary to such a discussion is, however, the asymptotic expansion for small values of m_0^2 of the basic ingredients in the evaluation of physical quantities, *i.e.* the effective propagators themselves. Due to their original definition as one-loop integrals over Feynman propagators, the inverse effective propagators $\Delta_{(\alpha)}^{-1}$ and $\Delta_{(\theta)\mu\nu}^{-1}$ will have formal asymptotic expansions in the form

$$(9.1)\qquad \sum_{n=0}^{\infty} [\tilde{A}_n(k) + \tilde{B}_n(k)\ln m_0^2]\,(m_0^2)^n .$$

We may however recognize from eq. (7.6) that the coupling constant itself admits such an asymptotic expansion in the form

$$(9.2)\qquad \beta \equiv \frac{1}{2f} = \sum_{n=0}^{\infty} (\alpha_n + \beta_n \ln m_0^2)\,(m_0^2)^n ,$$

and eq. (9.2) can in principle be inverted to

$$\ln m_0^2 = \sum_{n=0}^{\infty} (\gamma_n + \delta_n \beta)(m_0^2)^n . \tag{9.3}$$

As a consequence, a perfectly acceptable form of the asymptotic expansion of the inverse propagators is

$$\Delta_{(\alpha)}^{-1}(k) = \sum_{n=0}^{\infty} [A_n^{(\alpha)}(k) + \beta B_n^{(\alpha)}(k)](m_0^2)^n , \tag{9.4a}$$

$$\Delta_{(\theta)}^{-1}(k) = \sum_{n=0}^{\infty} [A_n^{(\theta)}(k) + \beta B_n^{(\theta)}(k)](m_0^2)^n . \tag{9.4b}$$

The expansion (9.4), in comparison with (9.1), involves no extra effort, but it shows the great advantage of expressing final results in a form that makes direct contact with the lattice weak-coupling expansion of the models, and allows explicit checks of commutativity between weak-coupling and $1/N$ expansion.

We obviously aim at a systematic way of generating the coefficients A_n and B_n. We shall discuss here a general technique that does not rely upon any specific feature of the one-loop integrals involved, and in the next section we shall present a different approach specific to the case discussed in sect. **8** when an integral representation is available.

9'1. *General technique.* – The general technique is worth a detailed analysis, because it will be very useful also in the evaluation of expectation values of physical quantities. It will also allow us to show the connection between sharp-momentum regularization and lattice formulation. It is therefore convenient for our purposes to discuss the case of a one-loop lattice integral with arbitrary propagators and vertices. The general form of the lattice integral is

$$I(k;\, m_0 a) = \int_0^{\pi} \frac{d^d p}{(2\pi)^d} F(k;\, m_0 a,\, p) , \tag{9.5}$$

where k is the external momentum (or a collection of external momenta) and we use the notation

$$\int_0^b \frac{d^d p}{(2\pi)^d} \equiv \prod_{i=1}^{d} \int_{-b}^{b} \frac{dp_i}{2\pi} , \qquad \int_a^b \frac{d^d p}{(2\pi)^d} \equiv \prod_{i=1}^{d} \int_{-b}^{b} \frac{dp_i}{2\pi} - \prod_{i=1}^{d} \int_{-a}^{a} \frac{dp_i}{2\pi} . \tag{9.6}$$

When comparing with the continuum formulation, we must keep in mind that powers of a have been included in the definitions of F and I in order to make them dimensionless. Formally we can rewrite eq. (9.5) in the form

$$(9.7)\quad I(k;\ m_0 a) = \int_0^{\pi} \frac{d^d p}{(2\pi)^d} F(k;\ m_0 a,\ p) + a^d \int_0^{\infty} \frac{d^d p}{(2\pi)^d} F(k;\ m_0 a,\ pa) - \\ - a^d \int_0^{\infty} \frac{d^d p}{(2\pi)^d} \mathrm{T} F(k;\ m_0 a,\ pa)\,,$$

where $\mathrm{T}F$ is the Taylor series expansion of the function F in powers of $m_0 a$ around $m_0 a = 0$. In order to turn eq. (9.7) into a mathematically rigorous statement, we must show that, order by order in a, all ultraviolet and infrared divergences are explicitly cancelled. Let us therefore introduce the truncated Taylor series expansion T_J (including all the powers of m_0 up to m_0^J) and notice that

$$(9.8)\quad \int_0^{\pi} \frac{d^d p}{(2\pi)^d} F(k;\ m_0 a,\ p) + \\ + a^d \int_0^{\infty} \frac{d^d p}{(2\pi)^d} F(k;\ m_0 a,\ pa) - a^d \int_0^{\infty} \frac{d^d p}{(2\pi)^d} \mathrm{T}_J F(k;\ m_0 a,\ pa) = \\ = \left[\int_0^{\pi} \frac{d^d p}{(2\pi)^d} F(k;\ m_0 a,\ p) - a^d \int_0^{\pi/a} \frac{d^d p}{(2\pi)^d} \mathrm{T}_J F(k;\ m_0 a,\ pa) \right] + \\ + a^d \int_0^{\pi/a} \frac{d^d p}{(2\pi)^d} F(k;\ m_0 a,\ pa) + a^d \int_{\pi/a}^{\infty} \frac{d^d p}{(2\pi)^d} (1 - \mathrm{T}_J) F(k;\ m_0 a,\ pa) = \\ = \int_0^{\pi} \frac{d^d p}{(2\pi)^d} (1 - \mathrm{T}_J)(F(k;\ m_0 a,\ p) + a^d \int_{\pi/a}^{\infty} \frac{d^d p}{(2\pi)^d} (1 - \mathrm{T}_J) F(k;\ m_0 a,\ pa) + \\ + I(k;\ m_0 a) = I(k;\ m_0 a) + O(a^{J+1})\,.$$

Indeed the first contribution to eq. (9.8) is trivially $O(a^{J+1})$, and the same can be shown to hold for the second contribution by noticing that

$$(9.9)\quad \begin{cases} F(k;\ m_0 a,\ pa) = \sum_j a^j F_j(k;\ m_0,\ p)\,, \\ F_j(k;\ \lambda m_0,\ \lambda p) = \lambda^j F_j(k;\ m_0,\ p)\,; \end{cases}$$

therefore we have

$$(9.10)\qquad a^d \int_{\pi/a}^{\infty} \frac{d^d p}{(2\pi)^d}(1 - T_J) F_j(k;\ m_0,\ p) = O(a^{J+1}).$$

Equation (9.8) paves the way to two major strategies in the regularization of lattice integrals. The first strategy, till now most popular, stems from the observation that the last term in the l.h.s. of eq. (9.8) vanishes in dimensional regularization. One may therefore adopt the definition

$$(9.11)\qquad I(k;\ m_0 a) = \lim_{d\to 2}\left[\int_0^{\pi} \frac{d^d p}{(2\pi)^d} F(k;\ m_0 a,\ p) + a^d \int_0^{\infty} \frac{d^d p}{(2\pi)^d} F(k;\ m_0 a,\ pa)\right].$$

When expanding in powers of a, the first contribution corresponds to infrared-divergent massless lattice integrals and the second to ultraviolet-divergent massive continuum integrals; in dimensional regularization infrared and ultraviolet poles cancel exactly and we are left with a finite result. However, one may also notice that the last term in the l.h.s. of eq. (9.8) is reminiscent of the structure of counterterms in the sharp-momentum regularization scheme. More precisely, we can decompose the expansion $T_J F$ into contributions which are ultraviolet divergent (relevant), contributions which are infrared divergent (irrelevant), and contributions which are ultraviolet and infrared divergent (marginal). Symbolically we may write

$$(9.12)\qquad T_J = T_J^{(UV)} + T_J^{(IR)} + T_J^{(0)}.$$

We can introduce an arbitrary cut-off M^2 and split the counterterm $T_J^{(0)}$ into two separate integrals. Finally we can express the original integral in terms of two separate contributions whose singularities are now independently regularized:

$$(9.13)\qquad I(k;\ m_0 a) = \left[\int_0^{\pi} \frac{d^2 p}{(2\pi)^2} F(k;\ m_0 a,\ p) - a^2 \int_0^{\infty} \frac{dp^2}{4\pi} T_J^{(IR)} F(k;\ m_0 a,\ pa) - \right.$$
$$\left. - a^2 \int_0^{M^2} \frac{dp^2}{4\pi} T_J^{(0)} F(k;\ m_0 a,\ pa)\right] + \left[a^2 \int_0^{\infty} \frac{dp^2}{4\pi} F(k;\ m_0 a,\ pa) - \right.$$
$$\left. - a^2 \int_0^{\infty} \frac{dp^2}{4\pi} T_J^{(UV)} F(k;\ m_0 a,\ pa) - a^2 \int_{M^2}^{\infty} \frac{dp^2}{4\pi} T_J^{(0)} F(k;\ m_0 a,\ pa)\right].$$

Equation (9.13) is the starting point for the series expansion in powers of a.

The first contribution in brackets is ultraviolet-regular. The lattice integral and the continuum integrals are separately infrared-singular, the infrared divergences cancelling out in the sum. In analogy with the standard notation for ultraviolet regularization (*cf.* eq. (5.1)), we always assume that such combinations of (massless) separately infrared-divergent integrals stand for, *e.g.*,

$$\int_0^\pi \frac{d^2p}{(2\pi)^2}\phi_{\rm latt}(p) - \int_0^{\Lambda^2}\frac{dp^2}{4\pi}\phi_{\rm cont}(p) \equiv \tag{9.14}$$

$$\equiv \int_0^\pi \frac{d^2p}{(2\pi)^2}(\phi_{\rm latt}(p)-\phi_{\rm cont}(p)) - \int_{\mathscr{R}}\frac{dp^2}{4\pi}\phi_{\rm cont}(p),$$

where $\mathscr{R}$ is the region comprised between the circle of radius Λ and the square of side 2π (assuming $\Lambda > \pi\sqrt{2}$). The integral of the difference is infrared-regular by construction.

The second contribution in brackets is infrared-regular and ultraviolet singularities are removed according to the prescriptions of the sharp-momentum scheme, which means that the marginal (scaling) component of $I(k;\, m_0 a)$ receives a contribution exactly equal to the one obtained in the corresponding continuum model: all lattice effects are included in the first term. Moreover, as we shall immediately show, by applying eq. (9.13) directly to the gap equation it is possible to fix the value of M^2 in such a way that the sharp-momentum coupling constant can be identified (for large N) with the lattice coupling.

9·2. *Expansion of the gap equation.* – For our purposes it will be sufficient to truncate the expansion in powers of a to the second non-trivial order, *i.e.* $J = 2$. The regularized form of eq. (7.6) is

$$\frac{1}{2f} \approx \int_0^\pi \frac{d^2p}{(2\pi)^2}\frac{1}{m_0^2a^2+\sum\limits_\mu 4\sin^2\frac12 p_\mu - c_2\sum\limits_\mu 4\sin^4\frac12 p_\mu} + \tag{9.15}$$

$$+ a^2\int_0^\infty \frac{dp^2}{4\pi}(1-\mathrm{T}_2)\frac{1}{m_0^2a^2+\sum\limits_\mu 4\sin^2\frac12 ap_\mu - c_2\sum\limits_\mu 4\sin^4\frac12 ap_\mu}$$

and, expanding to $O(a^2)$, we obtain

$$\frac{1}{2f} \approx \int_0^\pi \frac{d^2p}{(2\pi)^2}\left[\frac{1}{\hat p^2-\frac14 c_2\hat p^4} - \frac{a^2m_0^2}{(\hat p^2-\frac14 c_2 \hat p^4)^2}\right] + \tag{9.16}$$

$$+\int_0^\infty \frac{dp^2}{4\pi}\left[\frac{1}{p^2+m_0^2}+\frac14\frac{a^2(c_2+\frac13)\sum\limits_\mu p_\mu^4}{(p^2+m_0^2)^2}\right] -$$

$$-\int_0^\infty\frac{dp^2}{4\pi}\left[\frac{1}{p^2}-\frac{m_0^2}{(p^2)^2}+\frac14\frac{a^2(c_2+\frac13)\sum\limits_\mu p_\mu^4}{(p^2)^2}-\frac12\frac{a^2m_0^2(c_2+\frac13)\sum\limits_\mu p_\mu^4}{(p^2)^3}\right].$$

According to eq. (9.13), we can write

$$(9.17) \qquad \frac{1}{2f} \approx \int_0^\pi \frac{d^2p}{(2\pi)^2}\left[\frac{1}{\hat{p} - \frac{1}{4}c_2\hat{p}^4} - \frac{a^2 m_0^2}{(\hat{p}^2 - \frac{1}{4}c_2\hat{p}^4)^2}\right] +$$

$$+ \int_0^\infty \frac{dp^2}{4\pi}\frac{m_0^2}{(p^2)^2} - \int_0^{M^2}\frac{dp^2}{4\pi}\left[\frac{1}{p^2} - \frac{1}{2}\frac{a^2 m_0^2}{p^2}\left(c_2 + \frac{1}{3}\right)\frac{\sum_\mu p_\mu^4}{(p^2)^2}\right] +$$

$$+ \int_0^\infty \frac{dp^2}{4\pi}\left[\frac{1}{p^2 + m_0^2} + \frac{3}{16}\frac{a^2(c_2 + \frac{1}{3})(p^2)^2}{(p^2 + m_0^2)^2} - \frac{3}{16}a^2\left(c_2 + \frac{1}{3}\right)\right] -$$

$$- \int_{M^2}^\infty \frac{dp^2}{4\pi}\left[\frac{1}{p^2} - \frac{3}{8}\frac{a^2 m_0^2}{p^2}\left(c_2 + \frac{1}{3}\right)\right].$$

The continuum integrals can be explicitly evaluated and the final result is

$$(9.18) \qquad \frac{1}{2f} \approx \left[\int_0^\pi \frac{d^2p}{(2\pi)^2}\frac{1}{\hat{p}^2 - \frac{1}{4}c_2\hat{p}^4} - \int_0^{M^2}\frac{dp^2}{4\pi}\frac{1}{p^2}\right] + \frac{1}{4\pi}\ln\frac{M^2}{m_0^2} -$$

$$- a^2 m_0^2\left[\int_0^\pi \frac{d^2p}{(2\pi)^2}\frac{1}{(\hat{p}^2 - \frac{1}{4}c_2\hat{p}^4)^2} - \int_0^\infty \frac{dp^2}{4\pi}\frac{1}{(p^2)^2} - \int_0^{M^2}\frac{dp^2}{4\pi}\frac{1}{2}\left(c_2 + \frac{1}{3}\right)\frac{\sum_\mu p_\mu^4}{(p^2)^3}\right] +$$

$$+ a^2 m_0^2\frac{3}{64\pi}\left(c_2 + \frac{1}{3}\right)\left(1 - 2\ln\frac{M^2}{m_0^2}\right).$$

Equation (9.18) deserves some comments:

A rescaling $p \to p/a$ in the terms originated by $T_J^{(IR)}$ must be performed *after* the expansion in powers of a has been accomplished. It may sound arbitrary, because we are working with divergent quantities without an intrinsic scale, but it is mathematically sound as one may recognize by going back to the proof of regularization.

The effect of a Symanzik improvement is made apparent by the cancellation of the logarithmic dependence on m_0 in the first irrelevant contribution when we choose the value $c_2 = -\frac{1}{3}$.

In order to identify sharp-momentum and lattice couplings, we must choose M_L^2 such that

$$(9.19) \qquad \int_0^\pi \frac{d^2p}{(2\pi)^2}\frac{1}{\hat{p}^2 - \frac{1}{4}c_2\hat{p}^4} - \int_0^{M_L^2}\frac{dp^2}{4\pi}\frac{1}{p^2} = 0,$$

implying

$$\frac{1}{2f} = \frac{1}{4\pi} \ln \frac{M_L^2}{m_0^2} + a^2 m_0^2 \left[\alpha_1 (c_2) - \frac{1 + 3c_2}{32\pi} \ln \frac{M_L^2}{m_0^2} \right] + O(m_0^4) . \tag{9.20}$$

The choice implied by eq. (9.19) will allow us to express our results in a form immediately comparable to the lattice weak-coupling expansion, because the difference between the couplings is purely non-perturbative, as shown by eq. (9.20). Any other choice would correspond to a finite renormalization and would require a perturbative adjustment of f in order to recover standard perturbation theory. M_L^2 and α_1 are plotted as functions of c_2 in fig. 13. Let us notice the special values

$$M_L^2 |_{c_2=0} = 32, \qquad M_L^2 |_{c_2=-1/3} \cong 17.68967299 , \tag{9.21a}$$

$$\alpha_1 |_{c_2=0} = \frac{1}{32\pi}, \qquad \alpha_1 |_{c_2=-1/3} \cong -0.00479767 . \tag{9.21b}$$

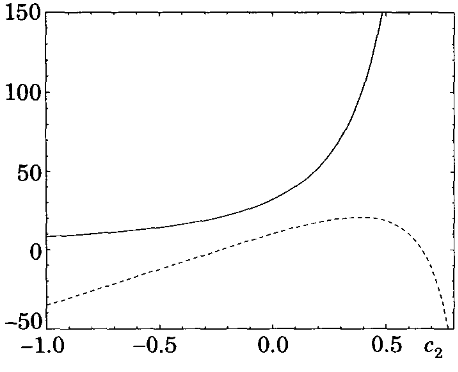


Fig. 13. – M_L^2 (solid line) and α_1 (dashed line) as functions of c_2 (α_1 is multiplied by 10^3).

9'3. *Expansion of the propagators.* – In practice, when willing to compute the expansion indicated in eqs. (9.4), much work can be saved by the following observations.

The functions $B_n(k)$ are related in a simple way to the coefficients of the $1/\varepsilon$ poles resulting from the continuum integration in the representation (9.11). These coefficients are related to essentially trivial one-loop continuum integrals that can all be computed in closed form. Therefore the functions $B_n(k)$ can be expressed in terms of elementary functions of $\hat{k}$ and do not require an integral representation.

Moreover, let us assume that the integrand in the representation (9.5) of an inverse propagator Δ^{-1} be a function $F_\Delta(k;\, m_0,\, p)$. By shifting the integration variable, it is possible to symmetrize F to $F_\Delta^{\rm sym}(k;\, m_0,\, p)$, where $F^{\rm sym}$ is an even function of k_μ with all singularities located in $p_\mu = \pm \frac{1}{2} k_\mu$. Let us now consider the function

$$(9.22)\qquad \int \frac{d^2p}{(2\pi)^2} F_\Delta^{\text{sym}}(k;\ m_0,\ p) - \beta \sum_{n=0}^{\infty} B_n(k)\,(m_0^2)^n \equiv$$
$$\equiv \int \frac{d^2p}{(2\pi)^2}\left[F_\Delta^{\text{sym}}(k;\ m_0,\ p) - \frac{1}{2}\left(\frac{1}{\overline{p+\frac{1}{2}k}^2} + \frac{1}{\overline{p-\frac{1}{2}k}^2}\right)\sum_{n=0}^{\infty} B_n(k)\,(m_0^2)^n\right].$$

By construction, according to eq. (9.4), this function is analytic in m_0 around $m_0 = 0$. Therefore, we immediately obtain the following integral representation of $A_n(k)$:

$$(9.23)\qquad A_n(k) = \frac{1}{n!}\int \frac{d^2p}{(2\pi)^2}\frac{d^n}{d(m_0^2)^n}\cdot$$
$$\cdot\left[F_\Delta^{\text{sym}}(k;\ m_0,\ p) - \frac{1}{2}\left(\frac{1}{\overline{p+\frac{1}{2}k}^2 + m_0^2} + \frac{1}{\overline{p-\frac{1}{2}k}^2 + m_0^2}\right)\sum_{l=0}^{n} B_l(k)\,(m_0^2)^l\right]\Bigg|_{m_0^2=0}.$$

The symmetrization procedure ensures us that all the singularities in the integrand are *locally* cancelled out. We are still left with singularities regularized by counterterms of the form

$$(9.24)\qquad \frac{\cos 4l\theta}{q^{2r}},$$

where q and θ are the polar coordinates in the p-plane with centre $\pm\frac{1}{2}k_\mu$ and $l \neq 0$; these counterterms are implicitly introduced whenever needed, or equivalently polar integration in two small circles around $p_\mu = \pm\frac{1}{2}k_\mu$ is understood as performed first. In practice, we shall only be interested in the functions $A_0(k)$ and $A_1(k)$, whose integral representations we explicitly quote:

$$(9.25a)\qquad A_0(k) = \int \frac{d^2p}{(2\pi)^2}\left[F_\Delta^{\text{sym}}(k;\ 0,\ p) - \frac{1}{2}\left(\frac{1}{\overline{p+\frac{1}{2}k}^2} + \frac{1}{\overline{p-\frac{1}{2}k}^2}\right)B_0(k)\right],$$

$$(9.25b)\qquad A_1(k) = \int \frac{d^2p}{(2\pi)^2}\left[\frac{d F_\Delta^{\text{sym}}(k;\ m_0,\ p)}{dm_0^2}\Bigg|_{m_0^2=0} +\right.$$
$$\left. + \frac{1}{2}\left(\frac{1}{\left(\overline{p+\frac{1}{2}k}^2\right)^2} + \frac{1}{\left(\overline{p-\frac{1}{2}k}^2\right)^2}\right)B_0(k) - \frac{1}{2}\left(\frac{1}{\overline{p+\frac{1}{2}k}^2} + \frac{1}{\overline{p-\frac{1}{2}k}^2}\right)B_1(k)\right].$$

Hence for our present purposes we shall only have to work out the functions $B_0(k)$ and $B_1(k)$. Let us illustrate the procedure by evaluating $B_0^{(\alpha)}(k)$ and $B_1^{(\alpha)}(k)$. Our starting point will be the expansion in powers of a of the continuum integral

$$(9.26)\qquad 2\int_0^{\infty} \frac{d^dp}{(2\pi)^d} a^d\left[m_0^2a^2 + \sum_\mu 4\sin^2\left(\frac{1}{2}ap_\mu\right) - c_2\sum_\mu 4\sin^4\left(\frac{1}{2}ap_\mu\right)\right]^{-1}\cdot$$
$$\cdot\left[m_0^2a^2 + \sum_\mu 4\sin^2\left(\frac{1}{2}ap_\mu + k_\mu\right) - c_2\sum_\mu 4\sin^4\left(\frac{1}{2}ap_\mu + k_\mu\right)\right]^{-1},$$

where the factor 2 has been included in order to take into account the effects of the (excluded) expansion around the second singularity in $p = -k$. Since we are interested in the expansion to second non-trivial order, we may replace integral (9.26) with

$$(9.27)\qquad 2a^{\varepsilon}\int_0^{\infty}\frac{\mathrm{d}^d p}{(2\pi)^d}\,\frac{1}{\overline{pa}^2+m_0^2a^2}\cdot$$

$$\cdot\left\{\frac{1}{\bar{k}^2}-\frac{a^2\sum_\mu p_\mu^2(c_1\cos k_\mu+c_2\cos 2k_\mu)+m_0^2}{(\bar{k}^2)^2}+\frac{a^2\left[\sum_\mu p_\mu(2c_1\sin k_\mu+c_2\sin 2k_\mu)\right]^2}{(\bar{k}^2)^3}\right\}.$$

Within the desired approximation, however,

$$(9.28)\qquad \int_0^{\infty}\frac{\mathrm{d}^d p}{(2\pi)^d}\,\frac{p_\mu p_\nu}{\bar{p}^2+m_0^2}\approx\frac{1}{d}\int_0^{\infty}\frac{\mathrm{d}^d p}{(2\pi)^d}\,\frac{\bar{p}^2\delta_{\mu\nu}}{\bar{p}^2+m_0^2}=-\frac{m_0^2}{d}\delta_{\mu\nu}\int_0^{\infty}\frac{\mathrm{d}^d p}{(2\pi)^d}\,\frac{1}{\bar{p}^2+m_0^2}.$$

Moreover, we are only interested in the pole art, and we can therefore replace integral (9.27) with

$$(9.29)\qquad 2\int_0^{\infty}\frac{\mathrm{d}^d p}{(2\pi)^d}\,\frac{1}{\bar{p}^2+m_0^2}\cdot$$

$$\cdot\left\{\frac{1}{\bar{k}^2}-m_0^2a^2\left[\frac{1-\frac{1}{2}\sum_\mu(c_1\cos k_\mu+c_2\cos 2k_\mu)}{(\bar{k}^2)^2}-\frac{1}{2}\,\frac{\sum_\mu(2c_1\sin k_\mu+c_2\sin 2k_\mu)^2}{(\bar{k}^2)^3}\right]\right\}.$$

However,

$$\int_0^{\infty}\frac{\mathrm{d}^d p}{(2\pi)^d}\,\frac{1}{\bar{p}^2+m_0^2}$$

is an exact representation of the pole part in the asymptotic expansion of β, and therefore we can read $B_0^{(\alpha)}$ and $B_1^{(\alpha)}$ directly off integral (9.29) and find

$$(9.30a)\qquad B_0^{(\alpha)}=\frac{2}{\bar{k}^2},$$

$$(9.30b)\qquad B_1^{(\alpha)}=-2\sum_\mu\frac{\sin^2\frac{1}{2}k_\mu(c_1+4c_2\cos^2\frac{1}{2}k_\mu)^2}{(\bar{k}^2)^2}-4\sum_\mu\frac{\sin^2 k_\mu(c_1+c_2\cos^2 k_\mu)^2}{(\bar{k}^2)^3}.$$

An important representation of eq. (9.30*b*) is

$$(9.31)\qquad B_1^{(\alpha)}=B_0^{(\alpha)}\left[\frac{1}{2}\sum_\mu\partial_\mu\partial_\mu\ln\bar{k}^2+\sum_\mu\frac{\sin^2\frac{1}{2}k_\mu(c_1+4c_2\cos^2\frac{1}{2}k_\mu)^2}{\bar{k}^2}-\frac{2}{\bar{k}^2}\right].$$

The computation of $B_0^{(\theta)}$ and $B_1^{(\theta)}$ is in no sense conceptually more involved than the computation we have just presented. However, many more terms have to be taken into account and a few algebraic tricks exploiting explicitly the fact that there are only two vector components in two dimensions have to be employed in order to simplify the result. Let us only quote the final result:

(9.32a) $$B_0^{(\theta)} = 2\sum_\mu (c_1 + c_2\cos^2\tfrac{1}{2}k_\mu) - 2\sum_\mu \frac{\hat{k}_\mu^2(c_1 + c_2\cos^2\frac{1}{2}k_\mu)^2}{\bar{k}^2} = 2\prod_\mu (c_1 + c_2\cos^2\tfrac{1}{2}k_\mu)\frac{\hat{k}^2}{\bar{k}^2},$$

(9.32b) $$B_1^{(\theta)} = B_0^{(\theta)}\left\{\sum_\mu\left[\frac{c_1 + 4c_2\cos^2\frac{1}{2}k_\mu}{\bar{k}^2} + \frac{8\cos^2\frac{1}{2}k_\mu(c_1 + c_2\cos k_\mu)^2}{(\bar{k}^2)^2}\right]\cdot\left[\frac{\sum_\nu \sin^2\frac{1}{2}k_\nu(c_1 + c_2\cos^2\frac{1}{2}k_\nu)}{c_1 + c_2\cos^2\frac{1}{2}k_\mu} - \sin^2\frac{1}{2}k_\mu\right]\right\} =$$
$$= B_0^{(\theta)}\left\{\frac{1}{2}\sum_\mu \partial_\mu\partial_\mu \ln\bar{k}^2 + \sum_\mu \frac{\sin^2\frac{1}{2}k_\mu(c_1 - 4c_2\cos^2\frac{1}{2}k_\mu)}{\bar{k}^2} + \frac{2}{\bar{k}^2} + \frac{1}{4}\sum_\mu \frac{1}{c_1 + c_2\cos^2\frac{1}{2}k_\mu}\left(c_1 + 4c_2\cos^2\frac{1}{2}k_\mu - \frac{8c_1\sin^2\frac{1}{2}k_\mu}{\bar{k}^2}\right)\right\}.$$

Finally we mention that it is possible to define an asymptotic expansion also for the functions $\Delta^{-1}_{(\theta)\mu}(k)$ defined in sect. **7**. The form of the expansion is

(9.33) $$\beta\Delta^{-1}_{(\theta)\mu}(k) = \sum_{n=0}^{\infty}[C_{n\mu} + \beta D_{n\mu}](m_0^2)^n,$$

and it is easy to obtain

(9.34a) $$D_{0\mu} = c_1 + c_2\cos^2\frac{1}{2}k_\mu,$$

(9.34b) $$C_{0\mu} = -\frac{1}{4}\int\frac{d^2p}{(2\pi)^2}\,\frac{c_1\hat{p}^2 + 4c_2\sum_\nu \hat{p}_\nu^2\cos^2\frac{1}{2}p_\nu\cos^2\frac{1}{2}k_\mu}{\bar{p}^2},$$

(9.34c) $$D_{1\mu} = \frac{1}{4}\left(c_1 + 4c_2\cos^2\frac{1}{2}k_\mu\right),$$

(9.34d) $$C_{1\mu} = -\frac{1}{4}\int\frac{d^2p}{(2\pi)^2}\left[\frac{c_1 + 4c_2\cos^2\frac{1}{2}k_\mu}{\bar{p}^2} - \frac{c_1\hat{p}^2 + 4c_2\sum_\nu \hat{p}_\nu^2\cos^2\frac{1}{2}p_\nu\cos^2\frac{1}{2}k_\mu}{(\bar{p}^2)^2}\right].$$

All these results take a particularly simple form in the case $c_2 = 0$; in this case we obtain

$$(9.35a)\qquad B_0^{(\alpha)}(k) = \frac{2}{\hat{k}^2},$$

$$(9.35b)\qquad B_1^{(\alpha)}(k) = \left(-\frac{1}{2} - \frac{4}{\hat{k}^2} + \frac{\hat{k}^4}{(\hat{k}^2)^2}\right)\frac{1}{\hat{k}^2} = B_0^{(\alpha)}(k)\left(\frac{1}{2}\sum_\mu \partial_\mu \partial_\mu \ln \hat{k}^2 + \frac{1}{4} - \frac{2}{\hat{k}^2}\right),$$

$$(9.35c)\qquad B_0^{(\theta)}(k) = 2\,,$$

$$(9.35d)\qquad B_1^{(\theta)}(k) = -\frac{1}{2} + \frac{4}{\hat{k}^2} + \frac{\hat{k}^4}{(\hat{k}^2)^2} = B_0^{(\theta)}(k)\left(\frac{1}{2}\sum_\mu \partial_\mu \partial_\mu \ln \hat{k}^2 + \frac{1}{4} + \frac{2}{\hat{k}^2}\right),$$

$$(9.35e)\qquad D_{0\mu} = 1\,,\qquad C_{0\mu} = -\frac{1}{4},\qquad D_{1\mu} = \frac{1}{4},\qquad C_{1\mu} = 0\,,$$

and one can easily construct the representations of $A_0(k)$ and $A_1(k)$ by a trivial application of eqs. (9.25). We must observe that these representations are well defined and they can be used to compute numerically $A_0(k)$ and $A_1(k)$; however, the integrands are plagued with a very singular behaviour, especially for small k, and it is very hard to perform accurate numerical integrations.

10. – Asymptotic expansions using integral representations.

The methods developed in the previous section apply perfectly well to the case $c_2 = 0$. However, this is not the most convenient way of performing the asymptotic expansion when we possess an integral representation like eqs. (8.19). In this case we start from the observation that

$$(10.1)\qquad I(k;\ m_0 a) = \int_0^1 \mathrm{d}x\, G(\hat{k}_\mu,\ m_0 a,\ x)\,.$$

Moreover, if we perform a homogeneous expansion of G in the powers of $\hat{k}_\mu$ and m_0, the x-integration of the resulting coefficients can be performed explicitly. As a consequence, we may write

$$(10.2)\qquad I(k;\ m_0 a) = \int_0^1 \mathrm{d}x\,[1 - \mathrm{T}_J^{(a)}]\, G(\hat{k}_\mu a,\ m_0 a,\ x) + \int_0^1 \mathrm{d}x\, \mathrm{T}_J^{(a)}\, G(\hat{k}_\mu a,\ m_0 a,\ x)\,,$$

where the unconventional notation $\hat{k}_\mu a$ indicates that the parameters of the Taylor expansion are $\hat{k}_\mu a$, not $k_\mu a$. The first integral is now regular up to $O(a^J)$, and the integrand can therefore be expanded in powers of m_0^2. The second integral in turn can be performed analytically and the result, which is a non-analytic function of m_0, can be expanded in an asymptotic series in the powers of m_0. This procedure is rather cumbersome, but the pay-offs are very high, as we shall see in the following.

Let us present the essential ingredients of the computation and sketch the derivation of the asymptotic expansion in the case of $\Delta^{-1}_{(\alpha)}$. Let us define

$$q_\mu^2 = \hat{k}_\mu^2 x(1-x)\,, \qquad q^2 = \sum_\mu q_\mu^2\,, \qquad q^4 = \sum_\mu q_\mu^4\,, \tag{10.3}$$

and expand all relevant quantities in powers of m_0^2 and $\hat{k}_\mu^2$:

$$a_\mu \approx 2 - q_\mu^2\,, \tag{10.4a}$$

$$\zeta^2 \approx 1 - \frac{1}{2}(m_0^2 + q^2)\,, \tag{10.4b}$$

$$E(\zeta) \approx 1 - \frac{m_0^2+q^2}{8}\left(\ln\frac{m_0^2+q^2}{32} + 1\right), \tag{10.4c}$$

$$K(\zeta) \approx -\frac{1}{2}\ln\frac{m_0^2+q^2}{32} - \frac{m_0^2+q^2}{16}\ln\frac{m_0^2+q^2}{32} + \frac{m_0^2-q^2}{16} - \frac{1}{8}\frac{q^4}{q^2+m_0^2}\,. \tag{10.4d}$$

The relevant integrals are

$$\int_0^1 \frac{\mathrm{d}x}{q^2+m_0^2} = \frac{2}{\hat{k}^2\hat{\xi}}\ln\frac{\hat{\xi}+1}{\hat{\xi}-1}\,, \tag{10.5a}$$

$$\int_0^1 \mathrm{d}x \ln\frac{q^2+m_0^2}{32} = \hat{\xi}\ln\frac{\hat{\xi}+1}{\hat{\xi}-1} - 2 - \ln\frac{32}{m_0^2}\,, \tag{10.5b}$$

$$\int_0^1 \mathrm{d}x\,\frac{q^4}{(q^2+m_0^2)^2} = \frac{\hat{k}^4}{(\hat{k}^2)^2}\left[\frac{1}{\hat{\xi}}\ln\frac{\hat{\xi}+1}{\hat{\xi}-1}\left(\frac{1}{4\hat{\xi}^2}+\frac{1}{2}-\frac{3}{4}\hat{\xi}^2\right) - \frac{1}{2\hat{\xi}^2} + \frac{3}{2}\right], \tag{10.5c}$$

where

$$\hat{\xi} \equiv \sqrt{1+\frac{4m_0^2}{\hat{k}^2}}\,. \tag{10.6}$$

Then, starting from eq. (8.19a), we obtain

$$\begin{aligned}\Delta^{-1}_{(\alpha)} &\approx \frac{1}{4\pi}\int_0^1 \mathrm{d}x\,\frac{1}{m_0^2+q^2}\left[1 - \frac{m_0^2+q^2}{8}\ln\frac{q^2+m_0^2}{32} - \frac{m_0^2}{4} - \frac{q^4}{4\,(m_0^2+q^2)}\right] = \\ &= \frac{1}{2\pi}\left[\frac{1}{\hat{k}^2\hat{\xi}}\ln\frac{\hat{\xi}+1}{\hat{\xi}-1} - \frac{1}{16}\frac{2\hat{\xi}^2-1}{\hat{\xi}}\ln\frac{\hat{\xi}+1}{\hat{\xi}-1} + \frac{1}{8} + \frac{1}{16}\ln\frac{32}{m_0^2} - \right. \\ &\qquad \left. - \frac{1}{8}\frac{\hat{k}^4}{(\hat{k}^2)^2}\left(\frac{(1-\hat{\xi}^2)(1+3\hat{\xi}^2)}{4\hat{\xi}^3}\ln\frac{\hat{\xi}+1}{\hat{\xi}-1} + \frac{3\hat{\xi}^2-1}{2\hat{\xi}^2}\right)\right].\end{aligned} \tag{10.7}$$

Therefore by expanding both sides of eq. (10.7) we can introduce the regulator in the form

$$(10.8)\quad -\frac{1}{2\pi}\int_0^1 \mathrm{d}x\,\frac{1}{2q^2}+\frac{1}{2\pi\hat{k}^2}\ln\frac{\hat{k}^2}{32}+2\beta\,\frac{1}{\hat{k}^2}+\frac{m_0^2}{2\pi}\int_0^1\mathrm{d}x\left[2\,\frac{1}{(q^2)^2}+\frac{3}{16}\frac{1}{q^2}-\frac{1}{4}\frac{\hat{k}^4}{(\hat{k}^2)^2}\frac{1}{q^2}\right]+$$

$$+\frac{m_0^2}{2\pi\hat{k}^2}\left[\left(-2\,\frac{1}{\hat{k}^2}-\frac{3}{8}+\frac{1}{2}\frac{\hat{k}^4}{(\hat{k}^2)^2}\right)\ln\frac{\hat{k}^2}{32}+2\,\frac{1}{\hat{k}^2}-\frac{1}{4}-\frac{1}{4}\frac{\hat{k}^4}{(\hat{k}^2)^2}\right]+\frac{m_0^2}{\hat{k}^2}\left(-\frac{4}{\hat{k}^2}-\frac{1}{2}+\frac{\hat{k}^4}{(\hat{k}^2)^2}\right)\beta.$$

As a consequence, the known values of $B_0^{(\alpha)}$ and $B_1^{(\alpha)}$ are reproduced, and we find the representations

$$(10.9a)\quad A_0^{(\alpha)}=\frac{1}{\pi}\int_0^1\mathrm{d}x\left\{\frac{\zeta_0^3}{(a_1a_2)^{3/2}}\frac{E(\zeta_0)}{1-\zeta_0^2}-\frac{1}{4q^2}\right\}+\frac{1}{2\pi\hat{k}^2}\ln\frac{\hat{k}^2}{32},$$

$$(10.9b)\quad A_1^{(\alpha)}=\frac{1}{4\pi}\int_0^1\mathrm{d}x\,\frac{\zeta_0^3}{(a_1a_2)^{3/2}}\left\{\frac{E(\zeta_0)}{1-\zeta_0^2}+\frac{4\zeta_0^2}{a_1a_2}\left[\frac{K(\zeta_0)-2E(\zeta_0)}{1-\zeta_0^2}-\frac{2E(\zeta_0)}{(1-\zeta_0^2)^2}\right]\right\}+$$

$$+\frac{1}{4\pi}\int_0^1\mathrm{d}x\left[\frac{1}{(q^2)^2}+\frac{3}{8}\frac{1}{q^2}-\frac{1}{2}\frac{q^4}{(q^2)^3}\right]+$$

$$+\frac{1}{2\pi}\left[\frac{2}{(\hat{k}^2)^2}-\frac{1}{4\hat{k}^2}-\frac{1}{4}\frac{\hat{k}^4}{(\hat{k}^2)^3}+\frac{1}{\hat{k}^2}\ln\frac{\hat{k}^2}{32}\left(-\frac{2}{\hat{k}^2}-\frac{3}{8}+\frac{1}{2}\frac{\hat{k}^4}{(\hat{k}^2)^2}\right)\right],$$

where

$$(10.10)\qquad \zeta_0\equiv\sqrt{\frac{4a_1a_2}{16-(a_1-a_2)^2}}.$$

By repeating the analysis for $\Delta_{(\theta)}^{-1}$, we find that

$$(10.11)\quad \Delta_{(\theta)}^{-1}\approx\frac{\hat{k}^2}{4\pi}\int_0^1\mathrm{d}x\,\frac{(1-2x)^2}{m_0^2+q^2}\cdot$$

$$\cdot\left[1+\frac{3}{4}m_0^2+\frac{3}{2}q^2+\frac{3}{8}(q^2+m_0^2)\ln\frac{q^2+m_0^2}{32}-\frac{1}{4}\frac{q^4}{m_0^2+q^2}\right]=$$

$$=\frac{1}{2\pi}\left[\hat{\xi}\ln\frac{\hat{\xi}+1}{\hat{\xi}-1}-2+\frac{1}{16}\hat{k}^2(3-2\hat{\xi}^2)\left(\hat{\xi}\ln\frac{\hat{\xi}+1}{\hat{\xi}-1}-2\right)+\frac{5}{24}\hat{k}^2-\frac{1}{16}\hat{k}^2\ln\frac{32}{m_0^2}+\right.$$

$$\left.+\frac{1}{8}\frac{\hat{k}^4}{\hat{k}^2}\left(\frac{(1-\hat{\xi}^2)(1-5\hat{\xi}^2)}{4\hat{\xi}}\ln\frac{\hat{\xi}+1}{\hat{\xi}-1}+\frac{13}{6}-\frac{5}{2}\hat{\xi}^2\right)\right],$$

and expanding both sides of eq. (10.11) we can introduce the regulator in the form

$$(10.12)\quad -\frac{1}{2\pi}\int_0^1 \mathrm{d}x\,\frac{\hat{k}^2}{2q^2}+\frac{1}{2\pi}\ln\frac{\hat{k}^2}{32}+2\beta+\frac{m_0^2}{2\pi}\int_0^1 \mathrm{d}x\left[\frac{\hat{k}^2}{2(q^2)^2}-\frac{2}{q^2}+\frac{3}{16}\frac{\hat{k}^2}{q^2}-\frac{1}{4}\frac{\hat{k}^4}{\hat{k}^2q^2}\right]+$$

$$+\frac{m_0^2}{2\pi}\left[\left(2\frac{1}{\hat{k}^2}-\frac{3}{8}+\frac{1}{2}\frac{\hat{k}^4}{(\hat{k}^2)^2}\right)\ln\frac{\hat{k}^2}{32}+2\frac{1}{\hat{k}^2}+\frac{1}{4}-\frac{1}{4}\frac{\hat{k}^4}{(\hat{k}^2)^2}\right]+m_0^2\left[4\frac{1}{\hat{k}^2}-\frac{1}{2}+\frac{\hat{k}^4}{(\hat{k}^2)^2}\right]\beta .$$

The values of $B_0^{(\theta)}$ and $B_1^{(\theta)}$ are reproduced and we obtain the representations

$$(10.13a)\quad A_0^{(\theta)}=\frac{4\hat{k}^2}{\pi}\int_0^1 \mathrm{d}x\,\frac{(1-2x)^2}{(a_1a_2)^{5/2}}\left[\frac{\zeta_0^3E(\zeta_0)}{1-\zeta_0^2}+2\zeta_0(E(\zeta_0)-K(\zeta_0))\right]-$$

$$-\frac{1}{2\pi}\int_0^1 \mathrm{d}x\,\frac{\hat{k}^2}{2q^2}+\frac{1}{2\pi}\ln\frac{\hat{k}^2}{32},$$

$$(10.13b)\quad A_1^{(\theta)}=\frac{\hat{k}^2}{\pi}\int_0^1 \mathrm{d}x\,\frac{(1-2x)^2}{(a_1a_2)^{5/2}}\zeta_0\left[\frac{E(\zeta_0)}{1-\zeta_0^2}+E(\zeta_0)-2K(\zeta_0)\right]-$$

$$-\frac{4\hat{k}^2}{\pi}\int_0^1 \mathrm{d}x\,\frac{(1-2x)^2}{(a_1a_2)^{7/2}}\zeta_0^3\left[2\frac{1-\zeta_0^2+\zeta_0^4}{(1-\zeta_0^2)^2}E(\zeta_0)-\frac{2-\zeta_0^2}{1-\zeta_0^2}K(\zeta_0)\right]+$$

$$+\frac{\hat{k}^2}{2\pi}\int_0^1 \mathrm{d}x\left[\frac{1}{2}\frac{1}{(q^2)^2}-2\frac{1}{q^2\hat{k}^2}+\frac{3}{16}\frac{1}{q^2}-\frac{1}{4}\frac{\hat{k}^4}{(\hat{k}^2)^2q^2}\right]+$$

$$+\frac{1}{2\pi}\left[\left(2\frac{1}{\hat{k}^2}-\frac{3}{8}+\frac{1}{2}\frac{\hat{k}^4}{(\hat{k}^2)^2}\right)\ln\frac{\hat{k}^2}{32}+2\frac{1}{\hat{k}^2}+\frac{1}{4}-\frac{1}{4}\frac{\hat{k}^4}{(\hat{k}^2)^2}\right].$$

The contour plots of $A_0^{(\alpha)}$ and $A_0^{(\theta)}$ in the k-plane are drawn in fig. 14.

We finally present the results of the evaluation of the inverse propagators along the principal diagonal of the momentum lattice, *i.e.* when $k_1=k_2\equiv l$. In this case direct manipulations of eqs. (8.7) lead to

$$(10.14a)\quad A_0^{(\alpha)}(l,\,l)=\frac{1}{16\pi\sin^2\frac{1}{2}l}\cos\frac{1}{2}l\ln\frac{1-\cos\frac{1}{2}l}{1+\cos\frac{1}{2}l},$$

$$(10.14b)\quad A_1^{(\alpha)}(l,\,l)=\frac{1}{64\pi\sin^4\frac{1}{2}l}\left[\cos^2\tfrac{1}{2}l-\left(2-\cos^2\frac{1}{2}l\right)\cos\frac{1}{2}l\ln\frac{1-\cos\frac{1}{2}l}{1+\cos\frac{1}{2}l}\right],$$

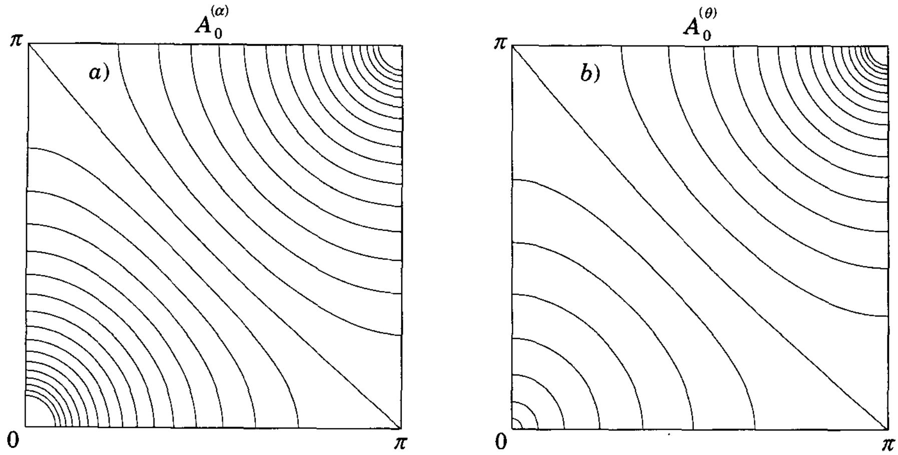


Fig. 14. – Contour plots of $A_0^{(\alpha)}(k)$ (a)) and $A_0^{(\theta)}(k) + 2/\pi$ (b)). Contour lines are in logarithmic scale, separated by a factor $\sqrt{2}$.

$$(10.14c)\qquad A_0^{(\theta)}(l,\ l) = \frac{1}{2\pi}\left[\frac{1}{\cos\frac{1}{2}l}\ln\frac{1-\cos\frac{1}{2}l}{1+\cos\frac{1}{2}l} - 2\right],$$

$$(10.14d)\qquad A_1^{(\theta)}(l,\ l) = \frac{1}{8\pi\sin^2\frac{1}{2}l}\left[2 - \cos^2\frac{1}{2}l + \cos\frac{1}{2}l\ln\frac{1-\cos\frac{1}{2}l}{1+\cos\frac{1}{2}l}\right].$$

11. – Evaluation of physical quantities in the scaling region.

The regularization (9.13) of lattice integrals in the scaling region and the asymptotic expansion (9.4) of the effective propagators are the essential ingredients for the evaluation of the scaling contributions to any physical quantity to $O(1/N)$, involving one-loop integrals over the effective fields propagators.

Actually there is no real need of evaluating separately every physical quantity: whenever two operators have the same (possibly anomalous) dimension, their ratio is a pure number that is scheme-independent, and therefore it can be computed in the simplest available scheme; only one of the two operators, possibly the easier to compute, must be evaluated on the lattice in the scaling region. These statements obviously rest on the existence of a renormalization group behaviour in the scaling regime of the model involved, but they can be explicitly verified in the $1/N$ expansion, as we shall see in the following.

11'1. *Lattice correlation length.* – A prototype dimensionful quantity is the inverse correlation length (mass gap), that sets the scale for all quantities having canonical dimension. For sufficiently large values of $\varkappa$, as discussed in sect. **3**, it is meaningful to extract the mass gap from the asymptotic behaviour of the two-point correlation function of the fundamental fields. This asymptotic behaviour is determined by the location of the pole of the two-point function, as in the continuum case. There is, however, a slight complication due to the absence of rotation invariance outside the scaling region. For the sake of completeness, let us discuss this point, that was solved in the $O(N)$ case by Müller, Raddatz and Ruhl [139] by applying saddle-point techniques to estimate the large-distance behaviour of $G(x)$ (see also Cristofano *et al.* [140]).

For the sake of definiteness and simplicity we shall now focus on the case $c_2 = 0$, where the inverse two-point functions is

$$G^{-1}(p) \equiv \hat{p}^2 + m_0^2 + \frac{1}{N}\,\Sigma(p)\,. \tag{11.1}$$

Let us introduce the function $\mu(\theta)$, indicating the coefficient of the large-distance exponential decay, which is dependent on the polar angle θ in the $(x,\, y)$-plane, and define

$$\mu(\theta) = \mu_x(\theta)\cos\theta + \mu_y(\theta)\sin\theta\,, \tag{11.2}$$

where μ_x and μ_y are functions of θ such that

$$G^{-1}(p_i = i\mu_i) = 0\,. \tag{11.3}$$

Equation (11.3) is not sufficient to determine $\mu_i(\theta)$, but, guided by the saddle-point analysis of ref. [139], we make the large-N Ansatz

$$\mu_i = \mu_{0i} + \frac{1}{N}\,\mu_{1i} + O\left(\frac{1}{N^2}\right), \tag{11.4a}$$

$$\sinh\mu_{0x} = v_0\cos\theta\,, \qquad \sinh\mu_{0y} = v_0\sin\theta\,. \tag{11.4b}$$

The generalization to $c_2 \neq 0$ would consist in the replacement

$$\sinh\mu_{0i} \to c_1\sinh\mu_{0i} + \frac{1}{2}\,c_2\sinh 2\mu_{0i}\,. \tag{11.5}$$

We can now solve the equation

$$\hat{p}^2 + m_0^2 = 0 \tag{11.6}$$

for $p_i = i\mu_{0i}$ with the Ansatz (11.4) and obtain

$$v_0 = m_0\sqrt{2 + \frac{1}{4}m_0^2}\left[1 + \sqrt{1 - m_0^2(8 + m_0^2)\left(\frac{\cos 2\theta}{4 + m_0^2}\right)^2}\,\right]^{-1/2}, \tag{11.7}$$

hence

$$\mu_0(\theta) = \cos\theta\,\mathrm{arsinh}(v_0\cos\theta) + \sin\theta\,\mathrm{arsinh}(v_0\sin\theta) = m_0 + O(m_0^3)\,. \tag{11.8}$$

Now replacing eq. (11.4a) in the condition (11.3), we immediately get

$$-2\,[v_0\mu_{1x}\cos\theta + v_0\mu_{1y}\sin\theta] + \Sigma_1(i\mu_{0i}) = 0\,, \tag{11.9}$$

implying

$$\mu_1(\theta) \equiv \mu_{1x}\cos\theta + \mu_{1y}\sin\theta = \frac{\Sigma_1(i\mu_{0i})}{2\nu_0} \tag{11.10}$$

and

$$\mu^2(\theta) = \mu_0^2(\theta) + \frac{1}{N}\frac{\mu_0(\theta)}{\nu_0(\theta)}\Sigma_1(i\mu_{0i}) + O\left(\frac{1}{N^2}\right). \tag{11.11}$$

These results are quite general and may be useful in detailed studies of lattice models outside the scaling region. One may check that in the appropriate (scaling) limit the dependence on θ disappears in $\mu_1(\theta)$ as well as in μ_0. However, in practice we may limit our attention to only two special values of θ, corresponding to the two extrema $\theta = 0$ (side correlation) and $\theta = \frac{1}{4}\pi$ (diagonal correlation). In both cases no special Ansatz is needed in order to find the expression for the mass gap: in the first case the lattice symmetry implies $\mu_y(0) = 0$ and therefore $\mu(0) = \mu_x(0)$; in the second case we have $\mu_x(\frac{1}{4}\pi) = \mu_y(\frac{1}{4}\pi) \equiv \mu_d$ and therefore $\mu(\frac{1}{4}\pi) = \sqrt{2}\,\mu_d$. Equation (11.3) reduces in both cases to a single-variable equation. It is easy to check that, when $c_2 = 0$,

$$\mu_0(0) = 2\,\mathrm{arsinh}\,\frac{1}{2}m_0, \tag{11.12a}$$

$$\mu_0\left(\frac{1}{4}\pi\right) = 2\sqrt{2}\,\mathrm{arsinh}\,\frac{m_0}{2\sqrt{2}}; \tag{11.12b}$$

$\mu_0^2(0)$ and $\mu_0^2(\frac{1}{4}\pi)$ are plotted as functions of f in fig. 15. It is also comforting to check that, in these two specific cases, the resulting definition of the mass gap coincides with that obtained from the so-called wall-wall correlations

$$G_s(y-x) = \frac{1}{L}\sum_{x_1, y_1} G(x_1,\ x;\ y_1,\ y), \tag{11.13a}$$

$$G_d\left(\frac{y-x}{\sqrt{2}}\right) = \frac{\sqrt{2}}{L}\sum_{x_1, y_1} G(x_1,\ x-x_1;\ y_1,\ y-y_1), \tag{11.13b}$$

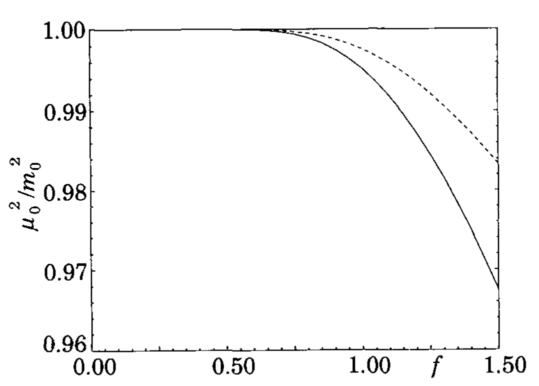


Fig. 15. – $\mu_0^2(0)$ (solid line) and $\mu_0^2(\frac{1}{4}\pi)$ (dashed line), normalized to m_0^2.

whose effective one-dimensional propagators in momentum space are

$$G_s^{-1} = G^{-1}(p_x = 0,\ p_y)\,, \tag{11.14a}$$

$$G_d^{-1} = \frac{1}{\sqrt{2}}\, G^{-1}(p_x = p_d,\ p_y = p_d)\,. \tag{11.14b}$$

Since side and diagonal correlations are the two extremal cases on the lattice, rotation invariance may be verified simply by checking that

$$G_s(x) = G_d(x) \tag{11.15}$$

for all values of x.

As long as we are interested only in the scaling behaviour, in order to minimize the effort we can focus on the case $\theta = 0$. We can be fairly general, because in the scaling region

$$\mu_0(\theta) = m_0 + O(m_0^3)\,, \tag{11.16a}$$

$$\nu_0(\theta) = m_0 + O(m_0^3)\,, \tag{11.16b}$$

independently of the choice of the lattice action. Therefore

$$\mu^2(\theta) = m_0^2 + \frac{1}{N}\,\Sigma_1(im_0) + O(m_0^4) + O\left(\frac{1}{N^2}\right), \tag{11.17}$$

and we need only to compute the lattice $O(1/N)$ contribution to the self-energy.

11'2. *Scaling behaviour of the free energy.* – Before attacking the abovementioned problem, let us, however, notice that, if we believe in standard renormalization group arguments, we may extract the scaling behaviour on the lattice directly and with lesser effort from the expression of the lattice free energy. The subtle point in the evaluation of the lattice free energy has to do with the existence of a perturbative tail that obscures the scaling behaviour of this physical quantity. However, this turns out to be a tractable problem, because the asymptotic expansion introduced in sect. **9** allows us to isolate and remove unambiguously the perturbative tail of the free energy. In practice, without belaboring in this point (*cf.* ref. [6] for more details), we must simply translate eq. (5.2) into the lattice language and write, in analogy with the continuum notation,

$$F^{(\mathrm{scal})} = N\,\mathrm{Tr}\ln(\bar{p}^2 + m_0^2) - N\,\mathrm{Tr}\ln\bar{p}^2 - N\beta m_0^2 + \frac{1}{2}\,\mathrm{Tr}\ln\Delta_{(\alpha)}^{-1}(p) - \\ -\frac{1}{2}\,\mathrm{Tr}\ln\Delta_{(\alpha)}^{-1}(p)\big|_{m_0^2=0} + \frac{1}{2}\,\mathrm{Tr}\ln\Delta_{(\theta)\mu\nu}^{-1}(p) - \frac{1}{2}\,\mathrm{Tr}\ln\Delta_{(\theta)\mu\nu}^{-1}(p)\big|_{m_0^2=0}\,, \tag{11.18}$$

where we need to recall that

$$\Delta_{(\theta)\mu\nu}^{-1}(p) = \varkappa\,\Delta_{(\theta)\mu}^{-1}(p)\,\delta_{\mu\nu} + \Delta_{(\theta)}^{-1}(p)\,\delta_{\mu\nu}^{t}(p) \tag{11.19}$$

and

$$\mathrm{Tr}\ln \Delta^{-1}_{(\theta)\mu\nu}(p) = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \ln\det \Delta^{-1}_{(\theta)\mu\nu}(p)\,. \tag{11.20}$$

Hence we only need to evaluate

$$\det \Delta^{-1}_{(\theta)\mu\nu}(p) = \varkappa\left[\sum_\mu \Delta^{-1}_{(\theta)\mu}\frac{\hat{p}_\mu^2}{\hat{p}^2}\Delta^{-1}_{(\theta)}(p) + \varkappa\prod_\mu \Delta^{-1}_{(\theta)\mu}(p)\right]. \tag{11.21}$$

The $m_0 \to 0$ limits of the relevant expressions are obtained by the replacements

$$\Delta^{-1}_{(\alpha)}(p) \to A_0^{(\alpha)} + \beta B_0^{(\alpha)}, \tag{11.22a}$$

$$\Delta^{-1}_{(\theta)}(p) \to A_0^{(\theta)} + \beta B_0^{(\theta)}, \tag{11.22b}$$

$$\beta\Delta^{-1}_{(\theta)\mu}(p) \to C_{0\mu} + \beta D_{0\mu}\,. \tag{11.22c}$$

Let us now notice that, according to the rules of the asymptotic expansions, we may evaluate

$$N\,\mathrm{Tr}\ln(\bar{p}^2 + m_0^2) - N\,\mathrm{Tr}\ln\bar{p}^2 - N\beta m_0^2 = N\int\frac{\mathrm{d}^2 p}{(2\pi)^2}\frac{m_0^2}{\bar{p}^2} - \tag{11.23}$$

$$- N\beta m_0^2 + N\int\frac{\mathrm{d}^2 p}{(2\pi)^2}\left[\ln\left(1 + \frac{m_0^2}{p^2}\right) - \frac{m_0^2}{p^2}\right] + O(m_0^4) = \frac{N}{4\pi}m_0^2 + O(m_0^4)\,.$$

Moreover, if we treat m_0^2 and β as formally independent variables in eqs. (9.4) and (9.33), we can introduce partial derivatives via the relationship

$$\frac{\mathrm{d}}{\mathrm{d}m_0^2} \equiv \frac{\partial}{\partial m_0^2} + \frac{\mathrm{d}\beta}{\mathrm{d}m_0^2}\frac{\partial}{\partial\beta} = \frac{\partial}{\partial m_0^2} - \Delta^{-1}_{(\alpha)}(0)\frac{\partial}{\partial\beta}; \tag{11.24}$$

this allows us to reformulate eq. (11.18), obtaining the following representation of the scaling lattice free energy to $O(1/N)$:

$$F^{(\mathrm{scal})} = \frac{N}{4\pi}m_0^2 + \tag{11.25}$$

$$+\frac{1}{2}m_0^2\frac{\partial}{\partial m_0^2}\left[\int\frac{\mathrm{d}^2 p}{(2\pi)^2}\ln\Delta^{-1}_{(\alpha)}(p) + \int\frac{\mathrm{d}^2 p}{(2\pi)^2}\ln\det\Delta^{-1}_{(\theta)\mu\nu}(p)\right]\Bigg|_{m_0^2=0} +$$

$$+ \text{continuum counterterms.}$$

The continuum counterterms in eq. (11.25) are to be introduced according to the usual rules and will generate a contribution akin to the counterterms used in the sharp-momentum cut-off evaluation of the free energy. We remind that the continuum conterparts of the relevant quantities appearing in eq. (11.25) are

$$\text{(11.26a)} \qquad \Delta^{-1}_{(\alpha)}(p) \to \frac{1}{2\pi p^2 \xi} \ln \frac{\xi+1}{\xi-1} + O(a^2),$$

$$\text{(11.26b)} \qquad \Delta^{-1}_{(\theta)}(p) \to \frac{\xi}{2\pi} \ln \frac{\xi+1}{\xi-1} - \frac{1}{\pi} + O(a^2),$$

$$\text{(11.26c)} \qquad \begin{cases} \Delta^{-1}_{(\theta)\mu}(p) \to 1 - \dfrac{1}{4\beta} \displaystyle\int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{c_1 \hat{q}^2 + 4c_2 \sum_\nu \hat{q}_\nu^2 \cos^2 \frac{1}{2} q_\nu}{c_1 \hat{q}^2 + c_2 \sum_\nu \hat{q}_\nu^2 \cos^2 \frac{1}{2} q_\nu} \equiv 1 + \dfrac{C_0}{\beta}, \\ C_0 = C_{0\mu}|_{p=0}. \end{cases}$$

The form of the continuum counterterms is therefore

$$\text{(11.27)} \qquad \frac{1}{2} \int \frac{\mathrm{d}^2 p}{(2\pi)^2} [1 - \mathrm{T}_2^{(m_0)}] \cdot \\ \cdot \left\{ \ln\left(\frac{1}{2\pi p^2 \xi} \ln \frac{\xi+1}{\xi-1} \right) + \ln\left[\frac{\xi}{2\pi} \ln \frac{\xi+1}{\xi-1} - \frac{1}{\pi} + \varkappa\left(1 + \frac{C_0}{\beta}\right) \right] \right\}.$$

We confirm the observation that, in order to get a better agreement with the corresponding continuum result, it is convenient to redefine $\varkappa$ by

$$\text{(11.28)} \qquad \varkappa = \frac{\tilde{\varkappa}}{1 + C_0/\beta},$$

where $\tilde{\varkappa}$ is to be kept constant along the renormalization group trajectory. The redefinition (11.28) is not going to affect eq. (11.25), because $\varkappa$ turns out to be a perturbative function of β and is, therefore, not affected by the partial derivative with respect to m_0^2. Non-perturbative redefinitions of $\varkappa$ are not allowed; as we shall see, they would spoil the scaling properties of the physical quantities.

Counterterm (11.27) can be analysed in the light of the results presented in sect. **5**, and we recognize that it can be rephrased in the form

$$\text{(11.29)} \qquad \frac{m_0^2}{4\pi} [\ln 4\pi\beta + (3 - 2\pi\tilde{\varkappa}) \ln (4\pi\beta + 2\pi\tilde{\varkappa} - 2) + c_F(\tilde{\varkappa})] - \\ - \int_0^{M_L^2} \frac{\mathrm{d}p^2}{4\pi} \frac{4\pi}{p^2} \left[\frac{1}{\ln (p^2/m_0^2)} + \frac{3 - 2\pi\tilde{\varkappa}}{\ln (p^2/m_0^2) + 2\pi\tilde{\varkappa} - 2} \right],$$

implying

$$\text{(11.30)} \qquad F^{(\mathrm{scal})} = F^{(\mathrm{SM})} + \delta F,$$

where

$$(11.31)\qquad \delta F = \frac{1}{2} m_0^2 \frac{\partial}{\partial m_0^2}\left[\int \frac{d^2p}{(2\pi)^2} \ln \Delta_{(\alpha)}^{-1}(p) + \int \frac{d^2p}{(2\pi)^2} \ln \det \Delta_{(\theta)\mu\nu}^{-1}(p)\right]\Bigg|_{m_0^2=0} - $$
$$- \int_0^{M_L^2} \frac{dp^2}{4\pi} \frac{4\pi}{p^2}\left[\frac{1}{\ln(p^2/m_0^2)} + \frac{3 - 2\pi\tilde{\varkappa}}{\ln(p^2/m_0^2) + 2\pi\tilde{\varkappa} - 2}\right]$$

and $F^{(\mathrm{SM})}$ is given by eq. (5.8) (with $\varkappa$ replaced by $\tilde{\varkappa}$).

11·3. *Scaling behaviour of the self-energy.* – The $O(1/N)$ contribution to the self-energy is obtained by computing the Feynman diagrams drawn in fig. 16. Evaluating the diagrams, according to the rules presented in fig. 12, leads to

$$(11.32)\qquad \Sigma_1(p) = \int \frac{d^2k}{(2\pi)^2} \frac{\Delta_{(\alpha)}(k)}{\overline{p+k}^2 + m_0^2} +$$
$$+ \frac{1}{2} \Delta_{(\alpha)}(0) \int \frac{d^2k}{(2\pi)^2} \Delta_{(\alpha)}(k) \frac{d}{dm_0^2} \Delta_{(\alpha)}^{-1}(k) + \int \frac{d^2k}{(2\pi)^2} \sum_{\mu\nu} \Delta_{(\theta)\mu\nu}(k) \cdot$$
$$\cdot \left\{(1 + \varkappa f)\delta_{\mu\nu} V_{4\mu}(p, k) - \exp\left[\frac{1}{2} i (k_\mu - k_\nu)\right] \frac{V_{3\mu}(p + \frac{1}{2}k, k) V_{3\nu}(p + \frac{1}{2}k, k)}{\overline{p+k}^2 + m_0^2}\right\} +$$
$$+ \frac{1}{2} \Delta_{(\alpha)}(0) \int \frac{d^2k}{(2\pi)^2} \sum_{\mu\nu} \Delta_{(\theta)\mu\nu}(k) \frac{d}{dm_0^2}\left[\Delta_{(\theta)\mu\nu}^{-1}(k) - 2(1 + \varkappa f)\delta_{\mu\nu} \int \frac{d^2q}{(2\pi)^2} \frac{V_{4\mu}(q, k)}{\bar{q}^2 + m_0^2}\right] +$$
$$+ (1 + \varkappa f)\Delta_{(\alpha)}(0) \int \frac{d^2k}{(2\pi)^2} \sum_{\mu\nu} \Delta_{(\theta)\mu\nu}(k) \delta_{\mu\nu} \frac{d}{dm_0^2} \int \frac{d^2q}{(2\pi)^2} \frac{V_{4\mu}(q, k)}{\bar{q}^2 + m_0^2}.$$

Let us now apply the decomposition of the total derivative with respect to the mass into partial derivatives, perform some simplifications, and rearrange the terms in

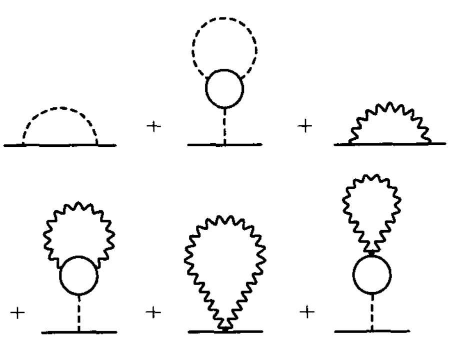


Fig. 16. – $O(1/N)$ contributions to the self-energy.

the form

(11.33) $$\Sigma_1(p) = \int \frac{d^2k}{(2\pi)^2} \Delta_{(\alpha)}(k) \left[\frac{1}{\overline{p+k}^2 + m_0^2} - \frac{1}{2}\frac{\partial}{\partial\beta}\Delta_{(\alpha)}^{-1}(k) \right] + $$
$$+ \frac{1}{2}\Delta_{(\alpha)}(0) \int \frac{d^2k}{(2\pi)^2}\frac{\partial}{\partial m_0^2} \ln \Delta_{(\alpha)}^{-1}(k) + \int \frac{d^2k}{(2\pi)^2} \sum_{\mu\nu} \Delta_{(\theta)\mu\nu}(k) \left\{ \delta_{\mu\nu} V_{4\mu}(p, k) - \right.$$
$$\left. - \exp\left[\frac{1}{2} i (k_\mu - k_\nu)\right] \frac{V_{3\mu}(p+\frac{1}{2}k, k) V_{3\nu}(p+\frac{1}{2}k, k)}{\overline{p+k}^2 + m_0^2} - \frac{1}{2}\frac{\partial}{\partial\beta}\Delta_{(\theta)}^{-1}(k)\,\delta_{\mu\nu}^t(k) \right\} + $$
$$+ \frac{1}{2}\Delta_{(\alpha)}(0) \int \frac{d^2k}{(2\pi)^2} \left[\frac{\partial}{\partial m_0^2} \ln\det \Delta_{(\theta)\mu\nu}^{-1}(k) - \sum_{\mu\nu} \Delta_{(\theta)\mu\nu}(k)\,\delta_{\mu\nu} \Delta_\mu^{-1}(k) \frac{\partial\varkappa}{\partial m_0^2} \right] + $$
$$+ \varkappa f \int \frac{d^2k}{(2\pi)^2} \sum_{\mu\nu} \Delta_{(\theta)\mu\nu}(k)\,\delta_{\mu\nu} \left[V_{4\mu}(p, k) - \frac{\partial}{\partial\beta}(\beta\Delta_\mu^{-1}(k)) \right].$$

Equations (7.10) imply the following relationship:

(11.34) $$\sum_\mu \exp\left[-\frac{1}{2} i k_\mu\right] \hat{k}_\mu \left\{ \delta_{\mu\nu} V_{4\mu}(p, k) - \right.$$
$$\left. - \exp\left[\frac{1}{2} i (k_\mu - k_\nu)\right] \frac{V_{3\mu}(p+\frac{1}{2}k, k) V_{3\nu}(p+\frac{1}{2}k, k)}{\overline{p+k}^2 + m_0^2} \right\} = \exp\left[-\frac{1}{2} i k_\nu\right] \cdot$$
$$\cdot \left[\frac{\bar{p}^2 + m_0^2}{\overline{p+k}^2 + m_0^2} V_{3\nu}\left(p + \frac{1}{2}k, k\right) - \frac{1}{2}\left(V_{3\nu}\left(p+\frac{1}{2}k, k\right) + V_{3\nu}\left(p - \frac{1}{2}k, k\right)\right)\right].$$

Eliminating terms vanishing under symmetric k-integration, we can perform the replacement

(11.35) $$\delta_{\mu\nu} V_{4\mu}(p, k) - \exp\left[\frac{1}{2} i (k_\mu - k_\nu)\right] \frac{V_{3\mu}(p+\frac{1}{2}k, k) V_{3\nu}(p+\frac{1}{2}k, k)}{\overline{p+k}^2 + m_0^2} \to$$
$$\to W(p, k)\,\delta_{\mu\nu}^t(k) + \frac{\bar{p}^2 + m_0^2}{\overline{p+k}^2 + m_0^2} \left[\frac{V_{3\mu}(p+\frac{1}{2}k, k)}{\hat{k}_\mu}\delta_{\mu\nu} - \sum_\rho \frac{V_{3\rho}(p+\frac{1}{2}k, k)}{\hat{k}_\rho}\delta_{\mu\nu}^t(k) \right],$$

where

(11.36) $$W(p, k) = \sum_\mu V_{4\mu}(p, k) - \sum_\mu \frac{[V_{3\mu}(p+\frac{1}{2}k, k)]^2}{\overline{p+k}^2 + m_0^2}.$$

In order to evaluate the function $\Sigma_1(p)$ in the scaling region, we must keep in mind that the formalism introduced in sect. **9** allows for a direct expansion of the lattice integrals in terms of the «external» variables m_0 and p, since all the non-analiticity is accumulated in the continuum integrals, which in turn have already been computed in the sharp-momentum regularization scheme. We now realize that, as far as the lattice contribution is concerned, we can make explicit use of the following relationship, holding for any regular (analytic) function f:

(11.37) $$f(k; p, m_0) \cong f(k; 0, 0) + p^2 \frac{\partial}{\partial p^2} f(k; 0, 0) + m_0^2 \frac{\partial}{\partial m_0^2} f(k; 0, 0).$$

In particular, when imposing the on-shell condition $p^2 + m_0^2 = 0$, we find

$$f(k;\ im_0,\ m_0) \cong f(k;\ 0,\ 0) - m_0^2 \frac{\partial}{\partial p^2} f(k;\ 0,\ 0) + m_0^2 \frac{\partial}{\partial m_0^2} f(k;\ 0,\ 0)\,, \tag{11.38}$$

and, as a consequence,

$$f(k;\ p,\ m_0) \cong f(k;\ im_0,\ m_0) + (p^2 + m_0^2) \frac{\partial}{\partial p^2} f(k;\ 0,\ 0)\,. \tag{11.39}$$

Applying eq. (11.39) to $\Sigma_1(p)$, we come to the conclusion that

$$\Sigma_1(p,\ m_0) \cong \Sigma_1^{(\mathrm{SM})} + \delta m_1^2 - (p^2 + m_0^2)\,\delta Z_1\,, \tag{11.40}$$

where $\Sigma_1^{(\mathrm{SM})}$ is the value of the self-energy in the sharp-momentum cut-off regularization scheme, and δm_1^2, δZ_1 are the $O(1/N)$ lattice contributions to mass and wave-function renormalization, and are amenable to finite lattice integrals whose infrared regularization is provided by the appropriate continuum counter-terms.

11'4. *Lattice contribution to mass renormalization.* – The $O(1/N)$ contribution to the mass (for arbitrary values of the coupling constant) can be obtained from eqs. (11.11) and (11.33):

$$\begin{aligned}
\mu_1^2 \equiv{}& \frac{\mu_0}{\nu_0}\,\Sigma_1(i\mu_0) = \\
={}& \frac{\mu_0}{\nu_0}\Bigg\{\int \frac{\mathrm{d}^2k}{(2\pi)^2}\,\Delta_{(\alpha)}(k)\left[\frac{1}{\overline{k + i\mu_0}^2 + m_0^2} + \frac{1}{2}\Delta_{(\alpha)}(0)\frac{\mathrm{d}}{\mathrm{d}m_0^2}\Delta_{(\alpha)}^{-1}(k)\right] + \\
&+ \int \frac{\mathrm{d}^2k}{(2\pi)^2} \sum_{\mu\nu} \Delta_{(\theta)\mu\nu}(k)\,\delta^t_{\mu\nu}(k)\left[W(i\mu_0,\ k) + \frac{1}{2}\Delta_{(\alpha)}(0)\frac{\mathrm{d}}{\mathrm{d}m_0^2}\Delta_{(\theta)}^{-1}(k)\right] + \\
&+ \varkappa f\int \frac{\mathrm{d}^2k}{(2\pi)^2} \sum_{\mu\nu} \Delta_{(\theta)\mu\nu}(k)\,\delta_{\mu\nu}\left[V_{4\mu}(i\mu_0,\ k) + \Delta_{(\alpha)}(0)\frac{\mathrm{d}}{\mathrm{d}m_0^2}(\beta\Delta_{(\theta)\mu}^{-1}(k))\right]\Bigg\} = \\
={}& \frac{\mu_0}{\nu_0}\Bigg\{\int \frac{\mathrm{d}^2k}{(2\pi)^2}\,\Delta_{(\alpha)}(k)\left[\frac{1}{\overline{k + i\mu_0}^2 + m_0^2} - \frac{1}{2}\frac{\partial}{\partial\beta}\Delta_{(\alpha)}^{-1}(k)\right] + \\
&+ \int \frac{\mathrm{d}^2k}{(2\pi)^2} \sum_{\mu\nu} \Delta_{(\theta)\mu\nu}(k)\,\delta^t_{\mu\nu}(k)\left[W(i\mu_0,\ k) - \frac{1}{2}\frac{\partial}{\partial\beta}\Delta_{(\theta)}^{-1}(k)\right] + \\
&+ \varkappa f\int \frac{\mathrm{d}^2k}{(2\pi)^2} \sum_{\mu\nu} \Delta_{(\theta)\mu\nu}(k)\,\delta_{\mu\nu}\left[V_{4\mu}(i\mu_0,\ k) - \frac{\partial}{\partial\beta}(\beta\Delta_{(\theta)\mu}^{-1}(k))\right] + \\
&+ \frac{1}{2}\Delta_{(\alpha)}(0)\int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\partial}{\partial m_0^2}[\ln\Delta_{(\alpha)}^{-1}(k) + \ln\det\Delta_{(\theta)\mu\nu}^{-1}(k)] - \\
&- \frac{1}{2}\Delta_{(\alpha)}(0)\int \frac{\mathrm{d}^2k}{(2\pi)^2} \sum_{\mu\nu} \Delta_{(\theta)\mu\nu}(k)\,\delta_{\mu\nu}\,\Delta_{(\theta)\mu}^{-1}(k)\frac{\partial\varkappa}{\partial m_0^2}\Bigg\}.
\end{aligned} \tag{11.41}$$

In order to evaluate δm_1^2, let us first complete the lattice computation of m_1^2 in the scaling region from eq. (11.17). From the definition of the asymptotic expansion presented in sect. **9** it is easy to check that, under symmetric k-integration, the following replacements are allowed:

$$\frac{1}{\overline{k+i\mu_0}^2+m_0^2} \to \frac{1}{2}[B_0^{(\alpha)}(k)+m_0^2 B_1^{(\alpha)}(k)] + O(m_0^4) = \frac{1}{2}\frac{\partial}{\partial\beta}\Delta_{(\alpha)}^{-1}(k) + O(m_0^4), \tag{11.42a}$$

$$W(i\mu_0,\ k) \to \frac{1}{2}\frac{\partial}{\partial\beta}\Delta_{(\theta)}^{-1}(k) + O(m_0^4), \tag{11.42b}$$

$$V_{4\mu}(i\mu_0,\ k) \to \frac{\partial}{\partial\beta}(\beta\Delta_{(\theta)\mu}^{-1}(k)) + O(m_0^4). \tag{11.42c}$$

We make now the crucial assumption

$$\frac{\partial\varkappa}{\partial m_0^2} = 0, \tag{11.43}$$

which corresponds to our choice of renormalization group trajectories and defines the class of theories we are going to generate in the continuum limit. The renormalization group flow trajectories defined by eq. (11.28), consistently with eq. (11.43), are plotted in fig. 17 (for $c_2 = 0$). Under such assumption and making use of the identity

$$\Delta_{(\alpha)}(0) = 4\pi m_0^2 + O(m_0^4), \tag{11.44}$$

we can conclude that

$$\Sigma_1(im_0) = 4\pi m_0^2 \times \frac{1}{2}\int\frac{d^2k}{(2\pi)^2}\frac{\partial}{\partial m_0^2}[\ln\Delta_{(\alpha)}^{-1}(k) + \ln\det\Delta_{(\theta)\mu\nu}^{-1}(k)]\,|_{m_0^2=0} + O(m_0^4) + \text{continuum counterterms}. \tag{11.45}$$

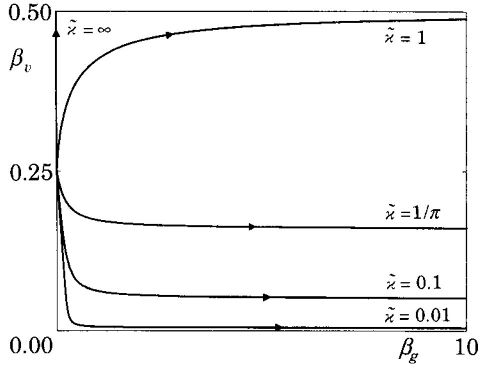


Fig. 17. – The renormalization group flow trajectories of the lattice models, as defined by eq. (11.28).

In order to write down explicitly the continuum counterterms appropriate to $\Sigma_1(im_0)$, it is convenient to go back to eq. (11.32) and perform the replacements indicated in eqs. (11.26), supplemented by the substitutions

$$\frac{1}{\overline{k + im_0}^2 + m_0^2} \to \frac{1}{k^2 \xi}, \tag{11.46a}$$

$$W(im_0,\ k) \to \xi, \tag{11.46b}$$

$$V_{4\mu}(im_0,\ k) \to 1. \tag{11.46c}$$

The resulting expression for the continuum counterterms is then

$$\int \frac{d^2k}{(2\pi)^2}[1 - T_2^{(m_0)}] \cdot \tag{11.47}$$

$$\cdot \left\{ \frac{2\pi}{\ln \dfrac{\xi + 1}{\xi - 1}} \left(1 - \frac{1}{\xi}\right) + \frac{2\pi}{\xi \ln \dfrac{\xi+1}{\xi-1} - 2 + 2\pi\tilde{\varkappa}} \left[\xi - \frac{1}{\xi^2} + \pi\tilde{\varkappa}\left(\frac{1}{\xi^2} - 1\right)\right]\right\}.$$

Let us notice that eq. (11.47) was derived from eq. (11.32) without making any assumption on the dependence of $\varkappa$ on β.

In the light of the results of sect. **5**, eq. (11.47) turns into

$$\frac{m_0^2}{4\pi}[\ln 4\pi\beta + (3 - 2\pi\tilde{\varkappa})\ln(4\pi\beta + 2\pi\tilde{\varkappa} - 2) + c_m(\tilde{\varkappa})] - \tag{11.48}$$

$$- \int_0^{M_L^2} \frac{dk^2}{4\pi}\frac{4\pi}{k^2}\left[\frac{1}{\ln(k^2/m_0^2)} + \frac{3 - 2\pi\tilde{\varkappa}}{\ln(k^2/m_0^2) + 2\pi\tilde{\varkappa} - 2}\right],$$

implying

$$m_1^2 = m_1^{2(\mathrm{SM})} + \delta m_1^2, \tag{11.49}$$

where

$$\delta m_1^2 = 4\pi\,\delta F \tag{11.50}$$

and $m_1^{2(\mathrm{SM})}$ is given by eq. (5.22*a*) (with $\varkappa$ replaced by $\tilde{\varkappa}$).

In conclusion, we can summarize the results contained in eqs. (11.30) and (11.50) into the relationship

$$F^{(\mathrm{scal})} = [N - c_m(\tilde{\varkappa}) + c_F(\tilde{\varkappa})]\frac{m^2}{4\pi} + O\left(\frac{1}{N}\right) + O(m_0^4). \tag{11.51}$$

Equation (11.51) is one of the main results of the present section. It shows that, performing a proper *perturbative* redefinition of the parameter $\varkappa$, it is possible to map the lattice theory in the scaling region into the corresponding continuum theory in such a way that the fundamental scaling relationships are preserved. Equation (11.51) is identical in form to the corresponding continuum relationship (5.29). Let us notice once more that non-perturbative redefinitions of $\varkappa$ would spoil the scaling relationship, because they would change $F^{(\mathrm{scal})}$ without affecting Σ_1 (the partial derivative $\partial\varkappa/\partial m_0^2$ cancels in eq. (11.41) with the corresponding term coming from $\partial\Delta^{-1}_{(\theta)\mu\nu}/\partial m_0^2$).

Recalling the explicit form of the asymptotic expansions presented in sect. **9**, we can now construct the following representation of the $O(1/N)$ contribution to the physical mass gap in the scaling region:

$$(11.52)\qquad \delta m_1^2 = 2\pi m_0^2 \left\{ \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{A_1^{(\alpha)} + \beta B_1^{(\alpha)}}{A_0^{(\alpha)} + \beta B_0^{(\alpha)}} + \int \frac{\mathrm{d}^2 k}{(2\pi)^2} \cdot \right.$$

$$\cdot \left[\sum_\mu (C_{0\mu} + \beta D_{0\mu}) \frac{\hat{k}_\mu^2}{\hat{k}^2} (A_1^{(\theta)} + \beta B_1^{(\theta)}) + \sum_\mu (C_{1\mu} + \beta D_{1\mu}) \frac{\hat{k}_\mu^2}{\hat{k}^2} (A_0^{(\theta)} + \beta B_0^{(\theta)}) + \right.$$

$$\left. + \left[(C_{01} + \beta D_{01})(C_{12} + \beta D_{12}) + (C_{02} + \beta D_{02})(C_{11} + \beta D_{11}) \right] \frac{\tilde{\varkappa}}{C_0 + \beta} \right] \cdot$$

$$\cdot \left[\sum_\mu (C_{0\mu} + \beta D_{0\mu}) \frac{\hat{k}_\mu^2}{\hat{k}^2} (A_0^{(\theta)} + \beta B_0^{(\theta)}) + \frac{\tilde{\varkappa}}{C_0 + \beta} \prod_\mu (C_{0\mu} + \beta D_{0\mu}) \right]^{-1} -$$

$$\left. - \int_0^{M_L^2} \frac{\mathrm{d}k^2}{4\pi} \frac{2}{k^2} \left[\frac{1}{\ln(k^2/m_0^2)} + \frac{3 - 2\pi\tilde{\varkappa}}{\ln(k^2/m_0^2) + 2\pi\tilde{\varkappa} - 2} \right] \right\}.$$

The integral representation is improper, since the denominators vanish on certain curves; the generalized principal-part prescription giving a precise meaning to eq. (11.52) will be discussed in subsect. **11·7** (*cf.* ref. [6]).

A rather straightforward consequence of eqs. (11.37) and (11.38) is the relationship

$$(11.53)\qquad f(k;\, im_0,\, m_0) \cong f(k;\, 0,\, m_0) - m_0^2 \frac{\partial}{\partial p^2} f(k;\, 0,\, m_0) + O(m_0^4),$$

implying that the lattice contribution to $\Sigma_1(im_0)$ and to $\Sigma_1(0) - m_0^2 \Sigma_1'(0)$ have the same scaling limit. As a consequence we should have obtained the same expression for δm_1^2 if we had computed the lattice counterpart of m_R^2 in the scaling region.

We are now ready to apply our knowledge of δm_1^2 to the problem of finding an explicit expression for any physical quantity with known scaling properties, when we analyse it on the lattice in the scaling regime. Let us assume Q to be a physical quantity with canonical scaling dimension $2\delta_Q$ in mass units (*i.e.* no anomalous dimension), and let us compute its large-N limit and $O(1/N)$ corrections:

$$(11.54)\qquad Q = (m_0^2)^{\delta_Q} \left[q_0 + \frac{1}{N} q_1(\beta) + O\left(\frac{1}{N^2}\right) \right],$$

evaluating the $O(1/N)$ corrections in the SM regularization scheme. Since q_0 is universal and β-independent, we can immediately predict the lattice scaling behaviour of Q to be

$$Q^{(L)} = (m_0^2)^{\delta_Q}\left[q_0 + \frac{1}{N}\,q_1(\beta) + \frac{1}{N}\,\delta_Q q_0\,\frac{\delta m_1^2(\beta)}{m_0^2} + O\left(\frac{1}{N^2}\right)\right] + O((m_0^2)^{\delta_Q+2}). \tag{11.55}$$

Equation (11.55) realizes the promise of evaluating every physical quantity in the scaling regime by means of a single-lattice computation.

11.5. *Lattice contribution to wave-function renormalization.* – The discussion of the wave-function renormalization goes along the same lines. We recall from eqs. (11.39) and (11.40) that

$$\delta Z_1 = -\frac{\partial}{\partial p^2}\,\Sigma_1(p,\, m_0)\Big|_{\substack{p=0\\ m_0=0}}. \tag{11.56}$$

From eq. (11.33) we obtain

$$\begin{aligned} -\delta Z_1 = {} & \int \frac{d^2k}{(2\pi)^2}\,\Delta_{(\alpha)}(k)\,\frac{\partial}{\partial p^2}\,\frac{1}{\overline{p+k}^2}\Big|_{p=0} + \\ & + \int \frac{d^2k}{(2\pi)^2}\sum_{\mu\nu}\Delta_{(\theta)\mu\nu}(k)\,\delta^t_{\mu\nu}(k)\,\frac{\partial}{\partial p^2}\,W(p,\, k)\Big|_{p=0} + \\ & + \int \frac{d^2k}{(2\pi)^2}\sum_{\mu\nu}\Delta_{(\theta)\mu\nu}(k)\,\frac{1}{\bar{k}^2}\left[D_{0\mu}(k)\,\delta_{\mu\nu} - \sum_\rho D_{0\rho}(k)\,\delta^t_{\mu\nu}(k)\right] - \\ & - \varkappa f \int \frac{d^2k}{(2\pi)^2}\sum_{\mu\nu}\Delta_{(\theta)\mu\nu}(k)\,\delta_{\mu\nu}D_{1\mu}(k) + \text{continuum conterterms}, \end{aligned} \tag{11.57}$$

where we made use of the relationships

$$V_{4\mu}(0,\, k) = \frac{V_{3\mu}(\frac{1}{2}k,\, k)}{\hat{k}_\mu} = D_{0\mu}(k), \tag{11.58a}$$

$$\frac{\partial}{\partial p^2}\,V_{4\mu}(p,\, k)\Big|_{p=0} = -D_{1\mu}(k). \tag{11.58b}$$

Let us further notice that

$$-\frac{\partial}{\partial p^2}\,\frac{1}{\overline{p+k}^2}\Big|_{p=0} = \frac{1}{2}B_1^{(\alpha)}(k) + \frac{1}{(\bar{k}^2)^2}, \tag{11.59a}$$

$$-\frac{\partial}{\partial p^2}\,W(p,\, k)\Big|_{p=0} = \frac{1}{2}B_1^{(\theta)}(k) - \frac{1}{(\bar{k}^2)^2}\sum_\mu \hat{k}_\mu^2\,[D_{0\mu}(k)]^2. \tag{11.59b}$$

We are finally ready to write down the complete explicit expression for δZ_1:

$$\text{(11.60)}\qquad \delta Z_1 = \int \frac{d^2k}{(2\pi)^2}\frac{1}{A_0^{(\alpha)}+\beta B_0^{(\alpha)}}\left[\frac{1}{2}B_1^{(\alpha)}+\frac{1}{(\bar{k}^2)^2}\right]+$$
$$+\int\frac{d^2k}{(2\pi)^2}\left[\sum_\mu (C_{0\mu}+\beta D_{0\mu})\frac{\hat{k}_\mu^2}{\hat{k}^2}(A_0^{(\theta)}+\beta B_0^{(\theta)})+\frac{\tilde{\varkappa}}{\beta+C_0}\prod_\mu (C_{0\mu}+\beta D_{0\mu})\right]^{-1}\cdot$$
$$\cdot\left\{\sum_\mu (C_{0\mu}+\beta D_{0\mu})\frac{\hat{k}_\mu^2}{\hat{k}^2}\left[\frac{1}{2}B_1^{(\theta)}-\frac{1}{(\bar{k}^2)^2}\sum_\mu \hat{k}_\mu^2[D_{0\mu}(k)]^2\right]-\right.$$
$$-\frac{1}{\bar{k}^2}\left[(C_{01}+\beta D_{01})D_{02}+(C_{02}+\beta D_{02})D_{01}-\sum_{\mu\nu}(C_{0\mu}+\beta D_{0\mu})\frac{\hat{k}_\mu^2}{\hat{k}^2}D_{0\nu}\right]-$$
$$-\frac{\beta+C_0}{\tilde{\varkappa}}\frac{1}{\bar{k}^2}\sum_\mu D_{0\mu}\frac{\hat{k}_\mu^2}{\hat{k}^2}(A_0^{(\theta)}+\beta B_0^{(\theta)})+\frac{1}{2}(A_0^{(\theta)}+\beta B_0^{(\theta)})\sum_\mu D_{1\mu}\frac{\hat{k}_\mu^2}{\hat{k}^2}+$$
$$\left.+\frac{1}{2}\frac{\tilde{\varkappa}}{\beta+C_0}[(C_{01}+\beta D_{01})D_{12}+(C_{02}+\beta D_{02})D_{11}]\right\}+$$
$$+\int_0^{M_L^2}\frac{dk^2}{4\pi}\left[\frac{2\pi}{\ln(k^2/m_0^2)}-\frac{4\pi}{\ln(k^2/m_0^2)+2\pi\tilde{\varkappa}-2}+\frac{1}{\tilde{\varkappa}}\right]\frac{1}{k^2},$$

and we may verify that

$$\text{(11.61)}\qquad \lim_{\beta\to\infty}\delta Z_1 = 0\,.$$

11·6. *CP^{N-1} and $O(2N)$ models.* – At the borders of the parameter space, CP^{N-1} ($\beta_v=0$) and $O(2N)$ ($\beta_g=0$) models require a separate discussion, because of some subtleties related to the order of the limiting procedures in the lattice formulation.

We know that, because of gauge invariance and confinement,

$$\text{(11.62)}\qquad \lim_{\tilde{\varkappa}\to\infty} m_1^2 = \infty\,.$$

However, the quantity δm_1^2 stays finite, and this is important because of its renormalization group interpretation: it allows us to define a renormalization-group–invariant, $1/N$-expandable scale in lattice CP^{N-1} models. When computing δm_1^2 for CP^{N-1} models, it is necessary to keep in mind the transversality of the vector propagator, due to gauge invariance. As a consequence, one may extract directly from eq. (11.41) the correct expression

$$\text{(11.63)}\qquad \delta m_1^2|_{\beta_v=0} = 2\pi m_0^2\left\{\int\frac{d^2k}{(2\pi)^2}\left[\frac{A_1^{(\alpha)}+\beta B_1^{(\alpha)}}{A_0^{(\alpha)}+\beta B_0^{(\alpha)}}+\frac{A_1^{(\theta)}+\beta B_1^{(\theta)}}{A_0^{(\theta)}+\beta B_0^{(\theta)}}\right]-\right.$$
$$\left.-\int_0^{M_L^2}\frac{dk^2}{4\pi}\frac{2}{k^2}\left[\frac{1}{\ln(k^2/m_0^2)}+\frac{3}{\ln(k^2/m_0^2)-2}\right]\right\}.$$

Equation (11.63) differs from the $\tilde{\varkappa}\to 0$ limit of eq. (11.52):

$$(11.64)\quad \lim_{\tilde{\varkappa}\to 0}\delta m_1^2 = $$

$$= 2\pi m_0^2\left\{\int\frac{d^2k}{(2\pi)^2}\left[\frac{A_1^{(\alpha)}+\beta B_1^{(\alpha)}}{A_0^{(\alpha)}+\beta B_0^{(\alpha)}}+\frac{A_1^{(\theta)}+\beta B_1^{(\theta)}}{A_0^{(\theta)}+\beta B_0^{(\theta)}}+\frac{\sum_\mu (C_{1\mu}+\beta D_{1\mu})\hat{k}_\mu^2}{\sum_\mu (C_{0\mu}+\beta D_{0\mu})\hat{k}_\mu^2}\right]-\right.$$

$$\left.-\int_0^{M_L^2}\frac{dk^2}{4\pi}\frac{2}{k^2}\left[\frac{1}{\ln(k^2/m_0^2)}+\frac{3}{\ln(k^2/m_0^2)-2}\right]\right\}.$$

On the other side, when we consider $O(2N)$ models, we find from eq. (11.41)

$$(11.65a)\quad \delta m_1^2|_{\beta_g=0}=2\pi m_0^2\left\{\int\frac{d^2k}{(2\pi)^2}\frac{A_1^{(\alpha)}+\beta B_1^{(\alpha)}}{A_0^{(\alpha)}+\beta B_0^{(\alpha)}}-\int_0^{M_L^2}\frac{dk^2}{4\pi}\frac{2}{k^2}\left[\frac{1}{\ln(k^2/m_0^2)}-1\right]\right\},$$

and

$$(11.65b)\quad \delta Z_1|_{\beta_g=0}=\int\frac{d^2k}{(2\pi)^2}\frac{1}{A_0^{(\alpha)}+\beta B_0^{(\alpha)}}\left[\frac{1}{2}B_1^{(\alpha)}+\frac{1}{(\bar{k}^2)^2}\right]+\int_0^{M_L^2}\frac{dk^2}{4\pi}\frac{1}{k^2}\frac{2\pi}{\ln(k^2/m_0^2)}.$$

Equations (11.65) ought to be compared with the $\tilde{\varkappa}\to\infty$ limit of eqs. (11.52) and (11.60):

$$(11.66a)\quad \lim_{\tilde{\varkappa}\to\infty}\delta m_1^2=2\pi m_0^2\left\{\int\frac{d^2k}{(2\pi)^2}\frac{A_1^{(\alpha)}+\beta B_1^{(\alpha)}}{A_0^{(\alpha)}+\beta B_0^{(\alpha)}}-\int_0^{M_L^2}\frac{dk^2}{4\pi}\frac{2}{k^2}\left[\frac{1}{\ln(k^2/m_0^2)}-1\right]+\right.$$

$$\left.+\int\frac{d^2k}{(2\pi)^2}\sum_\mu\frac{C_{1\mu}+\beta D_{1\mu}}{C_{0\mu}+\beta D_{0\mu}}\right\}$$

and

$$(11.66b)\quad \lim_{\tilde{\varkappa}\to\infty}\delta Z_1=\int\frac{d^2k}{(2\pi)^2}\frac{1}{A_0^{(\alpha)}+\beta B_0^{(\alpha)}}\left[\frac{1}{2}B_1^{(\alpha)}+\frac{1}{(\bar{k}^2)^2}\right]+$$

$$+\int_0^{M_L^2}\frac{dk^2}{4\pi}\frac{1}{k^2}\frac{2\pi}{\ln(k^2/m_0^2)}+\frac{1}{2}\int\frac{d^2k}{(2\pi)^2}\sum_\mu\frac{D_{1\mu}}{C_{0\mu}+\beta D_{0\mu}}.$$

We may observe that eqs. (11.63) and (11.65) differ from eqs. (11.64) and (11.66), respectively, because of a term depending on $C_{1\mu}$ and $D_{1\mu}$. Therefore this difference would be completely absent in the SM and in all other continuum regularization schemes. On the lattice these contributions are non-vanishing; however, they are perturbative in $1/\beta$, and in particular when $\tilde{\varkappa} \to \infty$ they do not affect the scaling relationship (11.51), where

$$\lim_{\tilde{\varkappa}\to\infty} [c_m(\tilde{\varkappa}) - c_F(\tilde{\varkappa})] = 1 . \tag{11.67}$$

Therefore in the scaling region they are amenable to a perturbative redefinition of the coupling f, *i.e.* to a different regularization scheme of the same theory. We shall come back to this phenomenon in sect. **14**, where we shall explicitly discuss the $\beta \to \infty$ limit of the above expressions.

11'7. *Evaluation of lattice integrals.* – The numerical evaluation of eqs. (11.52) and (11.60) is a fairly non-trivial task. The general structure of δm_1^2 and δZ_1 is that of a difference between an infrared-singular lattice integral and an ultraviolet–cut-off continuum integral, with the same singular infrared behaviour as the lattice integral.

By replacing $\ln(k^2/m_0^2)$ with $4\pi\beta + \ln(k^2/M_L^2)$, it is possible to perform the expansion of δm_1^2 in a power series in the powers of $f = 1/(2\beta)$. The coefficients of this weak-coupling expansion are individually infrared-regular combinations of lattice and continuum integrals. It is easy to generate the numerical values of the coefficients up to high orders, simply by expanding the integrands and integrating term by term. However, the full expressions themselves are not proper integrals, reflecting the fact that the series are not Borel-summable. As a consequence, the approximation by series expansion will fail for sufficiently large values of f.

In practice, we found that numerical predictions from series evaluation become unstable (with respect to different truncations) at the border of the scaling region. We do not mean that perturbative evaluations be intrinsically meaningless; we only stress that extrapolation to the intermediate coupling may be dangerous and need to be carefully tested against instabilities. In particular, disagreement between Monte Carlo results and low orders of perturbation theory cannot be automatically interpreted as absence of scaling and/or failure of field-theoretical predictions.

In any case, in the models we are discussing an explicit way out of these difficulties can be found, since a generalized principal-part prescription allows an unambiguous evaluation and resummation of (the scaling part of) the series (*cf.* ref. [6]). Our prescription is expressed as follows:

1) represent the integrals as sums of individual terms of the form

$$\int \frac{d^2k}{(2\pi)^2} \frac{M(k)}{[\beta + N(k)]^n}; \tag{11.68}$$

2) define complex variable functions

$$C(z) = \int_0^\infty \frac{dz'}{z - z'} \int \frac{d^2k}{(2\pi)^2} M(k)\, \delta(z' + N(k)); \tag{11.69}$$

3) identify

$$\int \frac{\mathrm{d}^2 k}{(2\pi)^2} \frac{M(k)}{[\beta + N(k)]^n} \equiv \frac{(-1)^{n-1}}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}\beta^{n-1}} \lim_{\varepsilon \to 0} \left[\frac{C(\beta + i\varepsilon) + C(\beta - i\varepsilon)}{2} \right]. \tag{11.70}$$

From a numerical point of view, a direct principal-part integration is possible but very unstable, since the line of vanishing denominator of the integrands has to be found numerically, and moreover it shrinks rapidly towards zero in the limit $f \to 0$. The solution is to adopt the method described in ref. [6]: the integral is split into the two regions $k^2 < \rho^2$ and $k^2 > \rho^2$, choosing ρ such that all the singularities are included into the first region. The integral over the second region is perfectly regular and it is easily computed numerically. In the first region, we expand the integrand in a power series in k and integrate analytically term by term, resulting in the exponential-integral and related functions. The typical analytic integration involved in the above-mentioned procedure has the form

$$\fint_0^y \mathrm{d}x \frac{x^n}{(\ln x)^{s+1}} = \frac{(n+1)^s}{s!} y^{n+1} \mathcal{E}_s((n+1)\ln y), \tag{11.71}$$

where

$$\mathcal{E}_s(z) = \exp[-z]\,\mathrm{Ei}(z) - \sum_{t=0}^{s-1} \frac{t!}{z^{t+1}} \sim \sum_{t=s}^{\infty} \frac{t!}{z^{t+1}}, \tag{11.72}$$

and Ei is the standard exponential-integral function. Tuning ρ as a function of f, the n-th order of this expansion approaches the exact result with an error decreasing as fast as $\exp[-4\pi(n+1)\beta]$. The series expansion is shown numerically to approximate extremely well the integrand in the relevant region.

It is worth noticing that the resummation procedure we have sketched turns out to be useful also in the evaluation of the so-called perturbative tails of composite operators, *i.e.* those non-scaling contributions to vacuum expectation values resulting from spin waves that must be explicitly subtracted in order to derive normal-ordered quantum expectation values [6].

We may notice that the ambiguity in $\lim_{z \to \beta} C(z)$ is related to the Borel ambiguity in the resummation of the perturbative series observed by David [80, 81]. However, the coefficients of the asymptotic expansion in powers of m_0^2 of a physical $1/N$-expandable quantity like the mass gap are real functions, whose expansion in powers of the coupling is the standard perturbative series.

12. – Evaluation of physical quantities for nearest-neighbour interactions.

As already discussed at the end of sect. **8**, the task of performing accurate numerical evaluations of the integrals entering the $O(1/N)$ contributions to physical quantities is greatly simplified in the case of nearest-neighbour interactions $c_2 = 0$, where we possess integral representations of the lattice propagators.

Notable simplifications occur in eq. (11.41) in this special case: the exact $O(1/N)$ contribution to the mass is

$$\mu_1^2 = \frac{\mu_0}{\nu_0}\left\{\int \frac{d^2k}{(2\pi)^2} \Delta_{(\alpha)}(k)\left[\frac{1}{\widehat{k+i\mu_0}^2 + m_0^2} + \frac{1}{2}\Delta_{(\alpha)}(0)\frac{d\Delta^{-1}_{(\alpha)}(k)}{dm_0^2}\right] + \right.$$
$$+ \frac{1}{8}\frac{\Delta_{(\alpha)}(0)}{\Delta_\mu^{-1}} + \int \frac{d^2k}{(2\pi)^2}[\varkappa\Delta_\mu^{-1} + \Delta^{-1}_{(\theta)}(k)]^{-1}\cdot$$
$$\left.\cdot\left[\frac{1}{2}(4+m_0^2) - \frac{\widehat{k+2i\mu_0}^2}{\widehat{k+i\mu_0}^2+m_0^2} + \frac{1}{2}\Delta_{(\alpha)}(0)\left(\frac{d\Delta^{-1}_{(\theta)}(k)}{dm_0^2} + \frac{\varkappa}{4}\right)\right]\right\}, \tag{12.1}$$

where

$$\Delta_\mu^{-1} = 1 - \frac{1}{4\beta} + \frac{m_0^2}{4}, \tag{12.2a}$$

$$\varkappa = \tilde{\varkappa}\,\frac{4\beta}{4\beta - 1}, \tag{12.2b}$$

and $\Delta_{(\alpha)}$, $\Delta_{(\theta)}$ can be computed from eqs. (8.19).

Equation (12.1) can be tested, for sufficiently small β, against existing strong-coupling results. Strong-coupling expansions for the mass gap in $O(N)$ and CP^{N-1} models were pioneered in refs. [131, 132, 141, 142], and in particular it was recognized that, in the CP^{N-1} case, strong-coupling and large-N limits cannot trivially commute: indeed the $U(1)$ gauge invariance is spontaneously broken at $N=\infty$ (states in the fundamental representation are deconfined free particles); on the other hand, it cannot be broken in strong coupling for any finite value of N. A signal for this phenomenon is the non-existence of a regular strong-coupling expansion of eq. (12.1) at $\varkappa = 0$. In the $O(N)$ case, extended strong-coupling series for the mass gap (up to $O(\beta^{11})$) were obtained by Butera and coworkers in ref. [143] and analysed in the large-N limit in ref. [144]. Results up to $O(\beta^{14})$ for the magnetic susceptibility and related quantities were obtained in refs. [145, 146]. Strong-coupling expansion of $O(N)$ models with improved action was discussed in ref. [147].

Equation (12.1) can be further simplified in the large-$\tilde{\varkappa}$ regime, where one can perform a $1/\tilde{\varkappa}$ expansion of the integral involving the vector field propagator, leading to an expression which can be evaluated exactly to $O(1/\tilde{\varkappa})$. We obtain the result

$$\mu_1^2 = \frac{\mu_0}{\nu_0}\left\{\int \frac{d^2k}{(2\pi)^2} \Delta_{(\alpha)}(k)\left[\frac{1}{\widehat{k+i\mu_0}^2 + m_0^2} + \frac{1}{2}\Delta_{(\alpha)}(0)\frac{d\Delta^{-1}_{(\alpha)}(k)}{dm_0^2}\right] + \right.$$
$$\left.+ \frac{1}{4}\frac{\Delta_{(\alpha)}(0)}{\Delta_\mu^{-1}} + \frac{1}{\tilde{\varkappa}}\frac{\beta^2}{2}\left(1 - \frac{1}{4\beta}\right)\frac{\Delta_{(\alpha)}(0)}{\Delta_\mu^{-1}}\left(\Delta_\mu^{-1} + \frac{1}{\Delta_\mu^{-1}}\right) + O\left(\frac{1}{\tilde{\varkappa}^2}\right)\right\}. \tag{12.3}$$

Equation (12.3) is a rather good approximation of eq. (12.1) even for quite small values of $\tilde{\varkappa}$, as long as we consider not too large values of β. Equation (12.3) should be compared with the exact $O(1/N)$ contribution to the mass of the $O(2N)$ model

$$\mu_1^2 = \frac{\mu_0}{\nu_0}\left\{\int \frac{d^2k}{(2\pi)^2}\Delta_{(\alpha)}(k)\left[\frac{1}{\widehat{k+i\mu_0}^2 + m_0^2} + \frac{1}{2}\Delta_{(\alpha)}(0)\frac{d\Delta_{(\alpha)}^{-1}(k)}{dm_0^2}\right]\right\}. \tag{12.4}$$

When $c_2 = 0$ significant simplifications occur also in the scaling region contributions obtained in eqs. (11.52) and (11.60). The explicit forms of δm_1^2 and δZ_1 are, respectively,

$$\delta m_1^2 = 2\pi m_0^2\left\{\int \frac{d^2k}{(2\pi)^2}\left[\frac{A_1^{(\alpha)} + \beta B_1^{(\alpha)}}{A_0^{(\alpha)} + \beta B_0^{(\alpha)}} + \frac{A_1^{(\theta)} + \beta B_1^{(\theta)} + \beta\tilde{\varkappa}/(4\beta-1)}{A_0^{(\theta)} + \beta B_0^{(\theta)} + \tilde{\varkappa}}\right] + \right.$$
$$\left. + \frac{\beta}{4\beta-1} - \int_0^{32}\frac{dk^2}{4\pi}\frac{2}{k^2}\left[\frac{1}{\ln(k^2/m_0^2)} + \frac{3-2\pi\tilde{\varkappa}}{\ln(k^2/m_0^2) + 2\pi\tilde{\varkappa} - 2}\right]\right\} \tag{12.5}$$

and

$$\delta Z_1 = \int \frac{d^2k}{(2\pi)^2}\frac{\frac{1}{2}B_1^{(\alpha)} + 1/(\hat{k}^2)^2}{A_0^{(\alpha)} + \beta B_0^{(\alpha)}} + \frac{1}{2}\int\frac{d^2k}{(2\pi)^2}\frac{B_1^{(\theta)} + \tilde{\varkappa}/(4\beta-1)}{A_0^{(\theta)} + \beta B_0^{(\theta)} + \tilde{\varkappa}} +$$
$$+ \frac{1}{2}\frac{1}{4\beta-1} + \int_0^{32}\frac{dk^2}{4\pi}\frac{1}{k^2}\left[\frac{2\pi}{\ln(k^2/m_0^2)} - \frac{4\pi}{\ln(k^2/m_0^2) + 2\pi\tilde{\varkappa} - 2}\right]. \tag{12.6}$$

It is easy to compute the first non-trivial orders in the weak-coupling expansion of eqs. (12.5) and (12.6), obtaining

$$\frac{\delta m_1^2}{m_0^2} = \left(\frac{3\pi}{2} - 2\right) + (c_1^{(\alpha)} + c_1^{(\theta)} + \tilde{\varkappa})f + O(f^2), \tag{12.7}$$

and

$$\delta Z_1 = \frac{f}{2\pi}\left[\left(\frac{3\pi}{2} - 2\right) + \left(c_1^{(\alpha)} + c_1^{(\theta)} + \tilde{\varkappa} + \frac{1}{\pi}\right)f + O(f^2)\right], \tag{12.8}$$

where

$$c_1^{(\alpha)} = 4\pi\int\frac{d^2k}{(2\pi)^2}\left[\frac{A_1^{(\alpha)}}{B_0^{(\alpha)}} - \frac{A_0^{(\alpha)}}{B_0^{(\alpha)}}\frac{B_1^{(\alpha)}}{B_0^{(\alpha)}} - \frac{1}{2\pi\hat{k}^2}\right], \tag{12.9a}$$

$$c_1^{(\theta)} = 4\pi\int\frac{d^2k}{(2\pi)^2}\left[\frac{A_1^{(\theta)}}{B_0^{(\theta)}} - \frac{A_0^{(\theta)}}{B_0^{(\theta)}}\frac{B_1^{(\theta)}}{B_0^{(\theta)}} + \frac{1}{16} - \frac{3}{2\pi\hat{k}^2}\right]. \tag{12.9b}$$

We can now make use of a number of identities holding for regulated lattice integrals, discussed in Appendix F, to obtain

$$c_1^{(\alpha)} = -\frac{\pi}{8} - \frac{1}{2\pi} + 4\pi G_1^{(\alpha)} \cong 0.0282552\,, \tag{12.10a}$$

$$c_1^{(\theta)} = -\frac{\pi}{8} + \frac{3}{2\pi} + 4\pi G_1^{(\theta)} \cong 1.7120726 \tag{12.10b}$$

(*cf.* eqs. (F.3)).

Equations (12.7) and (12.8) have a straightforward relationship with the three-loop computation of the lattice β- and γ-functions. In particular the $O(2N)$ results

$$\frac{\delta m_1^2}{m_0^2} = \left(\frac{\pi}{2} - 1\right) + c_1^{(\alpha)} f + O(f^2)\,, \tag{12.11a}$$

$$\delta Z_1 = \frac{f}{2\pi}\left[\left(\frac{\pi}{2} - 1\right) + \left(c_1^{(\alpha)} + \frac{1}{2\pi}\right) f + O(f^2)\right], \tag{12.11b}$$

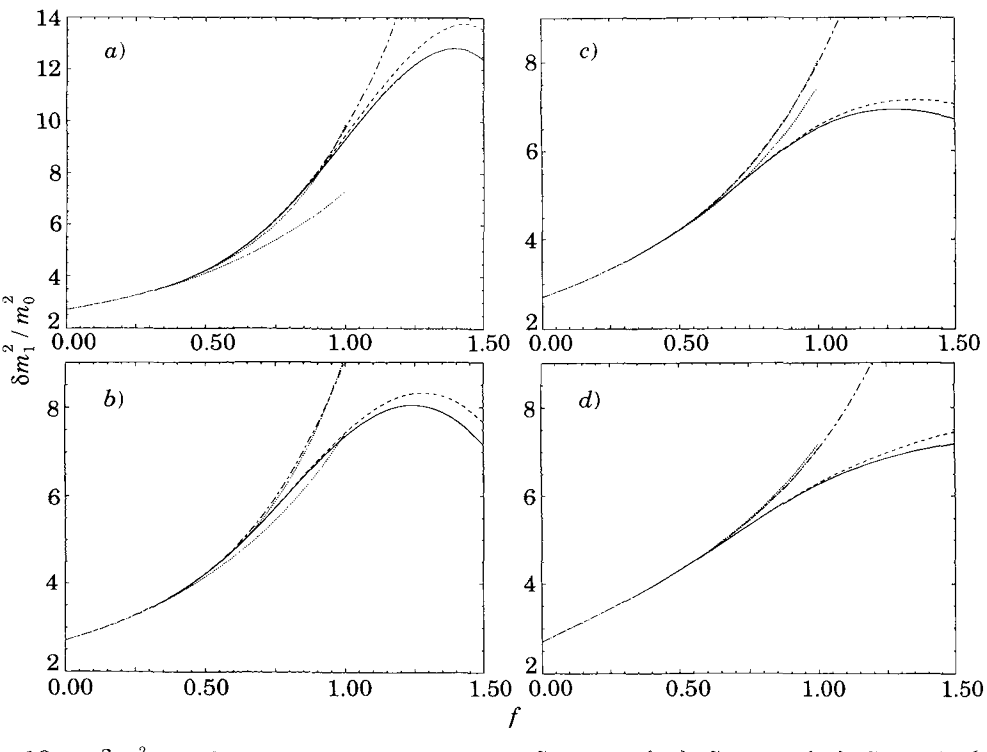


Fig. 18. – δm_1^2 as a function of f, for $c_2 = 0$ and $\tilde{\varkappa} = 0.01$ (a)), $\tilde{\varkappa} = 0.1$ (b)), $\tilde{\varkappa} = 1/\pi$ (c)) and $\tilde{\varkappa} = 1$ (d)). Solid and dashed lines are the results of eq. (12.1) for $\theta = 0$ and $\theta = \pi/4$, respectively; dot-dashed lines are the results of eq. (12.5); dotted lines are the results of a power series expansion of eq. (12.5) to 3rd and 6th order.

reproduce the original calculation by Falcioni and Treves [148], already confirmed in refs. [116, 149].

Higher-order coefficients of the weak-coupling series can be evaluated numerically with high precision, and different truncations of the series can be compared, checking for stability.

We can also compute numerically to high precision the difference between the exact lattice representation (12.1) of μ_1^2 and its SM continuum counterpart $m_{1\,\mathrm{SM}}^2$ as expressed by eq. (5.24), in the region $\beta \lesssim 1.5$.

Finally we may evaluate the representation (12.5) of the scaling contribution to the mass gap δm_1^2, as well as the representation (12.6) of the renormalization function δZ_1. We evaluated the integrals in eqs. (12.5) and (12.6) using the expansion in exponential-integral functions described in sect. **11** and ref. [6].

These different evaluations should all agree with each other in the very weak-coupling domain, where truncated perturbative series are accurate. Moreover the difference between the last two determinations of δm_1^2 is entirely due to scaling violations; therefore it can be compared with independent determinations of the scaling region, such as the study of rotation invariance properties of the mass gap.

All the relevant numerical results are presented in figs. 18 and 19, where δm_1^2 and δZ_1, respectively, are plotted as functions of f for different values of $\tilde{\varkappa}$.

In the case $c_2 = 0$ we can also perform a $1/\tilde{\varkappa}$ expansion of eq. (12.5). The

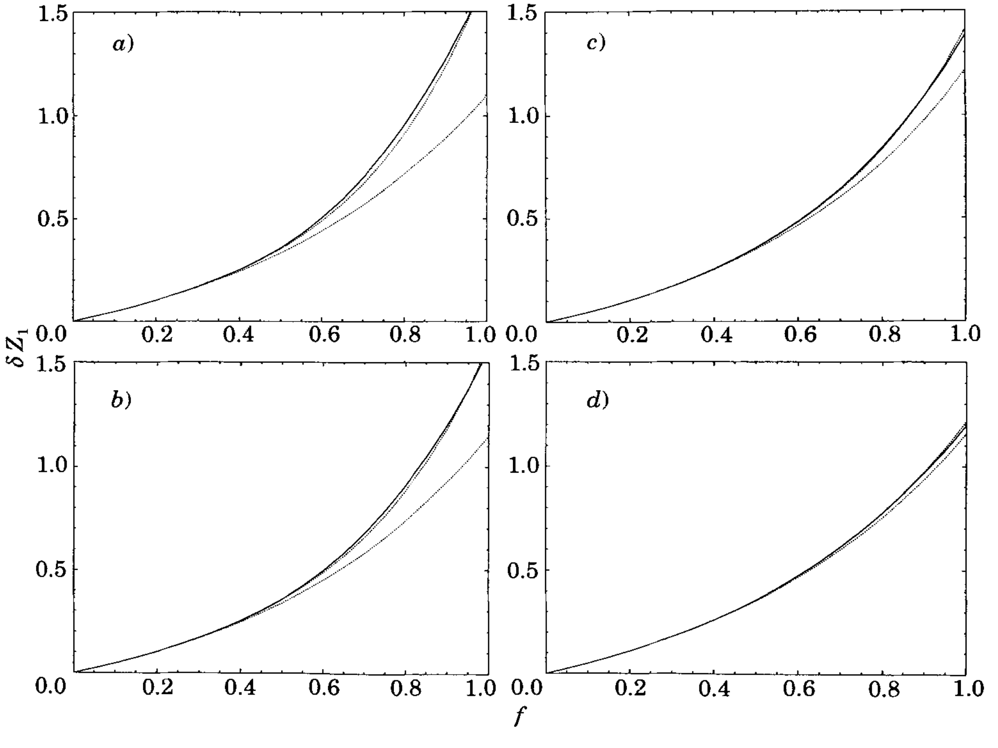


Fig. 19. – δZ_1 as a function of f, for $c_2 = 0$ and $\tilde{\varkappa} = 0.01$ (a)), $\tilde{\varkappa} = 0.1$ (b)), $\tilde{\varkappa} = 1/\pi$ (c)) and $\tilde{\varkappa} = 1$ (d)). Solid lines are the results of eq. (12.6); dotted lines are the results of a power series expansion of eq. (12.6) to 3rd and 6th order.

result is simply

$$(12.12)\qquad \delta m_1^2 = 2\pi m_0^2\left\{\int \frac{d^2k}{(2\pi)^2}\frac{A_1^{(\alpha)}+\beta B_1^{(\alpha)}}{A_0^{(\alpha)}+\beta B_0^{(\alpha)}} - \int_0^{32}\frac{dk^2}{4\pi}\frac{2}{k^2}\left[\frac{1}{\ln(k^2/m_0^2)}-1\right] + \right.$$
$$\left. + \frac{2\beta}{4\beta-1}+\frac{1}{\tilde{\varkappa}}\left[-\frac{\beta}{\pi}+\frac{\beta}{4}\frac{1}{4\beta-1}-\frac{1}{4\pi^2}\right]+O\left(\frac{1}{\tilde{\varkappa}^2}\right)\right\}.$$

It is possible to check directly the consistency of eq. (12.12) with the asymptotic expansion of eq. (12.3). As a byproduct, one may also verify that the asymptotic behaviour of $c_m(\tilde{\varkappa})$ is correctly represented by eq. (5.26). The corresponding result for the $O(2N)$ models is expressed by eq. (11.65a) and is plotted in fig. 20.

In the opposite limit, the result for CP^{N-1} models as expressed by eq. (11.63) is plotted in fig. 21.

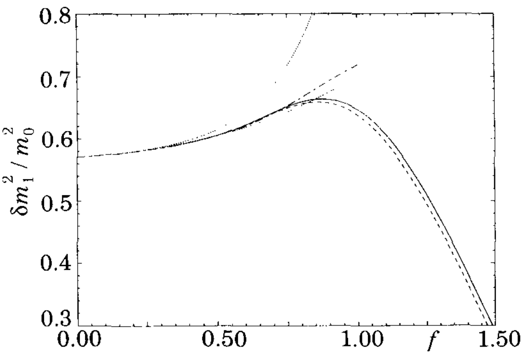


Fig. 20

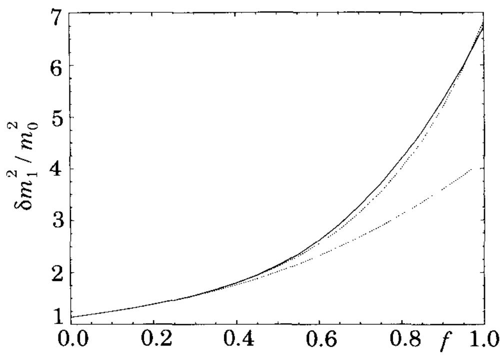


Fig. 21

Fig. 20. – δm_1^2 as a function of f, for the $O(2N)$ model. Solid and dashed lines are the results of eq. (12.4) for $\theta = 0$ and $\theta = \pi/4$, respectively; dot-dashed lines are the results of eq. (11.65a); dotted lines are the results of a power series expansion of eq. (11.65a) to 3rd and 6th order.

Fig. 21. – δm_1^2 as a function of f, for the CP^{N-1} model. The solid line is the result of eq. (11.63); dotted lines are the results of a power series expansion of eq. (11.63) to 3rd and 6th order.

13. – Topological operators.

We already mentioned at the end of sect. **5** the special rôle played by topological properties in the limit $\varkappa = 0$, corresponding to pure CP^{N-1} models and $U(1)$ gauge invariance. The problem of defining a sensible lattice counterpart of the topological charge density (5.66) has long been debated in the literature.

The geometrical definition originally proposed by Berg and Lüscher [150] amounts to defining

$$(13.1)\qquad q_n^g = \frac{1}{2\pi}\,\mathrm{Im}\,\{\ln\mathrm{tr}\,(P_{n+\mu+\nu}P_{n+\mu}P_n) + \ln\mathrm{tr}\,(P_{n+\nu}P_{n+\mu+\nu}P_n)\}\,,\qquad \mu\neq\nu\,,$$

where

$$P_{n,ij} = \bar{z}_{n,i} z_{n,j} . \tag{13.2}$$

q_n^g has the advantage of generating integer values of the topological charge for any given field configuration, and one can prove the absence of a perturbative tail in the $1/N$ expansion [134]. In formulations involving an explicit $U(1)$ gauge field $\lambda_{n,\mu} = \exp[i\theta_{n,\mu}]$, an alternative geometrical definition is obtained by defining

$$q_n = \frac{1}{4\pi} \sum_{\mu\nu} \varepsilon_{\mu\nu} (\theta_{n,\mu} + \theta_{n+\mu,\nu} - \theta_{n+\nu,\mu} - \theta_{n,\nu}) . \tag{13.3}$$

q_n enjoys the same properties of q_n^g and has the same $1/N$ expansion.

Unfortunately for finite N geometrical definitions are plagued by the so-called «dislocations», and therefore one cannot extract the correct scaling behaviour from numerical data [151, 152]. It is, however, possible to express the topological charge in terms of a local operator constructed from the gauge fields. Let us define the plaquette operator (elementary Wilson loop):

$$u_{n,\mu\nu} = \lambda_{n,\mu} \lambda_{n+\mu,\nu} \bar{\lambda}_{n+\nu,\mu} \bar{\lambda}_{n,\nu} , \qquad \mu \neq \nu , \tag{13.4a}$$

$$u_n = u_{n,12} = \bar{u}_{n,21} = \exp[2\pi i q_n] . \tag{13.4b}$$

Taking proper combinations of higher powers of the plaquette operator, it is possible to construct an infinite sequence of local operators

$$2\pi q_n^{(k)} = \sum_{l=1}^{k} \frac{(-1)^{l+1}}{l} \binom{2k}{k-l} \frac{2k}{\binom{2k}{k}} \operatorname{Im}\{(u_n)^l\} , \tag{13.5}$$

whose formal $k \to \infty$ limit is exactly eq. (13.3). One can then construct the sequence of topological susceptibilities

$$\chi_t^{(k)} = \left\langle \sum_n q_n^{(k)} q_0^{(k)} \right\rangle . \tag{13.6}$$

The perturbative evaluation of these quantities is obtained by considering that q_n is linear in the effective Lagrangian field $\theta_{n,\mu}$, and expanding $q_n^{(k)}$ in a power series in q_n. One may easily show that

$$2\pi q_n^{(k)} = 2\pi q_n - \frac{(k!)^2}{(2k+1)!} (2\pi q_n)^{2k+1} + O(q_n^{2k+3}) . \tag{13.7}$$

In standard perturbation theory, using lowest-order momentum space propagators

$$\langle q(p) q(-p) \rangle_0 = \frac{1}{(2\pi)^2} \frac{f}{N} \hat{p}^2 , \tag{13.8}$$

one can prove the relationship

$$(13.9)\quad \langle q^{(k)}(p)\,q^{(k)}(-p)\rangle \cong \langle q(p)\,q(-p)\rangle\left[1-2k!\left(\frac{2f}{N}\right)^{k}+O\left(\left(\frac{f}{N}\right)^{k+1}\right)\right].$$

A more refined analysis, based on the relationship

$$(13.10)\quad \chi_t^{(k)} = \lim_{p^2\to 0}\langle q^{(k)}(p)\,q^{(k)}(-p)\rangle$$

and the observation that

$$(13.11)\quad \lim_{p^2\to 0}\langle q(p)\,q(-p)\rangle_0 = 0\,,$$

allows us to prove that, in the leading order,

$$(13.12)\quad \chi_t^{(k)} \simeq \chi_t + c_k\left(\frac{2f}{N}\right)^{2k+1},$$

where

$$(13.13)\quad c_k \cong \frac{(k!)^4\,(4k+2)!}{((2k+1)!)^3} \sim (2k)!$$

for large k. Equation (13.12) shows that the perturbative tail of the topological susceptibility involves for high k only very high powers of f. The same property might easily be shown to hold also for other mixing coefficients. However, the corresponding numerical weights are growing so fast with k that the convergence to the geometrical definition χ_t cannot be uniform, *i.e.* the limit $k\to\infty$ does not commute with the continuum limit $(f\to 0)$. This phenomenon leads to a perturbative explanation of the observed discrepancy between geometrical and local-operator definitions of the topological susceptibility, and shows that, for fixed f/N, an optimal value of k should exist such that the mixing is minimized.

From the point of view of the $1/N$ expansion, the situation is, however, quite different: for fixed f the perturbative tail is depressed by a factor $(1/N)^{2k}$, and therefore the absence of a perturbative tail of the geometrical definition in the $1/N$ expansion is confirmed; the difference $\chi_t^{(k)} - \chi_t$ is calculable order by order in $1/N$ by generalizing the techniques described in the previous sections, while χ_t itself is simply obtained from the lattice counterpart of eq. (5.67):

$$(13.14)\quad \chi_t = \lim_{p^2\to 0}\frac{1}{(2\pi)^2}\,\hat{p}^2\tilde{\Delta}_{(\theta)}(p)\,,$$

where $\tilde{\Delta}_{(\theta)}(p)$ is the full lattice propagator of the field θ_μ. As a matter of illustration, let us consider the first contribution to the difference $\chi_t^{(1)} - \chi_t$, drawn in fig. 22. A factor of $\hat{p}^2\Delta_{(\theta)}(p)$ is associated with each wavy line; as a consequence, the infrared behaviour is regular.

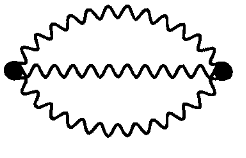
$\frac{1}{6}$ 

Fig. 22. – The leading contribution to the difference $\chi_t^{(1)} - \chi_t$.

14. – Ratio of Λ parameters and renormalization group functions.

It is possible to analyse the results presented in sect. **11**, and especially eq. (11.52), from the point of view of the perturbative renormalization group. Let us focus on the contributions to m_1^2 that depend on the specific lattice model adopted, *i.e.* the quantity δm_1^2 defined by eq. (11.49).

14'1. *Ratio of Λ parameters.* – Let us consider the $\beta \to \infty$ limit of eq. (11.52) and notice that even in this limit a ($\tilde{\varkappa}$-independent) contribution to δm_1^2 survives:

$$\lim_{\beta \to \infty} \delta m_1^2 = 2\pi m_0^2 \int \frac{d^2k}{(2\pi)^2} \left[\frac{B_1^{(\alpha)}}{B_0^{(\alpha)}} + \frac{B_1^{(\theta)}}{B_0^{(\theta)}} + \frac{\sum\limits_\mu D_{1\mu} \hat{k}_\mu^2}{\sum\limits_\mu D_{0\mu} \hat{k}_\mu^2} \right]. \tag{14.1}$$

Because of the non-commutativity of the limits, we must consider separately the borders of the parameter space. Indeed at $\tilde{\varkappa} = 0$, we find

$$\lim_{\beta \to \infty} \delta m_1^2 |_{\beta_v = 0} = 2\pi m_0^2 \int \frac{d^2k}{(2\pi)^2} \left[\frac{B_1^{(\alpha)}}{B_0^{(\alpha)}} + \frac{B_1^{(\theta)}}{B_0^{(\theta)}} \right]. \tag{14.2}$$

When $\tilde{\varkappa} \to \infty$ we must face an even more involved situation, because the limits $\tilde{\varkappa} \to \infty$ and $\beta \to \infty$ do not commute, as one may easily check directly from the standard perturbative expansion. From eq. (11.66*a*) we obtain

$$\lim_{\beta \to \infty} \lim_{\tilde{\varkappa} \to \infty} \delta m_1^2 = 2\pi m_0^2 \int \frac{d^2k}{(2\pi)^2} \left[\frac{B_1^{(\alpha)}}{B_0^{(\alpha)}} + \sum_\mu \frac{D_{1\mu}}{D_{0\mu}} + \frac{2}{\hat{k}^2} \right]. \tag{14.3}$$

Finally when $\tilde{\varkappa} = \infty$ we obtain from eq. (11.65*a*)

$$\lim_{\beta \to \infty} \delta m_1^2 |_{\beta_g = 0} = 2\pi m_0^2 \int \frac{d^2k}{(2\pi)^2} \left[\frac{B_1^{(\alpha)}}{B_0^{(\alpha)}} + \frac{2}{\hat{k}^2} \right]. \tag{14.4}$$

We observe that the difference between different regularizations of the same physical models is amenable in this limit to the condition $D_{1\mu} \neq 0$.

Let us now come to the physical interpretation of eqs. (14.1), (14.2), (14.3) and (14.4). These quantities are obviously related to the ratio of the so-called lattice Λ parameter to the continuum (SM) Λ-parameter in the models at hand [153, 154]. To be more precise, and recalling eqs. (9.19) and (9.20), we may write the relationship

$$\frac{\Lambda_{\rm SM}}{\Lambda_{\rm L}} \approx M_{\rm L}\left(1 + \frac{1}{2N} \lim_{\beta\to\infty} \frac{\delta m_1^2}{m_0^2}\right). \tag{14.5}$$

Actually we can do better than eq. (14.5); we can exploit the fact that the ratio of the Λ parameters is essentially a one-loop phenomenon and our knowledge of the first coefficient of the renormalization group β-function (N for all $\tilde{\varkappa} \neq \infty$, $N-1$ when $\tilde{\varkappa} \to \infty$) to exponentiate eq. (14.5) and obtain the exact relationships

$$\frac{\Lambda_{\rm SM}}{\Lambda_{\rm L}} = M_{\rm L} \exp\left[\frac{\pi}{N}\int \frac{{\rm d}^2k}{(2\pi)^2}\left[\frac{B_1^{(\alpha)}}{B_0^{(\alpha)}} + \frac{B_1^{(\theta)}}{B_0^{(\theta)}}\right]\right] \tag{14.6}$$

when $\tilde{\varkappa} = 0$ (CP^{N-1} models),

$$\frac{\Lambda_{\rm SM}}{\Lambda_{\rm L}} = M_{\rm L} \exp\left[\frac{\pi}{N}\int \frac{{\rm d}^2k}{(2\pi)^2}\left[\frac{B_1^{(\alpha)}}{B_0^{(\alpha)}} + \frac{B_1^{(\theta)}}{B_0^{(\theta)}} + \frac{\sum_\mu D_{1\mu}\hat{k}_\mu^2}{\sum_\mu D_{0\mu}\hat{k}_\mu^2}\right]\right] \tag{14.7}$$

when $\tilde{\varkappa} \neq 0$ and $\tilde{\varkappa} \neq \infty$, and finally

$$\frac{\Lambda_{\rm SM}}{\Lambda_{\rm L}} = M_{\rm L} \exp\left[\frac{\pi}{N-1}\int \frac{{\rm d}^2k}{(2\pi)^2}\left[\frac{B_1^{(\alpha)}}{B_0^{(\alpha)}} + \frac{2}{\hat{k}^2} + \sum_\mu \frac{D_{1\mu}}{D_{0\mu}}\right]\right] \tag{14.8}$$

when $\tilde{\varkappa} \to \infty$. Equation (14.8) is to be compared with

$$\frac{\Lambda_{\rm SM}}{\Lambda_{\rm L}} = M_{\rm L} \exp\left[\frac{\pi}{N-1}\int \frac{{\rm d}^2k}{(2\pi)^2}\left[\frac{B_1^{(\alpha)}}{B_0^{(\alpha)}} + \frac{2}{\hat{k}^2}\right]\right], \tag{14.9}$$

which we would obtain by setting $\beta_g = 0$ from the very beginning (standard $O(2N)$ models).

We can now obtain explicit representations of the quantities entering eqs. (14.6), (14.7), (14.8) and (14.9). First we notice that

$$\frac{\sum_\mu D_{1\mu}\hat{k}_\mu^2}{\sum_\mu D_{0\mu}\hat{k}_\mu^2} = \frac{1}{\hat{k}^2}\sum_\mu \sin^2\frac{1}{2}k_\mu\left(c_1 + 4c_2\cos^2\frac{1}{2}k_\mu\right), \tag{14.10a}$$

$$\sum_\mu \frac{D_{1\mu}}{D_{0\mu}} = \frac{1}{4}\sum_\mu \frac{c_1 + 4c_2\cos^2\frac{1}{2}k_\mu}{c_1 + c_2\cos^2\frac{1}{2}k_\mu}. \tag{14.10b}$$

Furthermore, using the results presented in sect. **9** we can show that

$$\frac{B_1^{(\alpha)}}{B_0^{(\alpha)}} + \frac{B_1^{(\theta)}}{B_0^{(\theta)}} = \sum_\mu \partial_\mu \partial_\mu \ln \bar{k}^2 - 2\,\frac{c_1 c_2}{\bar{k}^2} \sum_\mu \frac{\sin^4 \frac{1}{2} k_\mu}{c_1 + c_2 \cos^2 \frac{1}{2} k_\mu} + \sum_\mu \frac{D_{1\mu}}{D_{0\mu}}, \tag{14.11a}$$

$$\frac{B_1^{(\alpha)}}{B_0^{(\alpha)}} + \frac{2}{\bar{k}^2} = \frac{1}{2} \sum_\mu \partial_\mu \partial_\mu \ln \bar{k}^2 + \frac{\sum_\mu D_{1\mu} \hat{k}_\mu^2}{\sum_\mu D_{0\mu} \hat{k}_\mu^2}. \tag{14.11b}$$

We notice that terms proportional to $\sum_\mu \partial_\mu \partial_\mu \ln \bar{k}^2$ are total derivatives that can be integrated exactly for any physically acceptable form of $\bar{k}^2$:

$$\int \frac{d^2k}{(2\pi)^2} \sum_\mu \partial_\mu \partial_\mu \ln \bar{k}^2 = -\frac{1}{\pi}. \tag{14.12}$$

These terms are not really lattice artifacts: they are related to the ratio $\Lambda_{\text{SM}}/\Lambda_{\overline{\text{MS}}}$, where $\Lambda_{\overline{\text{MS}}}$ is the Λ parameter defined in the dimensional regularization scheme with minimal subtraction (notice that in dimensional regularization the integral of a total derivative vanishes exactly). Therefore we obtain

$$\frac{\Lambda_{\text{SM}}}{\Lambda_{\overline{\text{MS}}}} = \exp\left[-\frac{1}{N}\right] \qquad (\tilde{\varkappa} \neq \infty), \tag{14.13a}$$

$$\frac{\Lambda_{\text{SM}}}{\Lambda_{\overline{\text{MS}}}} = \exp\left[-\frac{1}{2(N-1)}\right] \qquad (\tilde{\varkappa} \to \infty). \tag{14.13b}$$

Equations (14.13) are crucial in finding the variable change from the SM to the $\overline{\text{MS}}$ scheme and verifying the perturbative consistency of the continuum results. The ratio of Λ parameters can now be expressed in the more conventional form

$$\frac{\Lambda_{\overline{\text{MS}}}}{\Lambda_{\text{L}}} = M_{\text{L}} \exp\left[\frac{\pi}{N} \int \frac{d^2k}{(2\pi)^2} \left[-\frac{2c_1 c_2}{\bar{k}^2} \sum_\mu \frac{\sin^4 \frac{1}{2} k_\mu}{c_1 + c_2 \cos^2 \frac{1}{2} k_\mu} + \sum_\mu \frac{D_{1\mu}}{D_{0\mu}}\right]\right] \tag{14.14}$$

when $\tilde{\varkappa} = 0$,

$$\frac{\Lambda_{\overline{\text{MS}}}}{\Lambda_{\text{L}}} = M_{\text{L}} \exp\left[\frac{\pi}{N} \int \frac{d^2k}{(2\pi)^2} \left[-\frac{2c_1 c_2}{\bar{k}^2} \sum_\mu \frac{\sin^4 \frac{1}{2} k_\mu}{c_1 + c_2 \cos^2 \frac{1}{2} k_\mu} + \sum_\mu \frac{D_{1\mu}}{D_{0\mu}} + \frac{\sum_\mu D_{1\mu} \hat{k}_\mu^2}{\sum_\mu D_{0\mu} \hat{k}_\mu^2}\right]\right] \tag{14.15}$$

when $\tilde{\varkappa} \neq 0$ and $\tilde{\varkappa} \neq \infty$,

$$(14.16) \qquad \frac{\Lambda_{\overline{\rm MS}}}{\Lambda_{\rm L}} = M_{\rm L} \exp\left[\frac{\pi}{N-1}\int \frac{d^2k}{(2\pi)^2}\left[\sum_\mu \frac{D_{1\mu}}{D_{0\mu}} + \frac{\sum_\mu D_{1\mu}\hat{k}_\mu^2}{\sum_\mu D_{0\mu}\hat{k}_\mu^2}\right]\right]$$

when $\tilde{\varkappa} \rightarrow \infty$, and

$$(14.17) \qquad \frac{\Lambda_{\overline{\rm MS}}}{\Lambda_{\rm L}} = M_{\rm L} \exp\left[\frac{\pi}{N-1}\int \frac{d^2k}{(2\pi)^2}\frac{\sum_\mu D_{1\mu}\hat{k}_\mu^2}{\sum_\mu D_{0\mu}\hat{k}_\mu^2}\right]$$

in the pure $O(2N)$ case.

When $c_2 = 0$ all integrals can be computed in closed form; we obtain

$$(14.18) \qquad \frac{\Lambda_{\overline{\rm MS}}}{\Lambda_{\rm L}} = \sqrt{32}\exp\left[\frac{\pi}{2N}\right] \qquad (CP^{N-1}),$$

$$(14.19) \qquad \frac{\Lambda_{\overline{\rm MS}}}{\Lambda_{\rm L}} = \sqrt{32}\exp\left[\frac{3\pi}{4N}\right] \qquad (\tilde{\varkappa} \neq 0)$$

$$(14.20) \qquad \frac{\Lambda_{\overline{\rm MS}}}{\Lambda_{\rm L}} = \sqrt{32}\exp\left[\frac{3\pi}{4(N-1)}\right] \qquad (\tilde{\varkappa} \rightarrow \infty),$$

$$(14.21) \qquad \frac{\Lambda_{\overline{\rm MS}}}{\Lambda_{\rm L}} = \sqrt{32}\exp\left[\frac{\pi}{4(N-1)}\right] \qquad (O(2N)).$$

Equations (14.14), (14.15), (14.16) and (14.17) have been all explicitly verified in standard perturbation theory. In particular, the difference between eq. (14.14) and eq. (14.15) can be traced to the contribution of the field associated to the phase of the last component of the field z_N, which cannot be eliminated by a gauge transformation when $\tilde{\varkappa} \neq 0$. In turn the difference between eq. (14.16) and eq. (14.17) is originated by the unsuppressed contribution of the tadpole graphs involving closed loops of vector propagators, shown in fig. 23. Trivial power counting arguments show that no inverse powers of $1 + \varkappa f$ appear in the perturbative evaluation of these diagrams.

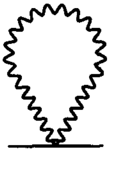


Fig. 23. – The tadpole graph contributing to the difference between eq. (14.16) and eq. (14.17).

The agreement between perturbative and $1/N$ evaluation of the ratios of Λ parameters is a strong confirmation of the commutativity of the $1/\beta$ and $1/N$ expansion with respect to renormalization group properties of the models. Even the apparent singularity of the $\tilde{\varkappa} \to 0$ and $\tilde{\varkappa} \to \infty$ limits has no consequences on the exchange of the $\beta \to \infty$ and $N \to \infty$ limits. Further confirmations can be obtained by the comparison of the perturbative β-function coefficients with those obtained by expanding the resummed $1/N$ lattice β-function, which can be easily obtained from δm_1^2.

14'2. *Lattice β- and γ-functions.* – Let us come to the evaluation of the $O(1/N)$ lattice renormalization group β-function. We may apply the homogeneous renormalization group equations to the expression of the mass gap

$$\left[M \frac{\partial}{\partial M} + \beta(f) \frac{\partial}{\partial f}\right] m^2(M, f) = 0 . \tag{14.22}$$

We can expand $\beta(f)$ in powers of $1/N$ in the form

$$\beta(f) = \beta_0(f) + \frac{1}{N} \beta_1(f) + O\left(\frac{1}{N^2}\right). \tag{14.23}$$

Our choice of lattice action allows us to use the relationship

$$m_0^2 = M^2 \exp\left[-\frac{2\pi}{f}\right] \tag{14.24}$$

in eq. (14.22) to obtain

$$\beta_0(f) = -\frac{f^2}{\pi}. \tag{14.25}$$

Further substitutions in eq. (14.22) lead to

$$\beta_1(f) = [\beta_0(f)]^2 \frac{1}{2} \frac{\partial}{\partial f} \frac{m_1^2}{m_0^2} \tag{14.26}$$

and, since we know β_1 in the SM scheme, we immediately obtain

$$\beta_1^{(\mathrm{L})}(f) = \beta_1^{(\mathrm{SM})}(f) + \frac{f^2}{\pi} \frac{f^2}{2\pi} \frac{\partial}{\partial f}\left(\frac{\delta m_1^2}{m_0^2}\right). \tag{14.27}$$

Equation (14.27) admits a natural interpretation. We must recognize that a change in the regularization scheme corresponds to a reparametrization of the model, *i.e.* $f' = f'(f)$. Convariance of the renormalization group equations under reparametrization implies

$$\beta'(f') = \beta(f(f')) \left(\frac{\partial f}{\partial f'}\right)^{-1}. \tag{14.28}$$

As a consequence eq. (14.27) implies

$$f^{(\mathrm{SM})} = f - \frac{1}{N}\,\beta_0(f)\frac{1}{2}\frac{\delta m_1^2}{m_0^2} + O\left(\frac{1}{N^2}\right). \tag{14.29}$$

The improper integral obtained by substituting eq. (11.52) into eq. (14.27) is defined according to the prescription (11.70). It is worth noticing that all the residues in the complex integration vanish, in contrast with eq. (11.52) itself. Taking the derivative with respect to $f \equiv 1/(2\beta)$ in eq. (11.52) is completely straightforward and we shall not write down the result in the most general case. We shall, however, consider a few interesting special cases.

For $\tilde{\varkappa} = 0$ (CP^{N-1} models) we have

$$\beta(f) = -\frac{f^2}{\pi}\left\{1 + \frac{1}{N}\frac{f}{2\pi}\left(1 + \frac{3}{1 - f/\pi}\right) - \frac{1}{N}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{2}\frac{A_1^{(\alpha)}B_0^{(\alpha)} - B_1^{(\alpha)}A_0^{(\alpha)}}{(A_0^{(\alpha)} + \beta B_0^{(\alpha)})^2} - \right.$$
$$- \frac{1}{N}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{2}\frac{A_1^{(\theta)}B_0^{(\theta)} - B_1^{(\theta)}A_0^{(\theta)}}{(A_0^{(\theta)} + \beta B_0^{(\theta)})^2} +$$
$$\left. + \frac{1}{N}\int^{M_{\mathrm{L}}^2}\frac{\mathrm{d}k^2}{4\pi}\frac{4\pi}{k^2}\left(\frac{1}{\ln^2(k^2/m_0^2)} + \frac{3}{(\ln(k^2/m_0^2) - 2)^2}\right)\right\}. \tag{14.30}$$

For $\tilde{\varkappa} \neq 0$ and $c_2 = 0$ we have

$$\beta(f) = -\frac{f^2}{\pi}\left\{1 + \frac{1}{N}\frac{f}{2\pi}\left[1 + \frac{3 - 2\pi\tilde{\varkappa}}{1 + f(\tilde{\varkappa} - 1/\pi)}\right] - \right.$$
$$- \frac{1}{N}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{2}\frac{A_1^{(\alpha)}B_0^{(\alpha)} - B_1^{(\alpha)}A_0^{(\alpha)}}{(A_0^{(\alpha)} + \beta B_0^{(\alpha)})^2} - \frac{1}{N}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{2}(A_0^{(\theta)} + \beta B_0^{(\theta)} + \tilde{\varkappa})^{-2}\cdot$$
$$\cdot\left[A_1^{(\theta)}B_0^{(\theta)} - B_1^{(\theta)}A_0^{(\theta)} + \tilde{\varkappa}\left(\frac{A_0^{(\theta)}}{(4\beta - 1)^2} + \frac{4\beta^2 B_0^{(\theta)}}{(4\beta - 1)^2} - B_1^{(\theta)}\right) + \frac{\tilde{\varkappa}^2}{(4\beta - 1)^2}\right] -$$
$$\left. - \frac{1}{2N}\frac{1}{(4\beta - 1)^2} + \frac{1}{N}\int^{32}\frac{\mathrm{d}k^2}{4\pi}\frac{4\pi}{k^2}\left(\frac{1}{\ln^2(k^2/m_0^2)} + \frac{3 - 2\pi\tilde{\varkappa}}{(\ln(k^2/m_0^2) + 2\pi\tilde{\varkappa} - 2)^2}\right)\right\}. \tag{14.31}$$

For $\tilde{\varkappa} \to \infty$ we have

$$\beta(f) = -\frac{f^2}{\pi}\left\{1 - \frac{1}{N} + \frac{1}{N}\frac{f}{2\pi} - \frac{1}{N}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{2}\frac{A_1^{(\alpha)}B_0^{(\alpha)} - B_1^{(\alpha)}A_0^{(\alpha)}}{(A_0^{(\alpha)} + \beta B_0^{(\alpha)})^2} + \right.$$
$$\left. + \frac{1}{N}\int\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{2}\sum_\mu\frac{D_{1\mu}C_{0\mu} - C_{1\mu}D_{0\mu}}{(C_{0\mu} + \beta D_{0\mu})^2} + \frac{1}{N}\int^{M_{\mathrm{L}}^2}\frac{\mathrm{d}k^2}{4\pi}\frac{4\pi}{k^2}\frac{1}{\ln^2(k^2/m_0^2)}\right\}. \tag{14.32}$$

Finally, for $O(2N)$ models we have

$$\beta(f) = -\frac{f^2}{\pi}\left\{1 - \frac{1}{N} + \frac{1}{N}\frac{f}{2\pi} - \frac{1}{N}\int \frac{d^2k}{(2\pi)^2}\frac{1}{2}\frac{A_1^{(\alpha)}B_0^{(\alpha)} - B_1^{(\alpha)}A_0^{(\alpha)}}{(A_0^{(\alpha)} + \beta B_0^{(\alpha)})^2} + \right.$$
$$\left. + \frac{1}{N}\int^{M_L^2} \frac{dk^2}{4\pi}\frac{4\pi}{k^2}\frac{1}{\ln^2(k^2/m_0^2)}\right\}. \tag{14.33}$$

The evaluation of the lattice renormalization group function γ follows essentially the same path. From eq. (5.41) we obtain the relationships

$$\gamma_0(f) = -\frac{\beta_0(f)}{f} = \frac{f}{\pi}, \tag{14.34a}$$

$$\gamma_1(f) = -\beta_0(f)\frac{\partial}{\partial f}Z_1(f) - \frac{\beta_1(f)}{f}. \tag{14.34b}$$

Since we know γ_1 in the SM scheme, we obtain

$$\gamma_1^{(L)}(f) = \gamma_1^{(SM)}(f) - \beta_0(f)\left[\frac{\partial}{\partial f}\delta Z_1 + \frac{\beta_0(f)}{f}\frac{1}{2}\frac{\partial}{\partial f}\frac{\delta m_1^2}{m_0^2}\right] =$$
$$= \gamma_1^{(SM)}(f) - \frac{\partial\gamma_0(f)}{\partial f}\beta_0(f)\frac{1}{2}\frac{\delta m_1^2}{m_0^2} - \beta_0(f)\frac{\partial}{\partial f}\left(\delta Z_1 - \gamma_0(f)\frac{1}{2}\frac{\delta m_1^2}{m_0^2}\right). \tag{14.35}$$

Equation (14.35) in turn is consistent with

$$\gamma^{(L)}(f) = \gamma^{(SM)}(f^{(SM)}(f)) - \beta(f)\frac{\partial}{\partial f}\ln\zeta(f), \tag{14.36}$$

where

$$\zeta(f) = 1 + \frac{1}{N}\left(\delta Z_1 - \gamma_0(f)\frac{1}{2}\frac{\delta m_1^2}{m_0^2}\right) + O\left(\frac{1}{N^2}\right) \tag{14.37}$$

is the additional finite field-amplitude renormalization due to the change of regularization scheme. We may appreciate the fact that $\zeta_1(f) = O(f^2)$, as expected.

15. – Finite-size scaling.

The study of finite-size effects is quite important, both from a purely theoretical point of view and in the context of controlling systematic deviations from the infinite-volume limit in numerical simulations. Finite-size scaling in the large-N

limit has been widely studied for different geometries and space dimensionalities by Brezin and collaborators [155, 156]. A finite-volume approach was applied to the problem of evalutating the low-lying spectrum in two-dimensional spin models by Lüscher [157] and extended by Floratos and coworkers [156-160]. The systematic analysis of $1/N$ finite-size effects is, however, rather recent [161]. Let us review the main results of this analysis.

Any coordinate-independent physical quantity Q defined in the context of the $1/N$-expandable finite-lattice model will in general depend on four different parameters:

$$Q = Q(f,\ a,\ L,\ N)\,, \tag{15.1}$$

where L^d is the physical volume in d dimensions and $a \sim 1/M_L$ is the lattice spacing. In the infinite-volume limit and in the scaling region (according to the discussion presented in sect. **14**) all separate dependence on f and a can be made disappear by parametrizing everything in terms of the physical mass gap $m^2(a, f, N)$, solution of eq. (14.22). The finite-size-scaling relation stems from the observation that one can reach the infinite-volume limit ($L/a \to \infty$), while simultaneously keeping a constant finite value of mL. As a consequence

$$\frac{Q(f,\ a,\ L,\ N)}{Q(f,\ a,\ \infty,\ N)} \xrightarrow[mL=\text{const}]{f\to 0} f^{(Q)}(mL,\ N)\,. \tag{15.2}$$

The $1/N$ expandability in turn implies that, assuming

$$Q(f,\ a,\ L,\ N) = Q_0(f,\ a,\ L) + \frac{1}{N} Q_1(f,\ a,\ L) + O\left(\frac{1}{N^2}\right) \tag{15.3}$$

and

$$m(f,\ N) = m_0(f) + \frac{1}{N} m_1(f) + O\left(\frac{1}{N^2}\right), \tag{15.4}$$

we may expand the finite-size functions $f^{(Q)}$ in the form

$$f^{(Q)}(mL,\ N) = f_0^{(Q)}(mL) + \frac{1}{N} f_1^{(Q)}(mL) + O\left(\frac{1}{N^2}\right). \tag{15.5}$$

Substituting eqs. (15.3), (15.4), and (15.5) into eq. (15.2), we obtain

$$f_0^{(Q)}(m_0 L) = \frac{Q_0(f,\ a,\ L)}{Q_0(f,\ a,\ \infty)}, \tag{15.6a}$$

$$\frac{f_1^{(Q)}(m_0 L)}{f_0^{(Q)}(m_0 L)} = \frac{Q_1(f,\ a,\ L)}{Q_0(f,\ a,\ L)} - \frac{Q_1(f,\ a,\ \infty)}{Q_0(f,\ a,\ \infty)} - (m_0 L)\left(\frac{m_1}{m_0}\right)\frac{f_0'^{(Q)}(m_0 L)}{f_0^{(Q)}(m_0 L)}. \tag{15.6b}$$

Equation (15.6) is the most general form of the $1/N$-expanded finite-size-scaling relation. In order to gain further insight, one must consider the specific properties of the quantity under investigation. In any case, a basic tool in the analysis is the knowledge of the finite-size-scaling properties of the finite-size mass parameter m_L defined by the gap equation:

$$\frac{1}{L^2}\sum_q \frac{1}{\bar{q}^2 + m_L^2} = \frac{1}{2f} = \int \frac{d^2 q}{(2\pi)^2}\frac{1}{\bar{q}^2 + m_0^2}, \tag{15.7}$$

where the sum runs over the momentum lattice modes, *i.e.* $q_\mu = 0, 2\pi/L, \ldots, 2\pi(L-1)/L$. In the case $c_2 = 0$ and in the scaling region, defining $z_L = m_L L$ and $z_0 = m_0 L$, we could establish the relationship

$$z_0 = z_c \exp\left[-\frac{1}{2}\omega(z_L)\right], \tag{15.8}$$

where $z_c \cong 4.163948$ and in the region $z_L \leq 2\pi$ the function ω may be defined by

$$\omega(z_L) = \frac{4\pi}{z_L^2} + 4\pi \sum_{n=1}^{\infty} (-1)^n z_L^{2n} d_{n+1}, \tag{15.9a}$$

$$d_n = \frac{1}{(2\pi)^{2n}} \sum_{\substack{n_1, n_2 = -\infty \\ (n_1, n_2) \neq (0,0)}}^{\infty} \frac{1}{(n_1^2 + n_2^2)^n}, \qquad n > 1. \tag{15.9b}$$

The function $z_0(z_L)$ is monotonic and invertible. Therefore all subsequent calculations can be performed making use of the auxiliary variable z_L, which simplifies many computations.

Without giving further technical details, we mention that finite-size-scaling functions were computed in the $1/N$ expansion of masses and magnetic susceptibilities, both in the pure $O(N)$ case [161] and in the pure CP^{N-1} models [162]. In $O(N)$ models the $1/N$ expansion of finite-size functions is an accurate description of finite-size functions is an accurate description of finite-size effects in all possible regimes: small volume ($mL \ll 2\pi$), where the results can also be compared, by asymptotic freedom, with those obtained from finite-volume weak-coupling perturbation theory [163-165]; large volume ($mL \gg 2\pi$) where, due to the existence of a physical mass gap, one expects exponentially fast convergence to the infinite-volume limit [166]; and intermediate volume.

In CP^{N-1} models at intermediate volume and for N not too large ($N < 100$), new phenomena occur: not every physical quantity is expandable in a $1/N$ series; moreover, even if we limit ourselves to $1/N$-expandable objects, we must observe that the scale of finite-size effects is not set by the mass gap, but instead, since we are in the presence of a «weak» confining potential, it depends on the (semi-classical) radius of the bound states; the radius in turn grows like $N^{1/3}$ and therefore does not have an analytic dependence on $1/N$, a result not unexpected in the light of the semi-classical results on the bound-state spectrum presented in Appendix E.

Another subtle point in the study of finite-size effects in CP^{N-1} models is related to the properties of the Abelian Wilson loop on finite lattices. We only mention here that, defining the Polyakov ratio, corresponding to the derivative of the static potential introduced in sect. **5**,

$$\chi_P(R) = \ln \frac{W(R-1,\, L)}{W(R,\, L)}, \tag{15.10}$$

in the infinite-volume limit and in the scaling region one should find the Abelian string tension

$$\chi_P(R) \xrightarrow[R\to\infty]{} \sigma \xrightarrow[N\to\infty]{} \sigma_0 \equiv \frac{6\pi m_0^2}{N}. \tag{15.11}$$

However, on finite lattices, even in the scaling region,

$$\lim_{m_0\to 0} \frac{\chi_P(R,\, L,\, a,\, f,\, N)}{\sigma} = f^{(P)}(N,\, m_0 L,\, R/L) \xrightarrow[N\to\infty]{} f_0^{(P)}(m_0 L,\, R/L), \tag{15.12}$$

and one may show that

$$\lim_{m_0 L\to\infty} f_0^{(P)}(m_0 L,\, R/L) = 1 - \frac{2R}{L}. \tag{15.13}$$

We may define a function measuring the deviations from the infinite (periodic) volume limit:

$$g_0^{(P)} = \frac{f_0^{(P)}(m_0 L,\, R/L)}{1 - 2R/L}; \tag{15.14}$$

$g_0^{(P)}$ in turn can be shown numerically to enjoy a factorization property: for large $m_0 L$ and $m_0 R$

$$g_0^{(P)}(m_0 L,\, R/L) \approx 1 + \phi(m_0 L)\, \psi(R/L), \tag{15.15}$$

which can be understood in terms of an effective Yukawa interaction replacing the Coulomb potential of finite lattices.

If we compare eq. (5.58) with the definition (13.14) of the topological susceptibility, we recognize that the latter quantity is strictly related in CP^{N-1} models to the Abelian string tension, *i.e.*

$$\frac{\sigma}{2\pi^2 \chi_t} \cong 1 - \frac{64\pi^2}{5N^2} \tag{15.16}$$

(*cf.* eq. (5.72)). Therefore, as a side effect of the above analysis, we are led to the (not unexpected) result that χ_t should vanish on any finite lattice. However, since the infinite-volume limit is reached smoothly, it should always be possible to devise an appropriate limiting procedure to extract infinite-volume information from finite-volume, finite-a measurements. Effects of topology in finite volumes were also studied, for different geometries, in ref. [167].

In closing the present section, it is relevant to observe that finite-size scaling in $O(N)$ models has been the subject of studies concerning the three-dimensional case, where a second-order phase transition occurs at a finite value β_c of the coupling. In ref. [168] finite three-dimensional lattices were studied in the context of the $1/N$ expansion. The method for treating near-critical behaviours in three dimensions, originally developed in ref. [169] for the infinite-volume limit, is very reminiscent of the asymptotic expansion techniques employed in the present work.

Another approach to finite-size effects in three-dimensional $O(N)$ models near criticality was developed by Hasenfratz and Leutwyler employing the techniques of chiral perturbation theory [170].

16. – Higher orders of the $1/N$ expansion on the lattice.

We have not seriously addressed the problem of evaluating $O(1/N^2)$ contributions in the scaling region. A finite-β, finite-lattice calculation can certainly be performed with some technical troubles in evaluating accurately two-loop lattice integrals involving dressed propagators.

This approach has been put forward in recent years by Flyvbierg and collaborators [171-173], who explicitly studied the case of $O(N)$ models. They evaluate the two-point function of the $O(N)$ non-linear σ-models up to $O(1/N^2)$, on finite square lattices and for fixed values of N, typically $N = 3$, 4, in order to compare with existing Monte Carlo simulations. From the two-point function they can extract the numerical value of such physical quantities as the mass gap and the magnetic susceptibility. The comparison with high-precision Monte Carlo results allows an estimate of the systematic errors involved in the series truncation to zeroth, first, and second order. These errors appear to be uniform and smaller than the expected magnitude of the neglected terms. Further insight is obtained by the use of Fourier-accelerated numerical evaluation of Feynman diagrams and extrapolations of finite-volume results to infinite volume by phenomenological finite-size scaling [174, 175].

These improved results lead to agreement with Monte Carlo data, within the expected errors, for $N \geq 3$, and give for $N = 3$, 4 extrapolated mass gap – Λ-parameter ratios consistent with the exact continuum results of ref. [112], reported in eq. (4.15).

However a fully analytic approach to higher orders in the $1/N$ expansion would require extracting the scaling contributions along the lines defined in principle in sect. **11**. This extraction in turn would involve a proper treatment of the regularization problem, which may not be straightforward in the presence of higher loops, if we want to stick to our favorite SM scheme: we might run into technical problems similar to those involved in generalizing the BPHZ scheme to massless theories. In any case, based on the proofs of renormalizability of the $1/N$ expansion, we believe there should be no general obstruction to such a calculation.

17. – A different approach: Schwinger-Dyson equations.

The approach to large N based on the effective action and effective Feynman rules is by no means the only way of generating an expansion that is naturally organized in powers of $1/N$. Writing down Schwinger-Dyson equations for $U(N)$ ($O(2N)$) invariant correlation functions, it is possible to recognize that N occurs only polynomially in the coefficients of the equations themselves. It is therefore possible to truncate the (*a priori* infinite) set of Schwinger-Dyson equations by keeping only terms and equations down to a chosen power of N. When the resulting finite set of equations is solved, the solution depends on $1/N$ through all powers, and it is equal to the sum of an infinite subseries of the exact $1/N$ series. It is therefore at least as accurate as the corresponding truncated $1/N$ series; in practice one can get sensibly higher accuracy, as shown in the original papers by Drouffe and Flyvbjerg, explicitly concerned with $O(N)$ models [165, 176, 177].

The derivation of the Schwinger-Dyson equations is essentially straightforward in the generating functional formalism, and we refer to [177] for all details. We just present in fig. 24 the graphical form of the equations involving the fundamental field and the α-field propagators in the $O(2N)$ model, truncated to $O(1/N^2)$. There is an ambiguity in the location of the bare and dressed vertices; we fixed it by the prescription that each α-field propagator should connect a bare and a dressed vertex.

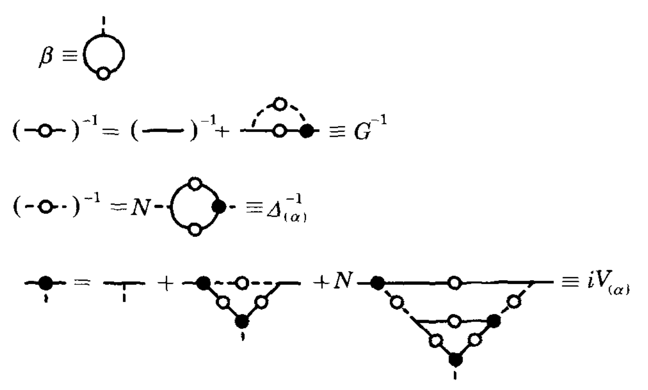


Fig. 24. – The graphical form of the Schwinger-Dyson equations truncated to $O(1/N^2)$. Open circles indicate dressed propagators; full circles indicate dressed vertices.

We tested our prescription by a very simple and illuminating example, the exactly-solvable continuum one-dimensional $O(2N)$ model. Since the $O(1/N)$ truncation amounts to identifying the bare and dressed vertex, we may try to solve the resulting equations by the Ansatz

$$G^{-1}(p) = A(p^2 + m^2). \tag{17.1}$$

Simple integrations (performed for convenience in dimensional regularization) lead to an explicit solution in the form

$$m = \frac{2N - 1}{4N\beta}, \tag{17.2a}$$

$$G^{-1}(p) = \frac{2N}{2N - 1}(p^2 + m^2), \tag{17.2b}$$

$$\Delta^{-1}(p) = \left(\frac{2N - 1}{2N}\right)^2 \frac{N}{m} \frac{1}{p^2 + 4m^2}. \tag{17.2c}$$

Actually eqs. (17.2*a*) and (17.2*b*) correspond to the *exact* solution of the one-dimensional $O(2N)$ model [178], thus showing the power of the approach: albeit truncated to $O(1/N)$, the equations enable us to resum the whole $1/N$ series.

We must now verify that the $O(1/N^2)$ truncation does not spoil the result. However, because of our prescription, this is simply achieved by the Ansatz

$$\Delta^{-1}(p) = \frac{N}{m}\left(\frac{2N - 1}{2N}\right)^2 \frac{1}{p^2 + 4m^2} V(p), \tag{17.3a}$$

$$V_{(\alpha)}(p, p') = V(p - p'), \tag{17.3b}$$

ensuring that the value of $G^{-1}(p)$ is unchanged, while by consistency one can find

$$V(p) = \frac{1}{1 - \dfrac{1}{2N}\left(1 + \dfrac{8m^2}{p^2 + 4m^2}\right)}. \tag{17.4}$$

Equation (17.4) can be verified in the context of the standard $1/N$ expansion.

Although rather promising, the Schwinger-Dyson approach has till now only be applied to the $O(1/N)$ truncation of the two-dimensional $O(N)$ models [177]; by comparing their results with high-precision Monte Carlo data for the $O(3)$ and $O(4)$ models, the authors find an apparent uniform systematic error of $O(1/N^3)$. Higher orders and more general models involve, besides the above-mentioned problem of locating the dressed vertices, technical difficulties related to performing the higher-loop integration with the needed accuracy.

18. – Fermionic models.

The next obvious extension of the $1/N$ approach in the study of model field theories is considering fermionic degrees of freedom. The natural counterparts of the bosonic models we have discussed are theories with four-Fermi interactions and $U(N)$ symmetry, power-counting renormalizable in two dimensions and $1/N$-expandable. These theories are ultraviolet renormalizable order by order in the

$1/N$ expansion in less than four dimensions, as shown by Rosenstein and coworkers [179-181] and by Kikukawa and Yamawaki [182], essentially because their ultraviolet limit has the same relevant operator content as the infrared limit of the superrenormalizable Yukawa model. These arguments were further developed in refs. [183-185], and the whole subject is carefully reviewed in the introduction of ref. [186]. Critical indices were computed by Gracey to $O(1/N^2)$ [187].

A sufficiently general $U(N)$-invariant two-dimensional continuum Euclidean Lagrangian depends on three couplings [188]:

$$\mathscr{L} = \bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}g_s(\bar{\psi}\psi)^2 - \frac{1}{2}g_p(\bar{\psi}\gamma_5\psi)^2 - \frac{1}{2}g_v(\bar{\psi}\gamma_\mu\psi)^2, \tag{18.1}$$

where ψ is an N-plet of Dirac fermions. This Lagrangian interpolates between the $O(2N)$-symmetric Gross-Neveu model ($g_p = g_v = 0$) and the $SU(N)$-symmetric chiral Gross-Neveu model ($g_p = g_v = Ng_v$), enjoying a global axial $U(1)$ invariance.

Gross-Neveu and chiral Gross-Neveu models possess factorized S-matrices in two dimensions, and other exact results can be obtained, in full analogy with their bosonic partners. The related literature has been steadily growing in the last twenty years, and we shall not even try to give references for the continuum formulation, addressing the interested reader to the (partial) bibliography appearing in ref. [189]. However, as already observed for bosonic models, only a minor effort was done in going beyond the leading large-N approximation, both in the continuum and in the lattice versions of the models.

Lattice formulations, as is well known, are plagued by the supplementary problem of fermion doubling. The problem is solved in principle in any dimension by the introduction of the Wilson term [190], which, however, leads to an unavoidable complication in both analytical and numerical computations. In the present context we only want to summarize the available results concerning the large-N limit and the $1/N$ expansion of lattice Gross-Neveu and the chiral Gross-Neveu models, and include a new result coming as a rather natural extension of our previous analysis.

The first lattice formulations of the Gross-Neveu models admitting the correct continuum limit were presented in ref. [191], and involved staggered fermions [192]; therefore they described models with $O(4N)$ symmetry, N being the number of «naïve» fermionic components (*cf.* also ref. [193]). A lattice formulation with Wilson fermions of the chiral Gross-Neveu model was introduced and discussed in the large-N limit in ref. [194] (see also ref. [195]). The Gross-Neveu model in the Wilson formulation was studied to $O(1/N)$ in the seminal paper by David and Hamber [196], where the notion of an asymptotic expansion of the effective propagator was introduced. In refs. [197, 198] the Symanzik improvement program was applied to the large-N Gross-Neveu model with Wilson fermions. A systematic analysis of the $1/N$ contributions to the lattice Gross-Neveu model was finally performed in the staggered version in ref. [199], and in the Wilson-Symanzik version in ref. [200]. For completeness we must mention that a lattice formulation of CP^{N-1} models coupled to fermions was discussed in ref. [201], and that the problem of formulating lattice versions of two-dimensional $1/N$-expandable supersymmetric models was addressed by a few authors [134, 202], but never systematically investigated.

Our new result concerns the possibility of obtaining an integral representation of the effective propagator $\Delta^{-1}_{(\sigma)}$ in a staggered version of the Gross-Neveu model, defined by the action

$$S = \sum_{x,\, y} \bar{\chi}_x D_{xy} \chi_y + \sum_x \bar{\chi}_x \chi_x \Sigma_x + N \sum_x \frac{\sigma_x^2}{2f}, \tag{18.2}$$

where χ_x is a N-plet of (one-component) fermionic fields, D_{xy} is the Susskind «differential» operator

$$D_{xy} = \frac{1}{2}[\delta_{x,\, y+\hat{1}} - \delta_{x,\, y-\hat{1}}] + \frac{1}{2}(-1)^{x_1}[\delta_{x,\, y+\hat{2}} - \delta_{x,\, y-\hat{2}}], \tag{18.3}$$

and the fermions are coupled to the Lagrange-multiplier field σ by

$$\Sigma_x = \frac{1}{4}(\sigma_x + \sigma_{x-\hat{1}} + \sigma_{x-\hat{2}} + \sigma_{x-\hat{1}-\hat{2}}). \tag{18.4}$$

In this case, we can obtain an integral representation of the fermionic integral

$$\begin{aligned} \Delta^{-1}_{(\sigma)} &= \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \operatorname{tr}\left\{\frac{1}{i\gamma_\mu \bar{p} + \frac{1}{2}k_\mu + m_0} \frac{1}{i\gamma_\mu \bar{p} - \frac{1}{2}k_\mu + m_0}\right\} = \\ &= 2\int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{m_0^2 - \sum_\mu \overline{p+\frac{1}{2}k}_\mu\, \overline{p-\frac{1}{2}k}_\mu}{\left(\overline{p+\frac{1}{2}k}^2 + m_0^2\right)\left(\overline{p-\frac{1}{2}k}^2 + m_0^2\right)}, \end{aligned} \tag{18.5}$$

where $\bar{p}_\mu = \sin p_\mu$, and the mass parameter m_0 is momentum-independent. By repeating the arguments of sect. **8**, we find that $\Delta^{-1}_{(\sigma)}$ can be computed in closed form along the principal diagonal of the momentum lattice:

$$\begin{aligned} \Delta^{-1}_{(\sigma)}(l,\, l) = \frac{4}{\pi}\frac{1}{1+m_0^2}\left(\frac{m_0^2 - \cos l}{m_0^2+1} + \frac{1}{\cos l}\right)\Pi\left(\frac{\cos^2 l}{(1+m_0^2)^2}, \frac{1}{1+m_0^2}\right) - \\ - \frac{4}{\pi}\frac{1}{1+m_0^2}\frac{1}{\cos l}K\left(\frac{1}{1+m_0^2}\right). \end{aligned} \tag{18.6}$$

More generally, using the standard Feynman parameter representation we obtain

$$\Delta^{-1}_{(\sigma)} = 2\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \frac{\bar{c} + \sum_\mu \bar{b}_\mu \cos q_\mu}{\left[1 + m_0^2 - \sum_\mu \bar{a}_\mu \cos q_\mu\right]^2}, \tag{18.7}$$

where

$$\bar{a}_\mu = \frac{1}{2}\sqrt{1 - x(1-x)\hat{k}_\mu^2}, \tag{18.8a}$$

$$\bar{b}_\mu = \frac{\cos k_\mu}{4\bar{a}_\mu}, \tag{18.8b}$$

$$\bar{c} = m_0^2 - \frac{1}{2}\sum_\mu \cos k_\mu. \tag{18.8c}$$

By essentially trivial algebraic manipulation we are led to the final result

$$\Delta_{(\sigma)}^{-1}(k) = \frac{m_0^2 + \bar{c}(1 + 2m_0^2)}{4\pi}\int_0^1 dx \frac{1}{(\bar{a}_1\bar{a}_2)^{3/2}} \frac{\bar{\zeta}^3 E(\bar{\zeta})}{1-\bar{\zeta}^2} + \frac{1}{2\pi}\int_0^1 dx \frac{\bar{\zeta}}{(\bar{a}_1\bar{a}_2)^{1/2}} \sum_\mu \frac{\bar{b}_\mu}{\bar{a}_\mu}\left[\frac{E(\bar{\zeta})}{1-\bar{\zeta}^2} + E(\bar{\zeta}) - 2K(\bar{\zeta})\right], \tag{18.9}$$

where

$$\bar{\zeta} = \sqrt{\frac{4\bar{a}_1\bar{a}_2}{(1+m_0^2)^2 - (\bar{a}_1 - \bar{a}_2)^2}}. \tag{18.10}$$

19. – Conclusions and outlook.

In our opinion the most important conclusions that can be drawn from our results are summarized by the following statements.

1) Non-trivial asymptotically-free two-dimensional Euclidean field theories can be constructed, in the context of the $1/N$ expansion, starting from a lattice formulation and exhibiting explicitly the existence of a scaling region. The accuracy of our construction is $O(1/N)$, but there is no obstruction to higher-order extensions.

2) In the scaling region, results that are expressible in terms of adimensional ratios of physical quantities are *universal*, *i.e.* they do not depend on the specific lattice model adopted as long as the physical parameters are kept fixed. Moreover, these results are unaffected by the pathologies of standard perturbation theory and can be unambiguously predicted.

3) The *width* of the scaling region, however, necessarily depends on the choice of a lattice action. In turn, it is widely independent of the $1/N$ corrections, since these depend on the effective propagators and vertices, whose scaling properties are fixed by the (large-N) effective action and are modeled upon the

scaling properties of the large-N lattice mass gap (*cf.* eq. (11.8) and fig. 15). For standard nearest-neighbour interactions, scaling within 10^{-3} is achieved starting from $2f \simeq 1.25$ ($\beta \simeq 0.8$), corresponding (for not too small N) to a correlation length $1/(ma) \simeq 27$.

4) «There ain't nothing like asymptotic scaling» in the real world (excluding very large N). The asymptotic scaling region is in our language the small-f region, where the behaviour of m_1^2/m_0^2 is well approximated by the two-loop perturbative renormalization group, *i.e.* its finite part (in the notation of sect. **5**) is very close to its value at $f = 0$. As an example, for the $O(N)$ models (the «best case») at $c_2 = 0$, with N as large as 20, the mass gap is approximated by the (two-loop) asymptotic formula within 10^{-3} for $\beta \gtrsim 3.3$, *i.e.* $1/(ma) > 10^8$.

5) Perturbation theory may however be a good guide to the physics of the models, in that it commutes order by order with the $1/N$ expansion and it leads to the same renormalization group functions and asymptotic behaviours. Moreover, by summing over a sufficient number of perturbative terms one may reproduce the correct lattice Λ parameter, renormalization constants and perturbative tails throughout the whole scaling region (*cf.* figg. 18-21).

As we have shown, the $1/N$ approach may be successfully extended in many different directions. The major limitation we could not, however, bypass is the restriction to models where the fields belong to the fundamental (vector) representation of the symmetry group. The problem of extension to fields in the adjoint (matrix) representation, like principal chiral models and gauge theories, has been the stumbling block of the $1/N$ expansion in the last decade. A breakthrough in this domain could turn the $1/N$ expansion from a toy in the theoretical playground into a major tool in the analysis of realistic physical models of the fundamental interactions.

* * *

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APPENDIX A.

Perturbative results.

In the literature on perturbative calculations, it is usual to report the results in terms of a rescaled renormalized coupling t, following the notation first adopted in ref. [14]. All four-loop-order β-functions of non-linear σ-models on symmetric spaces can be found in ref. [44]. Here we are only interested in two special cases:

1) $O(N)/O(N-1)$ *spaces*

$$\beta(t) \cong \varepsilon t - (N-2)\,t^2\Bigg[1 + t + \frac{1}{4}(N+2)\,t^2 + \left(-\frac{1}{12}N^2 + \frac{11}{6}N - \frac{17}{6} + \frac{3}{2}(N-3)\,\zeta(3)\right)t^3\Bigg]. \tag{A.1}$$

The result for $O(N)$ models is obtained by setting $t = 1/(2\pi N\beta_v)$.

2) *$U(N)/(U(N-1)\times U(1))$ spaces*

$$\beta(t) \cong \varepsilon t - Nt^2\left[1 + 2t + \left(\frac{3}{2}N + 2\right)t^2 + \left(\frac{1}{3}N^2 + \frac{13}{2}N + 1\right)t^3\right]. \tag{A.2}$$

The result for CP^{N-1} models is obtained by setting $t = 1/(2\pi N\beta_g)$.

The local non-derivative scaling operators can be expressed in terms of orthogonal polynomials in the variable $\sigma^2 = \bar{z}_1 z_1$ [40] (see also [122]). Their anomalous dimensions were computed to four-loop order:

1) *$O(N)$ models*

The scaling operators are the Gegenbauer polynomials $C_l^{(N/2-1)}(\sigma)$ and

$$\gamma_l(t) \cong l(N+l-2)\left\{t + \frac{3}{4}(N-2)t^3 + (N-2)\left[-\frac{1}{3}N + \frac{5}{3} + \frac{1}{2}\zeta(3)\left(1 - \frac{1}{2}l(N+l-2)\right)\right]t^4\right\}. \tag{A.3}$$

2) *CP^{N-1} models*

The scaling operators are the Jacobi polynomials $P_k^{(N-2,0)}(2\sigma^2 - 1)$ and

$$\gamma_k(t) \cong 2k(N+k-1)\left\{t + \frac{3}{2}Nt^3 + (N+6)\left[\frac{1}{3}N + \frac{1}{4}\zeta(3)(N - k(N+k-1))\right]t^4\right\}. \tag{A.4}$$

The ε-expansion of the critical exponents may be extracted from the above results by finding the critical point t^* defined by $\beta(t^*) = 0$ and applying the relationships

$$\eta = -\varepsilon + \gamma_1(t^*), \tag{A.5a}$$

$$\nu = -\frac{1}{\beta'(t^*)}, \tag{A.5b}$$

holding for $O(N)$ models.

APPENDIX B.

Effective propagators in d dimensions.

The mass-gap equation takes the form

$$\beta = \frac{\Gamma(1 - \frac{1}{2}d)}{(4\pi)^{d/2}}(m_0^2)^{d/2-1}. \tag{B.1}$$

The inverse propagator of the α-field is

$$\text{(B.2)}\quad \Delta^{-1}_{(\alpha)}(p) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{q^2 + m_0^2} \frac{1}{(p+q)^2 + m_0^2} =$$

$$= \frac{\Gamma(2-\frac{1}{2}d)}{(4\pi)^{d/2}} \int_0^1 \frac{\mathrm{d}x}{[p^2 x(1-x) + m_0^2]^{2-d/2}} =$$

$$= \frac{\Gamma(2-\frac{1}{2}d)}{(4\pi)^{d/2}} \left(\frac{1}{4}p^2 + m_0^2\right)^{d/2-2} F\left(2-\frac{1}{2}d,\ \frac{1}{2};\ \frac{3}{2};\ \frac{1}{\xi^2}\right),$$

where F is the hypergeometric function. For $p = 0$, the inverse propagator assumes the value

$$\text{(B.3)}\quad \Delta^{-1}_{(\alpha)}(0) = \frac{\Gamma(2-\frac{1}{2}d)}{(4\pi)^{d/2}} (m_0^2)^{d/2-2}.$$

Special values for the lowest integer dimensions are

$$\text{(B.4}a\text{)}\quad d=0:\quad \Delta^{-1}_{(\alpha)}(p) = \frac{8}{(p^2)^2} \frac{1}{\xi^2} \left[\frac{1}{\xi^2 - 1} + \frac{1}{2\xi} \ln \frac{\xi+1}{\xi-1}\right] =$$

$$= \frac{2\beta}{p^2 + 4m_0^2} + \frac{4\xi}{(p^2 + 4m_0^2)^2} \ln \frac{\xi+1}{\xi-1},$$

$$\text{(B.4}b\text{)}\quad d=1:\quad \Delta^{-1}_{(\alpha)}(p) = \frac{1}{m_0} \frac{1}{p^2 + 4m_0^2} = \frac{2\beta}{p^2 + 4m_0^2},$$

$$\text{(B.4}c\text{)}\quad d=2:\quad \Delta^{-1}_{(\alpha)}(p) = \frac{1}{2\pi p^2} \frac{1}{\xi} \ln \frac{\xi+1}{\xi-1} = \frac{1}{p^2 + 4m_0^2} \frac{\xi}{2\pi} \ln \frac{\xi+1}{\xi-1},$$

$$\text{(B.4}d\text{)}\quad d=3:\quad \Delta^{-1}_{(\alpha)}(p) = \frac{1}{4\pi p} \operatorname{arctg} \frac{p}{2m_0},$$

$$\text{(B.4}e\text{)}\quad d=4:\quad \Delta^{-1}_{(\alpha)}(p) = -\frac{1}{(4\pi)^2} \left[\xi \ln \frac{\xi+1}{\xi-1} - 1\right] - \frac{\beta}{m_0^2}.$$

The inverse propagator of the θ-field is

$$\text{(B.5)}\quad \Delta^{-1}_{(\theta)\mu\nu}(p) = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) \Delta^{-1}_{(\theta)}(p),$$

where

$$\text{(B.6)}\qquad \Delta_{(\theta)}^{-1}(p) = \int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{2}{q^2+m_0^2} - \frac{2}{d-1}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{q\cdot(p+q)-m_0^2}{[q^2+m_0^2]\,[(p+q)^2+m_0^2]} =$$

$$= 2\,\frac{\Gamma(1-\frac{1}{2}d)}{(4\pi)^{d/2}}\left[(m_0^2)^{d/2-1} - \int_0^1 \frac{\mathrm{d}x}{[p^2x(1-x)+m_0^2]^{1-d/2}}\right] =$$

$$= 2\,\frac{\Gamma(1-\frac{1}{2}d)}{(4\pi)^{d/2}}\left[(m_0^2)^{d/2-1} - \left(\frac{1}{4}p^2+m_0^2\right)^{d/2-1} F\left(1-\frac{1}{2}d,\ \frac{1}{2};\ \frac{3}{2};\ \frac{1}{\xi^2}\right)\right].$$

For $p=0$, we have

$$\text{(B.7)}\qquad \Delta_{(\theta)}^{-1}(0) = 0.$$

Appendix C.

Continuum integrals.

A number of continuum integrals occurring in the evaluation of the constants appearing in $O(1/N)$ results can be computed analytically. A typical dimensionless SM-regulated one-loop integral in the $1/N$ expansion, after angular integration has been performed, takes the form

$$\text{(C.1)}\qquad I^{\text{reg}} =$$

$$= m_0^{-\Delta-2}\left[\int_0^\infty \frac{\mathrm{d}p^2}{4\pi}F(p^2,\ m_0^2) - \int_0^\infty \frac{\mathrm{d}p^2}{4\pi}\mathrm{T}^{(\mathrm{UV})}F(p^2,\ m_0^2) - \int_{M^2}^\infty \frac{\mathrm{d}p^2}{4\pi}\mathrm{T}^0 F(p^2,\ m_0^2)\right],$$

where $\mathrm{T}^{(\mathrm{UV})}F$ are the terms in the series expansion of F in powers of m_0^2 that are only ultraviolet-divergent («perturbative tails»), while $\mathrm{T}^0 F$ are the terms that are both ultraviolet and infrared divergent; Δ is the canonical dimension of F. $F(p^2,\ m_0^2)$ may be represented in the form

$$\text{(C.2)}\qquad m_0^{\Delta}\hat{F}\left(\frac{p^2}{m_0^2}\right) = m_0^{\Delta} f(\xi)\,,$$

where $\xi = \sqrt{1+4m_0^2/p^2}$.

It is now convenient to perform the following changes of integration variable: in the first integral we set

$$\text{(C.3)}\qquad t = \frac{\xi+1}{\xi-1},\qquad \mathrm{d}p^2 = m_0^2\left(1-\frac{1}{t^2}\right)\mathrm{d}t;$$

in the last two integrals we set

$$u = \frac{p^2}{m_0^2} = \frac{(t-1)^2}{t}, \qquad dp^2 = m_0^2 du. \tag{C.4}$$

By observing that $t = u + 2 + O(1/u)$, we obtain the representation

$$4\pi I^{\text{reg}} = \lim_{\Lambda\to\infty}\left\{\int_1^{\Lambda+2} dt\left(1-\frac{1}{t^2}\right) f\left(\frac{t+1}{t-1}\right) - \int_0^{\Lambda} du T^{(\text{UV})}\hat{F}(u) - \int_{M^2/m_0^2}^{\Lambda} du T^0 \hat{F}(u)\right\}. \tag{C.5}$$

Now, in order to parametrize explicitly the regulator-dependent part of I^{reg} we can split the last integral into the two regions $M^2/m_0^2 < u < \exp[a]$ and $\exp[a] < u$, with $a = 1$ for integrals involving $\Delta_{(\alpha)}^{-1}$ and $a = 3 - 2\pi\varkappa$ for integrals involving $\Delta_{(\theta)}^{-1}$. The integration

$$\int_{M^2/m_0^2}^{\exp[a]} du T^0 \hat{F}(u) \tag{C.6}$$

is now trivial, and we are left with the task of evaluating

$$c \equiv 4\pi I^{\text{fin}} = \lim_{\Lambda\to\infty}\left\{\int_1^{\Lambda+2} dt\left(1-\frac{1}{t^2}\right) f\left(\frac{t+1}{t-1}\right) - \int_0^{\Lambda} du T^{(\text{UV})}\hat{F}(u) - \int_{\exp[a]}^{\Lambda} du T^0 \hat{F}(u)\right\}. \tag{C.7}$$

Exact analytic results have been obtained in the following instances:

$$f(\xi) = \ln\ln\frac{\xi+1}{\xi-1}, \tag{C.8}$$

$$c = \lim_{\Lambda\to\infty}\left\{\int_1^{\Lambda+2} dt\left(1-\frac{1}{t^2}\right)\ln\ln t - \int_0^{\Lambda} du \ln|\ln u| - \int_e^{\Lambda}\frac{du}{u}\frac{2}{\ln u}\right\} =$$

$$= -\int_1^{\infty}\frac{dt}{t^2}\ln\ln t - \int_0^1 du\ln|\ln u| = -2\int_1^{\infty}\frac{dt}{t^2}\ln\ln t =$$

$$= 2\gamma_E;$$

(C.9) $$f(\xi) = \frac{1 - \dfrac{1}{\xi}}{\ln\dfrac{\xi+1}{\xi-1}},$$

$$c = \lim_{\Lambda\to\infty}\left\{\int_1^{\Lambda+2} \mathrm{d}t\left(\frac{t-1}{t^2}\right)\frac{2}{\ln t} - \int_e^{\Lambda}\frac{\mathrm{d}u}{u}\frac{2}{\ln u}\right\} = -2\int_1^{\infty}\frac{\mathrm{d}t}{t^2}\ln\ln t =$$
$$= 2\gamma_E;$$

(C.10) $$f(\xi) = \frac{\dfrac{1}{\xi} - \dfrac{4\xi}{3+\xi^2}}{\ln\dfrac{\xi+1}{\xi-1}},$$

$$c = \lim_{\Lambda\to\infty}\left\{-\int_1^{\Lambda+2}\mathrm{d}t\,\frac{(t-1)^2}{t}\,\frac{3}{\ln t}\,\frac{1}{t^2-t+1} + \int_e^{\Lambda}\frac{\mathrm{d}u}{u}\frac{3}{\ln u}\right\} =$$
$$= -3\int_1^{\infty}\mathrm{d}t\,\frac{1-t^2}{(t^2-t+1)^2}\ln\ln t = -3(\gamma_E - c_1),$$

where c_1 is given by eq. (5.38);

(C.11) $$f(\xi) = \frac{\sqrt{1-\dfrac{1}{\xi^2}}}{2\ln\dfrac{\xi+1}{\xi-1}},$$

$$c = \lim_{\Lambda\to\infty}\left\{\int_1^{\Lambda+2}\mathrm{d}t\,\frac{t-1}{t\sqrt{t}}\,\frac{1}{\ln t} - \fint_0^{\Lambda}\frac{\mathrm{d}u}{\sqrt{u}}\frac{1}{\ln u}\right\} =$$
$$= \lim_{\varepsilon\to\infty}\left\{-\int_{1+\varepsilon}^{\infty}\frac{\mathrm{d}t}{t\sqrt{t}}\frac{1}{\ln t} - \int_0^{1-\varepsilon}\frac{\mathrm{d}u}{\sqrt{u}}\frac{1}{\ln u}\right\} =$$
$$= 0.$$

APPENDIX D.

Effective vertices in the continuum.

The effective vertices of the $1/N$ expansion are nothing but one-loop integrals over the fundamental field propagators with appropriate couplings to the external lines. The problem of evaluating the most general continuum one-loop integral in two dimensions is solved in principle in terms of elementary functions [203-205]. It is however convenient to derive explicit expressions for those special kinematic configurations entering the actual computations we would like to perform [114].

One basic ingredient is the three-point scalar vertex

$$\text{(D.1)} \qquad V_3(p_1, p_2) \equiv \int \frac{d^2 q}{(2\pi)^2} \frac{1}{q^2 + m_0^2} \frac{1}{(q+p_1)^2 + m_0^2} \frac{1}{(q+p_2)^2 + m_0^2},$$

a symmetric function of p_1, p_2 and $p_1 - p_2$. Two-dimensional identities allow a reduction of the integrand to a combination of terms involving only two fundamental field propagators. The integration is then straightforward, and the result is

$$\text{(D.2)} \qquad V_3(p_1, p_2) = D^{-1}(p_1, p_2)\,[p_1^2 (p_2 \cdot (p_2 - p_1)) \Delta_{(\alpha)}^{-1}(p_1) + \\ + p_2^2 (p_1 \cdot (p_1 - p_2))\, \Delta_{(\alpha)}^{-1}(p_2) + (p_1 - p_2)^2 (p_1 \cdot p_2)\, \Delta_{(\alpha)}^{-1}(p_1 - p_2)]\,,$$

where

$$\text{(D.3)} \qquad D(p_1, p_2) = p_1^2\, p_2^2 (p_1 - p_2)^2 + 4 m_0^2 [p_1^2\, p_2^2 - (p_1 \cdot p_2)^2]\,.$$

Let us now consider the four-point vertices: the exceptional configurations we are interested in are the cases when the external momenta are equal two by two. Let us define

$$\text{(D.4)} \qquad V_4^{(a)}(p_1, p_2) \equiv \int \frac{d^2 q}{(2\pi)^2} \frac{1}{[q^2 + m_0^2]^2} \frac{1}{(q+p_1)^2 + m_0^2} \frac{1}{(q+p_2)^2 + m_0^2}.$$

Again by applying algebraic identities we are led to an explicitly integrable expression. The final result is

$$\text{(D.5)} \qquad V_4^{(a)}(p_1, p_2) = D^{-1}(p_1, p_2) \left[\frac{(p_2^2 - (p_1 \cdot p_2))\, p_1^2}{p_1^2 + 4m_0^2} (\Delta_{(\alpha)}^{-1}(p_1) + \Delta_{(\alpha)}^{-1}(0)) + \right. \\ \left. + \frac{(p_1^2 - (p_1 \cdot p_2))\, p_2^2}{p_2^2 + 4m_0^2} (\Delta_{(\alpha)}^{-1}(p_2) + \Delta_{(\alpha)}^{-1}(0)) \right] + \\ + D^{-2}(p_1, p_2) \{ [(p_1 - p_2)^2 (p_1 \cdot p_2) + p_1^2\, p_2^2 - (p_1 \cdot p_2)^2] \cdot \\ \cdot [(p_2^2 - (p_1 \cdot p_2))\, p_1^2 \Delta_{(\alpha)}^{-1}(p_1) + (p_1^2 - (p_1 \cdot p_2))\, p_2^2 \Delta_{(\alpha)}^{-1}(p_2)] + \\ + [(p_1 - p_2)^2 (p_1 \cdot p_2)]^2\, \Delta_{(\alpha)}^{-1}(p_1 - p_2) \} - \\ - D^{-2}(p_1, p_2)\, [p_1^2 p_2^2 - (p_1 \cdot p_2)^2] \{ (p_1^2 + 4m_0^2)((p_1 \cdot p_2) - p_1^2) - \Delta_{(\alpha)}^{-1}(p_1) + \\ + (p_2^2 + 4m_0^2)((p_1 \cdot p_2) - p_2^2)\, \Delta_{(\alpha)}^{-1}(p_2) + \\ + [(p_1 - p_2)^2 + 4m_0^2]\, (p_1 - p_2)^2\, \Delta_{(\alpha)}^{-1}(p_1 - p_2) \}\,.$$

We must also evaluate

$$\text{(D.6)} \qquad V_4^{(b)}(p_1, p_2) \equiv \\ \equiv \int \frac{d^2 q}{(2\pi)^2} \frac{1}{q^2 + m_0^2} \frac{1}{(q+p_1)^2 + m_0^2} \frac{1}{(q+p_2)^2 + m_0^2} \frac{1}{(q+p_1+p_2)^2 + m_0^2}.$$

One can show that the result is expressible in the form

$$V_4^{(b)}(p_1,\ p_2) = \frac{1}{(p_1 \cdot p_2)} [V_3(p_1,\ p_2) - V_3(p_1,\ -p_2)]. \tag{D.7}$$

In order to compute the correlation function of the composite operator $P_{ij}(x)$, we also need the mixed four-point scalar-vector vertices in exceptional momentum configurations. We quote here the definitions:

$$V_{\mu\nu}^{(a)}(p,\ k) = \int \frac{d^2 q}{(2\pi)^2} \frac{1}{(q^2 + m_0^2)^2} \frac{(2q_\mu + k_\mu)(2q_\nu + k_\nu)}{[(q+p)^2 + m_0^2]\,[(q+k)^2 + m_0^2]}, \tag{D.8a}$$

$$V_{\mu\nu}^{(b)}(p,\ k) = \int \frac{d^2 q}{(2\pi)^2} \frac{1}{q^2 + m_0^2} \frac{2q_\mu + k_\mu}{(q+k)^2 + m_0^2} \frac{1}{(q+p)^2 + m_0^2} \frac{2q_\nu + 2p_\nu + k_\nu}{(q+p+k)^2 + m_0^2}. \tag{D.8b}$$

Actually we only need the combination of vertices appearing in fig. 7 and this can be shown to be a transverse tensor. Therefore we can limit ourselves to computing

$$\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) V_{\mu\nu}^{(a)}(p,\ k) = -(k^2 + 4m_0^2)\, V_4^{(a)}(p,\ k) + 2V_3(p,\ k) + \left(1 - \frac{4m_0^2}{k^2}\frac{(p\cdot k)}{p^2}\right)\frac{\Delta_{(\alpha)}^{-1}(0)}{p^2 + 4m_0^2} + \left[(p+k)^2 + 4m_0^2\left(1 + \frac{(p\cdot k)}{p^2}\right)\right]\frac{\Delta_{(\alpha)}^{-1}(p)}{k^2(p^2 + 4m_0^2)} - \frac{1}{k^2}\Delta_{(\alpha)}^{-1}(p-k) \tag{D.9a}$$

and

$$\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) V_{\mu\nu}^{(b)}(p,\ k) = -(k^2 + 2p^2 + 4m_0^2)\, V_4^{(b)}(p,\ k) - \frac{2}{k^2}\Delta_{(\alpha)}^{-1}(p) + \frac{1}{k^2}\Delta_{(\alpha)}^{-1}(p-k) + \frac{1}{k^2}\Delta_{(\alpha)}^{-1}(p+k) + 2V_3(p,\ k) + 2V_3(p,\ -k). \tag{D.9b}$$

APPENDIX E.

The bound-state equation in the large-N limit.

As discussed in sect. **5**, we must solve the following eigenvalue Schrödinger equation:

$$-\frac{d^2\psi}{d\rho^2} + \frac{6\pi}{N} A(x)(1 - \exp[-x\rho])\,\psi = \varepsilon\psi, \tag{E.1}$$

where $\rho = m_0 R$ and $x = m_\theta / m_0$. The resulting bound-state masses will be $m_B = m_0(2 + \varepsilon)$.

It is convenient to introduce new «natural» variables: a rescaled coordinate $y = x\rho$, a «weak» coupling

$$g = \left(\frac{6\pi}{N} xA(x)\right)^{1/3} \xrightarrow[x\to\infty]{} \left(\frac{6\pi}{N}\right)^{1/3}, \tag{E.2}$$

and a rescaled energy eigenvalue $\eta = \varepsilon/g^2$. Equation (E.1) now turns into

$$-\frac{d^2\psi}{dy^2} + t^3(1 - \exp[-y])\psi = \eta t^2 \psi, \tag{E.3}$$

where $t = g/x = gm_0/m_\theta$.

The general (unnormalized) solution of eq. (E.3) is a Bessel function:

$$\psi_\eta(t, y) = J_{2t\sqrt{t-\eta}}(2t^{3/2}\exp[-y/2]), \tag{E.4}$$

and the eigenvalue condition simply amounts to

$$J'_{2t\sqrt{t-\eta}}(2t^{3/2}) = 0 \quad \text{(even-parity levels)}, \tag{E.5a}$$

$$J_{2t\sqrt{t-\eta}}(2t^{3/2}) = 0 \quad \text{(odd-parity levels)}. \tag{E.5b}$$

More specifically, denoting by $j_{\nu,k}$ the k-th zero of $J_\nu(z)$ and by $j'_{\nu,k}$ the k-th zero of $J'_\nu(z)$, the n-th energy level η_n is determined by solving

$$j'_{2t\sqrt{t-\eta_n},\,(n+1)/2} = 2t^{3/2} \quad \text{(odd } n\text{)}, \tag{E.6a}$$

$$j_{2t\sqrt{t-\eta_n},\,n/2} = 2t^{3/2} \quad \text{(even } n\text{)}. \tag{E.6b}$$

The rescaled energies η_n are plotted as functions of t in fig. 25. The energy eigenvalues ε_n themselves and their dependence on the mass parameters can be easily derived from $\eta_n(t)$.

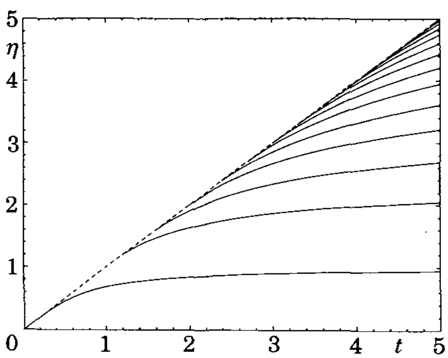

Fig. 25. – The rescaled energy levels η_n as a functions of t (solid lines). The dashed line corresponds to $\eta = t$, where the excited levels disappear.

This kind of analysis is especially suitable for discussing the large-t behaviour, corresponding to the condition

$$m_\theta \ll g m_0 \cong \left(\frac{6\pi}{N}\right)^{1/3} m_0 . \tag{E.7}$$

From the asymptotic formula for $j'_{l,1}$ at large order l [206]

$$j'_{l,1} \sim l\left[1 + \sum_{k=1}^{\infty} \alpha_k l^{-2k/3}\right], \tag{E.8}$$

one obtains an expansion in the form

$$\eta_1 = \sum_{k=0}^{\infty} \gamma_k t^{-k}, \tag{E.9}$$

where $\gamma_0 \cong 1.01879297$. Similar expansions can be derived for higher energy levels. As long as $t \gg 1$ these are good descriptions of the mass spectrum, and in particular when $t \to \infty$ they reproduce known results for the bound states of the CP^{N-1} models. In passing, we notice that the condition (E.7) can be rephrased in the form

$$m_\theta \langle R \rangle_B \ll 1 , \tag{E.10}$$

where $\langle R \rangle_B$ is the semiclassical bound-state radius. Equation (E.10) is an obvious consistency condition for the calculations.

Equations (E.5) indicate that, for sufficiently small values of t, higher bound states may disappear from the spectrum. More precisely, defining t_n by the condition

$$j'_{0,(n+1)/2} = 2t_n^{3/2} \qquad (\text{odd } n), \tag{E.11a}$$

$$j_{0,n/2} = 2t_n^{3/2} \qquad (\text{even } n), \tag{E.11b}$$

the corrisponding state satisfying $\eta_n = t_n$, the n-th level disappears for all $t < t_n$. In particular, when $t < t_2 \cong 1.130756402$ all excited bound states disappear. This phenomenon is illustrated in fig. 25.

We may appreciate that, as long as $m_\theta \sim g m_0$, one may show that the mass of the fundamental particle is well approximated by

$$m_F^2 = m_0^2 + \frac{1}{N}\,\Sigma_1(-m_0^2) \cong m_0^2 + \frac{1}{N}\,\frac{6\pi m_0^3}{m_\theta}, \tag{E.12}$$

implying

$$\frac{m_F}{m_0} = 1 + \frac{1}{2} g^2 t . \tag{E.13}$$

The treshold condition for the n-th bound state in turn has the form

$$\frac{m_{B(n)}}{m_0} = 2 + g^2 \eta_n = 2 + g^2 t_n , \tag{E.14}$$

and therefore it amounts to the condition

(E.15) $$m_{\mathrm{B}(n)}(t_n) = 2m_{\mathrm{F}}(t_n),$$

as one may expect on physical grounds.

Finally, the threshold condition for the first bound state is simply $t = 0$, which corresponds to $g = 0$, and from eqs. (E.2) and (5.63) this implies $m_\theta = 2m_0$, a result known by independent arguments.

APPENDIX F.

Lattice integrals.

We know from general theorems that the perturbative expectation values of quantities invariant under the full symmetry groups of the models must be infrared-finite. Therefore in principle it must be possible to represent finite lattice expectation values in terms of lattice integrals only. However our regularization technique has led to the introduction of continuum counterterms. There must therefore exist (infinitely many) simple identities connecting lattice integrals and their continuum counterparts.

We have found all the identities that might be relevant in a three-loop perturbative computation of lattice renormalization group functions (when $c_2 = 0$):

(F.1a) $$\int \frac{\mathrm{d}^2k}{(2\pi)^2} A_0^{(\alpha)}(k) - \int_0^{32} \frac{\mathrm{d}k^2}{4\pi} \frac{1}{2\pi k^2} \ln\frac{k^2}{32} = 0,$$

(F.1b) $$\int \frac{\mathrm{d}^2k}{(2\pi)^2} \hat{k}^2 A_1^{(\alpha)}(k) + \int_0^{32} \frac{\mathrm{d}k^2}{4\pi} \frac{1}{\pi k^2}\left(\ln\frac{k^2}{32} - 1\right) = -\frac{1}{4\pi^2},$$

(F.1c) $$\int \frac{\mathrm{d}^2k}{(2\pi)^2} \frac{A_0^{(\theta)}(k)}{\hat{k}^2} - \int_0^{32} \frac{\mathrm{d}k^2}{4\pi} \frac{1}{2\pi k^2}\left(\ln\frac{k^2}{32} - 2\right) = -\frac{1}{2\pi^2},$$

(F.1d) $$\int \frac{\mathrm{d}^2k}{(2\pi)^2} A_1^{(\theta)}(k) - \int_0^{32} \frac{\mathrm{d}k^2}{4\pi} \frac{1}{\pi k^2}\left(\ln\frac{k^2}{32} + 1\right) = -\frac{1}{4\pi^2}.$$

At the same order of approximation, a number of intrinsically finite lattice integrals must be computed. When $c_2 = 0$, some integrals can be evaluated analytically:

(F.2a) $$\int \frac{\mathrm{d}^2k}{(2\pi)^2} \hat{k}^2 A_0^{(\alpha)}(k) = -\frac{1}{4},$$

(F.2b) $$\int \frac{\mathrm{d}^2k}{(2\pi)^2} A_0^{(\theta)}(k) = -\frac{3}{4}.$$

However, some computations can only be, to the best of our knowledge, performed numerically [116, 148]:

$$G_1^{(\alpha)} = -\frac{1}{4}\int \frac{d^2k}{(2\pi)^2}\frac{\hat{k}^4}{\hat{k}^2} A_0^{(\alpha)}(k) \cong 0.04616363\,, \tag{F.3a}$$

$$G_1^{(\theta)} = -\frac{1}{4}\int \frac{d^2k}{(2\pi)^2}\frac{\hat{k}^4}{(\hat{k}^2)^2} A_0^{(\theta)}(k) = G_1^{(\alpha)} + \frac{1}{12}. \tag{F.3b}$$

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