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The invariant master field approach to $U(\infty)$ lattice chiral models

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Abstract. A modified strong coupling expansion allows for the introduction of an invariant master field in lattice theories involving unitary matrices. We analyze the corresponding saddle point equations in the case of chiral models and show how to identify the perturbative value of the critical point corresponding to the large N transition. Numerical results for $d=2, 3, 4$ are presented and their relevance beyond the perturbative regime is discussed.

A new approach to the large N limit of lattice gauge models has been recently proposed by Kazakov, Kozhamkulov and Migdal [1] (KKM).

In this formulation, a modified strong coupling expansion allows for the introduction of a gauge invariant master field. The corresponding saddle point equations can be established by expanding in the strong coupling parameter β and can be solved numerically. The solutions obtained in this way give an indication on the nature and location of the large N phase transition.

We extended and improved this method considering the case of $U(N)\times U(N)$ principal chiral models in d dimensions.

These models are a useful laboratory in order to test ideas and methods for matrix models on the lattice because their rich structure is not obscured by the presence of the gauge degrees of freedom, and their continuum properties are non trivial and well established.

In Sect. 1 we describe the model and the method.

In Sect. 2 we discuss the modified strong coupling expansion.

In Sect. 3 we establish the saddle point equations and show how to identify (the perturbative value of) the critical point and how to reduce the computations to the standard strong coupling expansion.

In Sect. 4 we present and discuss our analytical and numerical results, commenting on the reliability and efficiency of the method.

Appendices are devoted to the presentation of technical details.

1 The model and its effective action formulation

We start from the lattice partition function

$$Z_N(\beta)=\int\prod_n dU_n \exp\{N\beta\sum_{n,\mu}\mathrm{tr}(U_n U_{n+\mu}^+)\} \tag{1.1}$$

where U_n are $U(N)$ unitary matrices defined on the sites of a d-dimensional hypercubic lattice, μ are the $2d$ directions on the lattice and dU_n is the Haar measure of integration over unitary groups.

For our purposes we only need to evaluate thermodynamical properties, mainly the free energy per degree of freedom

$$F_N(\beta)=\frac{1}{VN^2}\ln Z_N(\beta)\xrightarrow[N\to\infty]{} F(\beta) \tag{1.2}$$

where V is the lattice volume, and the internal energy per degree of freedom

$$E_N(\beta)=\frac{\partial}{\partial\beta}F_N(\beta)\xrightarrow[N\to\infty]{} E(\beta). \tag{1.3}$$

The extension of the KKM method amounts to the following essential steps:

(i) We replace the unitary measure dU with an integration over complex matrices by the introduction of a Lagrange multiplier matrix α

$$\begin{aligned} dU_n &= d^2U_n\,\delta(U_n^+U_n-1)\\ &= N\frac{d\alpha_n}{2\pi}\exp\{-N\,\mathrm{tr}[\alpha_n(U_n^+U_n-1)]\}\,d^2U_n. \end{aligned} \tag{1.4}$$

(ii) We perform the integration over $d^2 U_n$. For chiral models this can be done exactly (in contrast with gauge models where the action is quartic in U_n) and leaves us with the following representation of the partition function (apart from irrelevant multiplicative constants)

$$Z_N(\beta)=\int\prod_n d\alpha_n \exp[N\sum_n \operatorname{tr}\alpha_n-\operatorname{tr}\ln(\alpha_n\delta_{nm} -\sum_\mu \beta\,\delta_{n,m+\mu})]. \tag{1.5}$$

In the strong coupling (small β) regime the logarithm can be formally expanded, and the effective action can be represented as a sum over closed loops

$$Z_N(\beta)=\int\prod_n d\alpha_n \exp\left\{N\sum_n \operatorname{tr}\left[\alpha_n-\ln\alpha_n + \sum_{k=1}^{\infty}\frac{\beta^k}{k}\sum_{\substack{\mu_1\ldots\mu_k\\ \Sigma\mu_i=0}}\alpha_n^{-1}\alpha_{n+\mu_1}^{-1}\ldots\alpha_{n+\mu_1+\ldots+\mu_{k-1}}^{-1}\right]\right\} \tag{1.6}$$

where the condition $\sum\mu_i=0$ corresponds to the request that the loops be closed.
(iii) We diagonalize the α fields by the equations

$$\alpha_{ij}=\sum_k \Omega_{ik}^{+}\lambda_k\Omega_{kj} \tag{1.7}$$

$$d\alpha=\prod_i d\lambda_i\prod_{i<j}(\lambda_i-\lambda_j)^2\,d\Omega \tag{1.8}$$

where Ω are unitary matrices and λ_k are the eigenvalues of α.
(iv) We perform (at least formally) the integration over the angular variables Ω and express the partition function as an integral over the eigenvalues $\lambda_i^{(n)}$ of α_n

$$\begin{aligned}Z_N(\beta)&=\int\prod_{n,i}d\lambda_i^{(n)}\exp\left\{N^2\sum_n\left[\frac{1}{N}\sum_i(\lambda_i^{(n)}-\ln\lambda_i^{(n)}) +\frac{1}{N^2}\sum_{i\neq j}\ln(\lambda_i^{(n)}-\lambda_j^{(n)})+A_{\text{eff}}^{(n)}(\lambda^{(m)},\beta)\right]\right\}\\ &=\int\prod_{n,i}d\lambda_i^{(n)}\exp\{N^2\sum_n S_{\text{eff}}^{(n)}(\lambda^{(m)},\beta)]\}.\end{aligned} \tag{1.9}$$

(v) In the large N limit, fluctuations of the λ fields are suppressed, and the saddle point value

$$\lambda_i^{(n)}=\lambda_i \tag{1.10}$$

is translation invariant.
We can therefore define the translation invariant function

$$S_{\text{eff}}(\lambda_k,\beta)=\frac{1}{N}\sum_i(\lambda_i-\ln\lambda_i) +\frac{1}{N^2}\sum_{i\neq j}\ln(\lambda_i-\lambda_j)+A_{\text{eff}}(\mu_k,\beta) \tag{1.11}$$

where

$$\mu_k=\frac{1}{N}\sum_i\lambda_i^{-k} \tag{1.12}$$

are the moments of the saddle point matrix α^{-1}.

The saddle point condition is then

$$N\frac{\partial}{\partial\lambda_i}S_{\text{eff}}(\lambda_k,\beta) \equiv 1-\frac{1}{\lambda_i}+\frac{2}{N}\sum_{j\neq i}\frac{1}{\lambda_i-\lambda_j}+N\frac{\partial}{\partial\lambda_i}A_{\text{eff}}(\mu_k,\beta)=0 \tag{1.13}$$

where

$$N\frac{\partial\mu_k}{\partial\lambda_i}=-k\lambda_i^{-k-1}. \tag{1.14}$$

The large N free energy is the value taken by the function $S_{\text{eff}}(\lambda,\beta)$ at the saddle point value of λ.

2 The strong coupling expansion

No approximation was involved in deriving the results presented in Sect. 1. However in practice we cannot determine the exact value of

$$\sum_n A_{\text{eff}}^{(n)}(\lambda_i^{(m)},\beta)\equiv\frac{1}{N^2}\ln\int\prod_n d\Omega_n \cdot\exp\left\{N\sum_n\left[\operatorname{tr}\sum_{k=1}^{\infty}\frac{\beta^k}{k}\sum_{\mu_i}\alpha_n^{-1}\ldots\alpha_{n+\mu_1+\ldots+\mu_{k-1}}^{-1}\right]\right\} \tag{2.1}$$

because the Ω integration is not trivial and cannot be performed in closed form.

As a consequence we must resort to an expansion of the integrand in powers of β.

Taking the logarithm in the rhs ensures us that the lhs can still be represented as a sum over closed connected loops, weighted by a power β^k (where k is the length of the loop) and by a function of μ_i depending on the structure of the loop. One of our tasks was to derive the rules leading to the determination of these weight functions.

The most relevant parameter in the description of a loop is the number of times each lattice site is visited. Whenever a site is visited only once, the result of the corresponding Ω integration is straightforward

$$\int d\Omega(\alpha^{-1})_{ij}=\mu_1\,\delta_{ij}. \tag{2.2}$$

Therefore the contribution of self-avoiding loops to the effective action $A_{\mathrm{eff}}(\lambda, \beta)$ is just a factor $\mu_1^k \beta^k/k$ for every different loop of length k starting from a given point.

Moreover, every self-avoiding section (of length h) of a more general loop can be shrinked to a single link joining its endpoints after extraction of a factor $\mu_1^{h-1} \beta^{h-1}$.

We can therefore focus on loops whose sites are visired more than once. Among these, we identify single-site loops, one-dimensional loops and "potentials".

The weight of single-site loops is obtained trivially as a consequence of the previous analysis and for sites that are visited p times is just

$$\beta^p \mu_p. \tag{2.3}$$

One dimensional loops (that is loops enclosing no area) are obtained as a slight generalization of the previous, and their weight is simply

$$\prod_{p=1}^{\infty} (\mu_p)^{n_p} \beta_p^{\sum p n_p} \tag{2.4}$$

where n_p is the number of sites visited p times.

The real difficulty comes from "potentials" [3], i.e. contributions from two (or more) self-avoiding subloops sharing two or more links. We evaluated the contribution of all two-loop potentials, both for parallel and antiparallel loops.

The connected contribution of a couple of loops sharing n links ($n \geq 3$) is for finite N

$$\frac{1}{2}(N+1)^2 \left(\frac{N\mu_1^2+\mu_2}{N+1}\right)^n + \frac{1}{2}(N-1)^2 \left(\frac{N\mu_1^2-\mu_2}{N-1}\right)^n - N^2 \mu_1^{2n} \tag{2.5}$$

for parallel loops and

$$\frac{2N^2-1}{N^2} \mu_2^n + n\mu_1^2 \mu_2^{n-1} + \left(\frac{N^2-1}{N^2}\right)^2 \left(\frac{N\mu_1^2-\mu_2}{N-1}\right)^n - N^2 \mu_1^{2n} \tag{2.6}$$

for antiparallel loops.

The large N limit of these expressions is respectively

$$\mu_2^n + (\mu_1^2-\mu_2)^3 \sum_{k=0}^{n-3} \frac{(k+1)(k+2)}{2} \mu_1^{2k} \mu_2^{n-k-3} \tag{2.7}$$

and

$$n\mu_2^n + (\mu_1^2-\mu_2)^3 \sum_{k=0}^{n-3} (n-k-2)(k+1) \mu_1^{2k} \mu_2^{n-k-3}. \tag{2.8}$$

In Appendix A we discuss some technical details of the Ω integration presenting a few steps that are essential in order to derive the abovementioned results and their possible extensions.

In Appendix B we exhibit the explicit form of the effective action to 8th order in β and for arbitrary d.

Here we would like to make a few comments about the structure of the effective action $A_{\mathrm{eff}}(\lambda, \beta)$.

It is immediate to check that the substitution $\alpha_n^{-1} = \mu 1$, turning $S_{\mathrm{eff}}(\lambda, \beta)$ into the free energy of the gaussian model expressed as a sum over closed random paths, has the same effect on $A_{\mathrm{eff}}(\lambda, \beta)$, where it corresponds to the choice $\mu_n = (\mu)^n$.

Our results concerning the Ω integration can be easily be shown to be completely consistent with this result; it is especially pleasant to observe that the "potentials" converge to their gaussian value (μ^{2n} and $n\mu^{2n}$ respectively) after the substitutions.

From a computational point of view, this result is a very important and easy check of accuracy for the perturbative evaluation of A_{eff}.

3 The saddle point evaluations and solutions

We have already established the general form of the saddle point equation. Let's just mention that, by the introduction of an eigenvalue density function $\rho(\lambda)$ describing the λ distribution, we can recast it into the form [1]

$$2\int d\lambda' \frac{\rho(\lambda')}{\lambda-\lambda'} = Q\left(\frac{1}{\lambda}\right) \tag{3.1}$$

where

$$Q(x) \equiv \sum_{k=0}^{\infty} Q_k x^k \tag{3.2}$$

and

$$Q_0 = -1 \qquad Q_1 = 1 \qquad Q_{k+1} = k \frac{\partial}{\partial \mu_k} A_{\mathrm{eff}}(\mu_i, \beta). \tag{3.3}$$

In this formalism, the relationship

$$\mu_k = \int \frac{\rho(\lambda')}{(\lambda')^k} d\lambda' \tag{3.4}$$

holds.

Now one might in principle follow KKM and write down a set of $M+2$ equations determining consistently the integration domain and the first M moments μ_k when $A_{\rm eff}$ (and hence Q_{k+1}, $k \leqq M$) has been evaluated to order β^{2M}.

However for our purposes a different approach will prove more convenient.

Our starting point is the crucial observation that the substitution

$$\mu_k = \delta_{k1} \tag{3.5}$$

into $A_{\rm eff}(\mu_k, \beta)$ turns $S_{\rm eff}(\lambda, \beta)$ into the exact strong coupling value of the free energy, to all orders in the $1/N$ expansion.

Otherwise stated, $\mu_k = \delta_{k1}$ is the exact strong coupling master field for the chiral model (and for unitary matrix models in general) in this formulation, as well as for all truncated models obtained from the perturbative expansion of the effective action.

This is just a property of the α representation of the integration over unitary matrices: let's indeed consider the generating functional for $U(N)$ integrals [4–6]

$$\begin{aligned} Z(AA^+) &\equiv \int dU \exp[N \operatorname{tr}(A^+ U + AU^+)] \\ &= \int d\alpha \exp[N \operatorname{tr}\alpha - N \operatorname{tr}\ln\alpha + N \operatorname{tr}(\alpha^{-1} AA^+)] \end{aligned} \tag{3.6}$$

$Z(A, A^+)$ appears to be the also generating functional for the moments of α^{-1}.

From the definitory properties of $Z(A, A^+)$ one can deduce that the expectation values of the moments of α^{-1} are exactly

$$\mu_k = \delta_{k1} \qquad k < N. \tag{3.7}$$

The $U(N)$ integrals can be performed by first integrating over $d^2 U$, $d\Omega$ and then replacing the moments μ_k with their expectation values, which do not depend on the details of the interaction because their evaluation is purely local.

If we substitute (3.5) into the saddle point equation, the consistency requirement leaves us with an equation determining the branch point r where the integration domain starts.

The resulting equation is

$$Q(r) + 2 = 0 \tag{3.8}$$

and we must remember that

$$Q_{k+1} \equiv k \frac{\partial}{\partial \mu_k} A_{\rm eff}(\mu_i = \delta_{i1}, \beta). \tag{3.9}$$

In Appendix C we show explicitly that (3.8) also follows from KKM treatment, and the condition for the existence of a double solution (bifurcation point) can be recast into the form

$$\frac{\partial}{\partial r} Q(r) = 0. \tag{3.10}$$

Therefore, since the solution of (3.8) is some function $r = r(\beta)$, (3.10), after application of the chain rule, corresponds to the condition

$$\frac{dr}{d\beta} \to \infty \qquad \text{for } \beta \to \beta_c. \tag{3.11}$$

The bifurcation condition, when it can be satisfied, is a rigorous definition of the large N critical point for chiral models: one branch is the unphysical continuation of the strong coupling solution, while the other is the weak coupling solution, obviously satisfying (3.8) only at the critical point.

Equation (3.11) does also correspond to our intuition that, in the presence of a transition whose order is greater than one, there is no metastable phase and the criticality coincides with the endpoint of acceptability for the strong coupling solution (turning point for $r(\beta)$).

When we consider the perturbative expansion of the effective action we may still want to apply our criterion for the (perturbative) determination of the critical point. The kind of information thus obtained is of a non perturbative nature, because it leads to a determination of the critical point (better and better with increasing orders of perturbative theory) in a region of values of β detached from the region $\beta \approx 0$ where the expansion is defined.

Establishing (3.8) perturbatively presents enormous semplifications in comparison with the determination of the effective action.

Actually because of the condition $\mu_k = \delta_{1k}$ the only terms that are relevant to this problem are those depending only on μ_1 or at most linearly on μ_k, $k > 1$.

As a simple example, let's consider the evaluation of Q_2: it follows from its definition and (1.3) and (3.5) that

$$Q_2 = \beta E(\beta). \tag{3.12}$$

Equation (3.12) is just a special case of a more general relationship between the coefficients Q_k and a class of correlation functions that can be computed in the standard strong coupling expansion of the model.

One must first recognize that the saddle point equations imply [6]

$$1 + Q(x) = \left\langle \frac{1}{N} \operatorname{tr} \frac{x}{1 - x\alpha} \right\rangle. \tag{3.13}$$

However the classical equations of motions of the model expressed in the variables U_n, α_n allow us to relate the expectation values of moments of α_n to expectation values of functions of U_n. The derivation of the equations of motion is straightforward

$$\beta \sum_\mu U_{n+\mu}^+ = \alpha_n U_n^+ \quad (3.14)$$

$$U_n^+ U_n = 1 \quad (3.15)$$

and implies the classical relationship

$$\alpha_n = \beta \sum_\mu U_{n+\mu}^+ U_n. \quad (3.16)$$

As a consequence, one obtains

$$Q(x) = \left\langle \frac{1}{N} \operatorname{tr} \frac{x}{1 - \beta x \sum_\mu U_{n+\mu}^+ U_n} \right\rangle - 1 \quad (3.17)$$

$$Q_{k+1} = \left\langle \frac{1}{N} \operatorname{tr} (\beta \sum_\mu U_{n+\mu}^+ U_n)^k \right\rangle \qquad k \geqq 1 \quad (3.18)$$

implying in particular (3.12).

Therefore Q_k can be computed directly in the strong coupling expansion of the chiral model.

4 Analytical and numerical results

The main practical difficulty in this approach is related to the fact that not all truncations of the full model exhibit the critical behavior we have outlined; actually one can show that such a behavior is obtained if and only if M is odd and the highest k for which Q_k is nonzero is even.

The typical pattern for even M shows a minimum of $r(\beta)$ for some $\beta \neq 0$

$$\left.\frac{dr}{d\beta}\right|_{\beta=\bar\beta} = 0. \quad (4.1)$$

Therefore for $\beta > \bar\beta$ there is no acceptable value of $r(\beta)$ (the function is no longer monotonic) and $\bar\beta$ is an endpoint for the strong coupling solution. This does not imply it is the critical point of the model (a first order transition might occur for some $\beta < \bar\beta$) and its value can only be used as an upper bound for β_c (see however Appendix D for a discussion of the one dimensional model).

We performed the strong coupling expansion for the coefficients Q_k up to order β^{10} and for arbitrary d. Our results are presented in Appendix E.

An analysis of results shows that, when $M=5$ and $d \geqq 3$ a bifurcation exists and the corresponding numerical value of β_c can be easily determined.

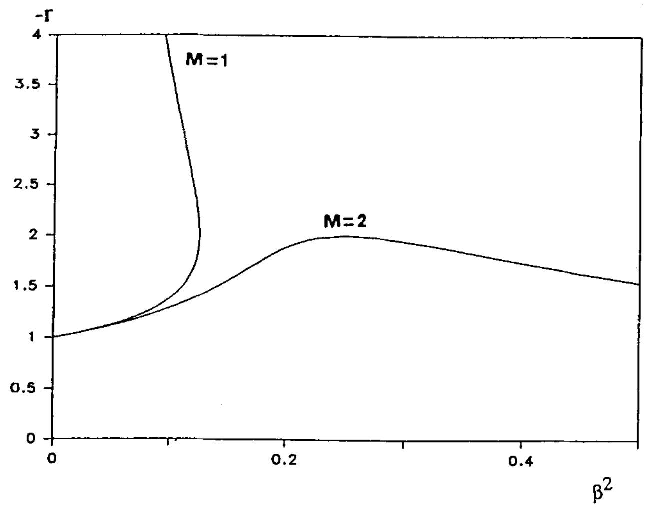

Fig. 1. The function $-r(\beta)$ for $M=1$, 2 and $d=1$

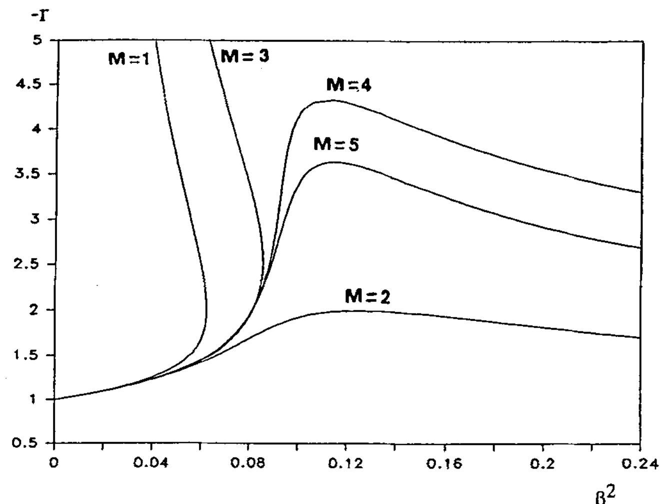

Fig. 2. The function $-r(\beta)$ for $M=1 \div 5$ and $d=2$

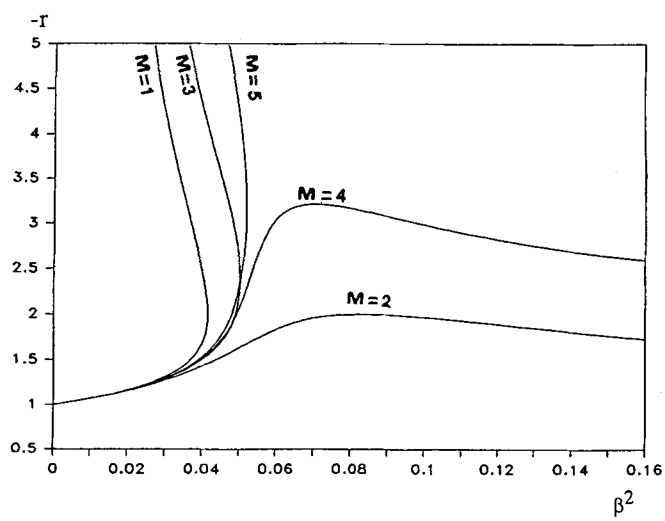

Fig. 3. The function $-r(\beta)$ for $M=1 \div 5$ and $d=3$

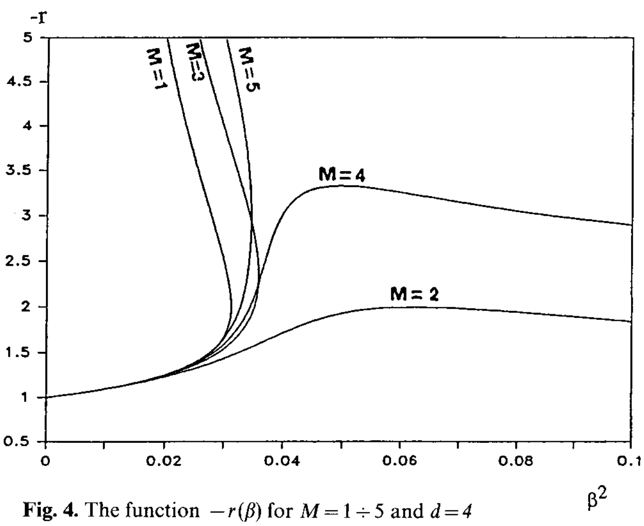

Fig. 4. The function $-r(\beta)$ for $M=1\div 5$ and $d=4$

Table 1. Numerical values of β_c as a function of d and M

d	M				
	1	2	3	4	5
1	0.3536	$\overline{0.5}$			
2	0.25	$\overline{0.3536}$	0.2925	$\overline{0.3340}$	$\overline{0.3381}$
3	0.2041	$\overline{0.2887}$	0.2242	$\overline{0.2651}$	0.2276
4	0.1768	$\overline{0.25}$	0.1893	$\overline{0.2227}$	0.1858

When $d=2$ one has to face the fact that the first nontrivial contribution to Q_6 occurs at order β^{16}, and therefore for $4 \leq M \leq 8$ there is no bifurcation.

However for higher and higher M the function $r(\beta)$ becomes closer and closer to its exact value for the chiral model for most values of β, and we can take its endpoint $\bar{\beta}$ as an indicator for the location of criticality.

Table I shows the values of β_c for various choices of M and d. The bar denotes values corresponding to $\bar{\beta}$ instead of β_c.

Different attempts to determine the critical point of chiral models in large N have been made.

Mean field analysis [7–9] leads to the gaussian value $\beta_c = 1/2d$, with no possibility of evaluating the corrections to this result in $1/d$ perturbation theory.

Mixed methods involving weak coupling, mean field and strong coupling β expansion do not seem very reliable.

Montecarlo results are too insensitive to make an higher order transition apparent, and moreover only a limited number of results have been presented in the literature.

Finally Green and Samuel [3, 10] have derived an order parameter that should indicate the presence and location of a critical point in chiral and gauge models in the large N limit.

Their result for $d=2$ is $\beta_c = 0.324$, not inconsistent with our numerical values.

A final comment concerns the possible existence, for d sufficiently large, of a double transition, such that the large N transition we are observing might occur in a metastable phase and therefore be completely irrelevant. This phenomenon happens in lattice gauge theories in large N for $d \geq 4$ [12].

We argue that this is not the case for chiral models. Indeed we can show that, for all $M \leq d$ the leading powers of d in the strong coupling expansion form a series in the variable $\beta^2 d$ and enjoy the property (related to (3.8))

$$Q_{k+1}^{\text{lead}}(\beta^2 d) = [Q_2^{\text{lead}}(\beta^2 d)]^k. \tag{4.2}$$

As a consequence the power series solution for $r(\beta)$ takes the form

$$r^{\text{lead}} = -\sum_{k=0}^{M} Q_{k+1}^{\text{lead}}(\beta^2 d) = -\sum_{k=0}^{M} [Q_2^{\text{lead}}(\beta^2 d)]^k \tag{4.3}$$

and we found [12] that

$$Q_2^{\text{lead}}(x) = \sum_{n=1}^{M} q_n x^n \tag{4.4}$$

$$q_n = (n-1) \sum_{p=1}^{n-1} q_p q_{n-p} \qquad q_1 = 2. \tag{4.5}$$

However when $M, d \to \infty$, the coefficients q_n grow as fast as $(2n)!/3(n!)$ and therefore the convergence radius of Q_2^{lead} is zero and there is no nontrivial value x_c such that the large d asymptotic behavior of the critical point be: $\beta_c \to \sqrt{x_c}/\sqrt{d}$ as expected [11] if the large N phase transition were different from the mean field phase transition.

In conclusion we think ours is a sensible and reliable method for the computation of the critical point in the large N chiral models.

Numerical improvement of our results requires however a better control on the higher orders of the large N strong coupling series.

Appendix A

The essential ingredient in the evaluation of one-dimensional loops is the computation of the Ω integrated connected contribution of the single link loop run k times

$$\operatorname{tr}(\alpha_1^{-1} \alpha_2^{-1})^k. \tag{A.1}$$

The contribution of such a loop to the partition function can be expressed in the form

$$\sum_{\{n_m\}} \prod_{m=1}^{k} \frac{1}{n_m!} \left[N \operatorname{tr} \frac{(\alpha_1^{-1} \alpha_2^{-1})^m}{m} \right]^{n_m} \delta\left(\sum m n_m - k\right). \quad (A.2)$$

This expression admits a character expansion

$$\prod_{m=1}^{k} [\operatorname{tr}(\alpha_1^{-1} \alpha_2^{-1})^m]^{n_m} = \sum_{\{r\}} C_{\{n_m\}}^{\{r\}} \chi_{\{r\}}(\alpha_1^{-1} \alpha_2^{-1}) \quad (A.3)$$

where $\{r\}$ are the irreducible representations of $U(N)$.

Trivially one obtains

$$\prod_{m=1}^{k} N^{n_m} = \sum_{\{r\}} C_{\{n_m\}}^{\{r\}} d_{\{r\}} \quad (A.4)$$

where $d_{\{r\}}$ is the dimension of the representation.

From the orthogonality of characters one obtains

$$\sum_{\{n_m\}} \prod_{m=1}^{k} \frac{1}{n_m!(m)^{n_m}} C_{\{n_m\}}^{\{r\}} C_{\{n_m\}}^{\{s\}} = \delta_{rs}. \quad (A.5)$$

Therefore the contribution to the partition function is

$$\sum_{\{r\}} \chi_{\{r\}}(\alpha_1^{-1} \alpha_2^{-1}) d_{\{r\}}. \quad (A.6)$$

The Ω integration over characters is straightforward and leads to the general relationship

$$\int d\Omega_1 \ldots d\Omega_n \frac{\chi_{\{r\}}(\alpha_1^{-1} \ldots \alpha_n^{-1})}{d_{\{r\}}} = \prod_i \frac{\chi_{\{r\}}(\alpha_i^{-1})}{d_{\{r\}}}. \quad (A.7)$$

Therefore the Ω-integrated contribution to the partition function is

$$\sum_{\{r\}} \chi_{\{r\}}(\alpha_1^{-1}) \chi_{\{r\}}(\alpha_2^{-1})$$

$$= \sum_{\{n_m\}} \prod_{m=1}^{k} \frac{1}{n_m m^{n_m}} [\operatorname{tr}(\alpha_1^{-m}) \operatorname{tr}(\alpha_2^{-m})]^{n_m}. \quad (A.8)$$

By comparison of the integral and the integrand one recognize that the contribution to the effective action is

$$\frac{1}{m} (\operatorname{tr} \alpha_1^{-m})(\operatorname{tr} \alpha_2^{-m}). \quad (A.9)$$

It may be interesting to compare this result to the effective action for the two matrix model found by KKM: the coincidence of results is not unespected but not even obvious.

The evaluation of potentials is achieved by establishing recursive equations relating the $n+1$ link potential to the n link potential by integration over a single site. Integrations can be performed by systematic use of (A.7). The algebraic details of our derivation of (2.5) and (2.6) are not especially illuminating and we shall not present them here.

Appendix B

The strong coupling expansion of the effective action to 8th order is

$$\begin{aligned}
S_{\text{eff}} &= \mu_1^2 \binom{d}{1} \beta^2 \\
&+ \left[(\mu_1^2 \mu_2 + \tfrac{1}{2} \mu_2^2) \binom{d}{1} + (2 \mu_1^2 + 4 \mu_1^2 \mu_2) \binom{d}{2} \right] \beta^4 \\
&+ \left[(\mu_1^2 \mu_2^2 + 2 \mu_1 \mu_2 \mu_3 + \tfrac{1}{3} \mu_3^2) \binom{d}{1} + (24 \mu_1^2 \mu_2^2 + 8 \mu_1^3 \mu_3 \right. \\
&+ 16 \mu_1^4 \mu_2 + 8 \mu_1 \mu_2 \mu_3 + 4 \mu_1^6) \binom{d}{2} + (24 \mu_1^2 \mu_2^2 + 16 \mu_1^3 \mu_3 \\
&\left. + 48 \mu_1^4 \mu_2 + 32 \mu_1^6) \binom{d}{3} \right] \beta^6 + \left[(\mu_1^2 \mu_2^3 + 2 \mu_1^2 \mu_3^2 + 2 \mu_1 \mu_2^2 \mu_3 \right. \\
&+ 2 \mu_1 \mu_3 \mu_4 + \tfrac{3}{2} \mu_2^2 \mu_4 + \tfrac{1}{4} \mu_4^2) \binom{d}{1} + (19 \mu_1^8 + 36 \mu_1^6 \mu_2 \\
&+ 16 \mu_1^5 \mu_3 + 138 \mu_1^4 \mu_2^2 + 6 \mu_1^4 \mu_4 + 152 \mu_1^3 \mu_2 \mu_3 + 84 \mu_1^2 \mu_2^3 \\
&+ 36 \mu_1^2 \mu_2 \mu_4 + 40 \mu_1^2 \mu_3^2 + 48 \mu_1 \mu_2^2 \mu_3 + 8 \mu_1 \mu_3 \mu_4 + 6 \mu_2^4 \\
&+ 6 \mu_2^2 \mu_4) \binom{d}{2} + 12(31 \mu_1^8 + 80 \mu_1^6 \mu_2 + 20 \mu_1^5 \mu_3 + 86 \mu_1^4 \mu_2^2 \\
&+ 6 \mu_1^4 \mu_4 + 52 \mu_1^3 \mu_2 \mu_3 + 26 \mu_1^2 \mu_3^2 + 6 \mu_1^2 \mu_2 \mu_4 + 4 \mu_1^2 \mu_3^2 \\
&+ 4 \mu_1 \mu_2^2 \mu_3) \binom{d}{3} + 48(27 \mu_1^8 + 36 \mu_1^6 \mu_2 + 8 \mu_1^5 \mu_3 \\
&\left. + 20 \mu_1^4 \mu_2^2 + 2 \mu_1^4 \mu_4 + 8 \mu_1^3 \mu_2 \mu_3 + 4 \mu_1^2 \mu_2^3) \binom{d}{4} \right] \beta^8 \quad (B.1)
\end{aligned}$$

Appendix C

Following KKM we can define a function

$$F(x) = 2 \int \frac{\rho(\lambda)}{1 - \lambda x} d\lambda x \quad (C.1)$$

analytic in the whole complex plane except for a cut corresponding to the support of the eigenvalue distribution.

Equation (3.1) is now equivalent to

$$\operatorname{Re} F(x) = Q(x). \quad (C.2)$$

A solution is to be found in the form

$$F(x) = Q(x) + L(x) \sqrt{r^2 - 2rzx + x^2} \quad (C.3)$$

where $L(x)$ is a polynomial.

Using the expansion

$$\frac{1}{\sqrt{r^2-2rzx+x^2}}=\sum_{m=0}^{\infty} r^m P_m(z)\, x^{-m-1} \tag{C.4}$$

where $P_m(z)$ are Legendre polynomials, one obtains the set of equations

$$1+r+\sum_{m=1}^{M} P_m(z)\, Q_{m+1}\, r^{m+1}=0 \tag{C.5}$$

$$z+r+\sum_{m=1}^{M} P_{m-1}(z)\, Q_{m+1}\, r^{m+1}=0 \tag{C.6}$$

$$2\sum_{m=1}^{M} \mu_m P_{k-m}(z)\, r^{1-m}-P_{k-1}(z)+P_k(z)\, r$$

$$+\sum_{m=1}^{M} P_{k+m}(z)\, Q_{m+1}\, r^{m+1}=0 \qquad k=1, \ldots, M. \tag{C.7}$$

By substituting $\mu_k=\delta_{1k}$ into this set of equations one finds

$$1+r+\sum_{m=1}^{M} P_m(z)\, Q_{m+1}\, r^{m+1}=0 \tag{C.8}$$

$$z+r+\sum_{m=1}^{M} P_{m-1}(z)\, Q_{m+1}\, r^{m+1}=0 \tag{C.9}$$

$$P_{k-1}(z)+P_k(z)+\sum_{m=1}^{M} P_{k+m}(z)\, Q_{m+1}\, r^{m+1}=0$$

$$k=1, \ldots, M. \tag{C.10}$$

The choice

$$z=1 \tag{C.11}$$

implying $P_k(z)=1$, reduces this set of equations to the single equation

$$2+Q(r)=0. \tag{C.12}$$

By subtracting the first equation from the second and dividing by $z-1$ we obtain the condition for the existence of a second solution

$$1+\sum_{m=1}^{M} \left.\frac{P_{m-1}(z)-P_m(z)}{z-1}\right|_{z=1} Q_{m+1}\, r^{m+1}=0 \tag{C.13}$$

that is

$$1-\sum_{m=1}^{M} m\, Q_{m+1}\, r^{m+1}\equiv -r^2\frac{\partial}{\partial r}\left[\frac{1}{r}(2+Q(r))\right]=0 \tag{C.14}$$

corresponding to the condition

$$\frac{dr}{d\beta}\to\infty \quad \text{for } \beta\to\beta_c. \tag{C.15}$$

Appendix D

In the one dimensional model the coefficients Q_k are easily determined to be (see also Appendix E)

$$Q_2=2\beta^2 \tag{D.1}$$

$$Q_3=2\beta^4 \tag{D.2}$$

$$Q_k=0 \quad k>3 \tag{D.3}$$

to all orders in strong coupling perturbative theory.

The strong coupling saddle point condition is therefore

$$1+r+2\beta^2 r^2+2\beta^4 r^3=0 \tag{D.4}$$

and it admits no bifurcation point for the function $r(\beta)$.

However by solving in β^2

$$\beta^2(r)=(\sqrt{-1-2/r}-1)/2r \tag{D.5}$$

one finds that the end point for the strong coupling regime is determined to be

$$\bar\beta=\tfrac{1}{2} \quad \bar r=-2. \tag{D.6}$$

This result coincides with the exact solution for the critical point of the model, thus indicating that in this case $\bar\beta=\beta_c$, which also justifies the absence of bifurcations.

Appendix E

The coefficients Q_k computed up to the 10th order in the strong coupling expansion are

$$Q_2=2\binom{d}{1}\beta^2+8\binom{d}{2}\beta^4+\left[24\binom{d}{2}+192\binom{d}{3}\right]\beta^6$$
$$+\left[152\binom{d}{2}+2976\binom{d}{3}+10368\binom{d}{4}\right]\beta^8$$
$$+\left[960\binom{d}{2}+49920\binom{d}{3}+462720\binom{d}{4}\right.$$
$$\left.+952320\binom{d}{5}\right]\beta^{10} \tag{E.1}$$

representing also the strong coupling internal energy of large N chiral models in arbitrary d expanded to order β^{10} and

$$Q_3=\left[2\binom{d}{1}+8\binom{d}{2}\right]\beta^4+\left[32\binom{d}{2}+96\binom{d}{3}\right]\beta^6$$

$$+\left[72\binom{d}{2}+1920\binom{d}{3}+3456\binom{d}{4}\right]\beta^8$$

$$+\left[384\binom{d}{2}+26016\binom{d}{3}+191232\binom{d}{4}\right.$$

$$\left.+238080\binom{d}{5}\right]\beta^{10} \tag{E.2}$$

$$Q_4=\left[24\binom{d}{2}+48\binom{d}{3}\right]\beta^6+\left[48\binom{d}{2}+720\binom{d}{3}\right.$$

$$\left.+1152\binom{d}{4}\right]\beta^8+\left[-48\binom{d}{2}+8928\binom{d}{3}\right.$$

$$+51840\binom{d}{4}$$

$$\left.+57600\binom{d}{5}\right]\beta^{10} \tag{E.3}$$

$$Q_5=\left[24\binom{d}{2}+288\binom{d}{3}+384\binom{d}{4}\right]\beta^8$$

$$+\left[2304\binom{d}{3}+13824\binom{d}{4}+15360\binom{d}{5}\right]\beta^{10} \tag{E.4}$$

$$Q_6=\left[720\binom{d}{3}+3840\binom{d}{4}+3840\binom{d}{5}\right]\beta^{10}. \tag{E.5}$$

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