ON THE ONE-LINK INTEGRALS OVER COMPACT GROUPS

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ABSTRACT. For O(*N*), U(*N*) and SU(*N*) groups, we study the weak coupling behaviour of the one-link integrals using their Schwinger–Dyson equations. Special attention is paid to the perturbative corrections to the large *N* limit. In the case of unitary groups, the $1/N^2$ correction is obtained explicitly.

0. INTRODUCTION

A few years ago, one-link integrals over the unitary groups received much attention; indeed these integrals are especially important in the mean field approach to U(*N*) or SU(*N*) lattice gauge theories. A relevant result was obtained by Brower *et al.* [1, 2], who using the technique of Schwinger–Dyson equations, discovered a closed form for the one-link integrals over U(*N*) and SU(*N*). Their result is valid for any finite *N* and with arbitrary sources [see (1.1)].

More recently, it was stressed that similar integrals with matrix variables in the adjoint representation of SU(*N*) also play a role, namely in the study of reduced chiral models and in the mean-field approach to reduced gauge models [3].

This new problem seems very different from the previous one and we do not know how to treat it at the moment; however, since the adjoint representation of SU(*N*) is a subgroup of SO($N^2 - 1$), it could be relevant to study the one-link integral over orthogonal groups. Furthermore, lattice gauge theories with orthogonal groups were also considered by several authors [4, 5] so that the knowledge of the one-link integral over SO(*N*) presents some interest for anyone who might want to study the mean-field approach of these last models.

In the first part of the paper, we establish the Schwinger–Dyson (SD) equations for the general O(*N*) group integral. We do not succeed in finding a closed-form solution apart from the special cases $N = 1, 2, 3$ (treated in Appendix) and in the large *N*-limit which receives special attention. We also present the first terms of the weak coupling expansion of the integral using a modified SD equation.

The second section is devoted to U(*N*) [and SU(*N*)] one-link group integrals. Using the same technique, we study this weak coupling expansion and we point out the main difference: in the latter case we have been able to compute the $O(1/N^2)$ and $O(1/N^4)$ corrections to the large *N* limit. This is interesting because, despite its beauty, the closed form of [2], cannot be easily used to study the behaviour of the solution.

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1. THE O(N) CASE

The one-link integral on the group O(N) is defined by

$$Z(c) = \int_{\mathrm{O}(N)} \mathrm{d}V \exp N\, C_{\alpha\beta} V_{\beta\alpha} \tag{1.1}$$

where V denotes an arbitrary matrix in the group, dV is the Haar measure and C is an arbitrary real matrix. This last matrix can be decomposed in the form

$$C = HO, \tag{1.2}$$

where H is a real symmetric matrix and O an orthogonal matrix

$$H = \sqrt{CC^t}, \tag{1.3}$$

$$O = H^{-1}C, \qquad O^t = C^t H^{-1}. \tag{1.4}$$

H can be diagonalized by an orthogonal transformation $H = QDQ^t$ and, therefore, by changing variables to

$$K \to O^t Q K' Q^t \tag{1.5}$$

and exploiting the invariance of the Haar measure, one shows that the integral (1.1) depends only on the eigenvalues λ_i of the matrix CC^t

$$\begin{aligned} N^2 CC^t &= Q \operatorname{diag}(\lambda_1, \ldots, \lambda_N) Q^t \\ &= Q \operatorname{diag}(C_1^2, \ldots, C_N^2) Q^t. \end{aligned} \tag{1.6}$$

We now apply the standard technique based on writing down an appropriate SD equation for Z. The orthogonality condition takes the form

$$\frac{\delta}{\delta C_{\alpha\beta}} \frac{\delta}{\delta C_{\gamma\beta}} Z = \delta_{\alpha\gamma} Z \tag{1.7}$$

and since Z depends only on λ_i, one obtains the differential equation

$$\lambda_i \frac{\partial^2 Z}{\partial \lambda_i^2} + \frac{N}{2} \frac{\partial Z}{\partial \lambda_i} + \frac{1}{2} \sum_{j \neq i} \frac{\lambda_j}{\lambda_j - \lambda_i} \left(\frac{\partial Z}{\partial \lambda_j} - \frac{\partial Z}{\partial \lambda_i} \right) = \tfrac{1}{4} Z. \tag{1.8}$$

This equation is similar to that derived by Brower and Nauenberg [6] and Brezin and Gross [7] for U(N); the only difference is in the numerical coefficients. However, while the equation for U(N) could be explicitly integrated for all N, in this case we found solutions only for $N = 1, 2$ and in the large N limit. In this limit, if one rescales the eigenvalues according to $\lambda_i \to N^2\lambda_i$, one can write an equation for $W = \ln Z/N$; when $N \to \infty$ this equation reduces to

$$\lambda_i\left(\frac{\partial W}{\partial\lambda_i}\right)^2 + \frac{1}{2}\left(\frac{\partial W}{\partial\lambda_i}\right) + \frac{1}{2N}\sum_{j\neq i}\frac{\lambda_j}{\lambda_j-\lambda_i}\left(\frac{\partial W}{\partial\lambda_j}-\frac{\partial W}{\partial\lambda_i}\right) = \frac{1}{4}. \tag{1.9}$$

It is trivial to check that this equation is solved by

$$W_\infty = \tfrac{1}{2}\overline{W}(\lambda_i) \tag{1.10}$$

$$= \frac{1}{2}\left[\frac{2}{N}\sum_i \lambda_i^{1/2} - \frac{1}{2N}\sum_{i,j}\log\left(\frac{\lambda_i^{1/2}+\lambda_j^{1/2}}{N}\right) - \tfrac{3}{4}N\right]. \tag{1.11}$$

The $\overline{W}$ defined here is the solution found by Brower and Nauenberg [6] and Gross and Brezin [7] for the group U(N). The large N solution can be checked explicitly at least in the case when C is proportional to the identity matrix

$$\begin{aligned}\int \mathrm{d}V \exp C \operatorname{tr} V &= \sum_m \frac{C^m}{m!}\int \mathrm{d}V\,(\hbar V)^m \\ &= \sum_{m\ \text{even}} \frac{C^m}{m!}(m-1)!! = \sum_n \frac{C^{2n}}{2n!!} = \exp\frac{c^2}{2}\end{aligned} \tag{1.12}$$

We can re-express Equation (1.8) in terms of the eigenvalues c_i [see (1.6)] of the matrix H

$$\frac{\partial Z}{\partial c_i^2} + \sum_{j\neq i}\frac{1}{c_j^2-c_i^2}\left(c_j\frac{\partial Z}{\partial c_j} - c_i\frac{\partial Z}{\partial c_i}\right) = Z. \tag{1.13}$$

This equation can be used to compute the $1/c_i$-correction to its large N approximation (1.10). To this end, we factorize out of Z the large N limit contribution:

$$Z = X\cdot \exp NW_\infty \tag{1.14}$$

Then the new equation obtained for the function X takes the form

$$\frac{\partial X}{\partial c_k} = -\frac{1}{2}\left[\frac{\partial^2}{\partial c_k^2} + \sum_{l\neq k}\frac{c_l}{c_l^2-c_k^2}\left(\frac{\partial}{\partial c_l}-\frac{\partial}{\partial c_k}\right) + \frac{1}{4}\sum_{l\neq k}\frac{1}{c_l^2+c_k^2}\right]X \tag{1.15}$$

which is suitable for an expression in $1/c_l$, as can be seen by naive power counting.

Here we present the first few terms of the weak coupling expansion of X (neglecting an irrelevant multiplicative constant)

$$X = 1 + \frac{1}{8} \sum_{i<j} \frac{1}{c_i + c_j} + \frac{1}{2^7} \left[g \sum_{i<j} \frac{1}{(c_i + c_j)^2} + \right.$$

$$\left. + 6 \sum_{\substack{k \neq i \neq j \\ j > k}} \frac{1}{(c_i + c_j)(c_i + c_k)} + 2 \sum_{\substack{i<k<l \\ i<j}} \frac{1}{(c_i + c_j)(c_k + c_l)} \right]. \tag{1.16}$$

This leads to

$$\frac{W}{N} = \frac{1}{N^2} \left[\sum_i c_i - \frac{1}{4} \sum_{i,j} \log (c_i + c_j) - \frac{3}{8} N^2 + \frac{1}{8} \sum_{i<j} \frac{1}{c_i + c_j} + \right.$$

$$\left. + \frac{1}{16} \sum_{i<j} \frac{1}{(c_i + c_j)^2} + \frac{1}{32} \sum_{\substack{k \neq i \neq j \\ k<j}} \frac{1}{(c_i + c_j)(c_i + c_k)} + O\left(\frac{1}{c_i^3}\right) \right] \tag{1.17}$$

The different coefficients appearing in the expansion can be checked either by direct comparison with the $N = 3$ case given in the Appendix, or by direct asymptotic expansion of the integral (1.1). The latter can be done by expanding V in (1.1) in terms of a generator ($V = \exp i\omega$) and by performing Gaussian integration in ω. The quadratic form of the exponent is, for instance

$$S_{(2)} = -\frac{1}{2} \sum_{i<j} (\omega_{ij})^2 (c_i + c_j). \tag{1.18}$$

After rescaling the eigenvalues c_i according to $c_i = N\tilde{c}_i$, it is obvious to see that the correction to the large N part in (1.17) is of order $1/N$ and that the quadruple summation which appeared in (1.16) disappears from the free energy W as expected from our large N limit calculation (since this term is of order N^2 as well).

To get more information requires a lot of work; furthermore, at the moment it is not obvious how to resum the terms which contribute to the $1/N$ correction of the large N limit. This is unlike the U(N) analog, as we shall see in the next section.

At least we want to show how the strong coupling expansion of (1.1) begins. It can be determined from Equation (1.8) but it is crucial here that the equation determines the odd terms independently of the even ones.

Imposing the boundary condition (we suppose $N > 2$)

$$Z(0) = 1, \left. \frac{\partial Z}{\partial c_i} \right|_{c_i = 0} = 0 \tag{1.19}$$

we have obtained

$$Z(c_i) = 1 + \frac{1}{N} \hbar CC^t + \frac{3}{N(N-1)(N+2)} [(N+1)(\hbar CC^t)^2 - 2\hbar (CC^t CC^t)] + O(c^6). \tag{1.20}$$

From the elementary group integrations

$$\int \mathrm{d}V\, V_{ij} V_{kp} \dots V_{mn} \tag{1.21}$$

it is known [8] that, up to order N, only even powers of c_l appear in the development (1.20). At orders larger than N, the integrals depend on $\epsilon_{i_1 \dots i_N}$ as well and odd powers appear.

2. U(N) CASE

The one link integral over the group U(N) is defined to be

$$Z(A, A^\dagger) = \int_{\mathrm{U}(N)} \mathrm{d}U \exp N\hbar(UA^\dagger + U^\dagger A) \tag{2.1}$$

with the same convention as before. According to Brower *et al.* [2], $Z(A, A^+)$ depends only on the eigenvalues of the matrix AA^+; it is useful to define z_i by

$$2N\sqrt{AA^+} = R \,\mathrm{diag}\,(z_1, z_2, \dots, z_N)R^+. \tag{2.2}$$

With these notations it was shown [2] that Z is given by the following ratio:

$$Z(z_i) = \frac{\det z_i^{j-1} I_{j-1}(z_i)}{\Delta} \tag{2.3}$$

where

$$\Delta \equiv \det Z_i^{j-1} = \prod_{i<j} (z_i - z_j) \tag{2.4}$$

and I_n denotes the modified Bessel functions. Solution (2.3) was also confirmed in [9], using different methods.

So far, the knowledge of the solution (2.3) does not obviously provide an idea of the behaviour of the function Z. Of course, one can try to use the Taylor expansion (or asymptotic expansion) of the Bessel functions but the task is rather hard.

A strong coupling expansion of Z (small values of z_i) was given by Bars [10] who, using a character expansion, was able to compute explicitly the integral (2.1) up to order (z_i) [10]. An analogous result was also given by Samuel [11].

Here, our purpose is to present a weak coupling expansion; to this end we will use again the SD equation of the integral (2.1).

The idea is the same as in the first section, i.e., to consider the SD equation to factorize the large-N limit [see (1.10)]

$$Z = X \exp NW\left(\frac{z_i^2}{4}\right) \tag{2.5}$$

and to study the equation in X. For instance, it takes the form

$$\frac{\partial}{\partial z^a} X = -\left[\frac{1}{2}\frac{\partial^2}{\partial z_a^2} + \sum_{b \neq a} \frac{z_b}{z_b^2 - z_a^2}\left(\frac{\partial}{\partial z_b} - \frac{\partial}{\partial z_a}\right) + \frac{1}{8 z_a^2}\right] X. \tag{2.6}$$

This is the analog of Equation (1.15) for O(N) but here the situation is more agreeable. Indeed, by expanding X in powers of $1/z_a$, we have been able to resum all terms contributing to the same order in N. We mean that for the first few terms we have obtained the following form (neglecting an irrelevant multiplicative constant)

$$X = 1 + [(1-t)^{-1/8} - 1] + \frac{3}{2^7} t^{(3)} (1-t)^{-25/8} + \mathrm{O}\left(\frac{1}{N^4}, z_i^{-5}\right) \tag{2.7}$$

where

$$t^{(i)} = \sum_a \frac{1}{z_a^i}, \quad t \equiv t^{(1)} \tag{2.8}$$

and

$$\mathrm{O}\left(\frac{1}{N^4}; z_i^{-5}\right) = \frac{45}{2^{10}} t^{(5)} + \cdots . \tag{2.9}$$

For the quantity $W = \ln Z/N$, the formula (2.7) leads to

$$\frac{W}{N} = \frac{1}{N^2}\left[\sum_a z_a - \frac{1}{2}\sum_{a,b} \ln\left(\frac{z_a + z_b}{2N}\right) - \frac{3N^2}{4} + \ln(1-t)^{-1/8} + \frac{3}{2^7} t^{(3)} (1-t)^{-3} + \cdots\right]. \tag{2.10}$$

In the right-hand side of formula (2.10), it is apparent that the first two terms are of order 1 (after rescaling $z_a = N\tilde{z}_a$) and that the third term is of order $1/N^2$. Furthermore, this term contains all terms of order $1/N^2$. The same is true for the fourth term in the order $1/N^4$. This is a crucial difference with the case of orthogonal groups treated in Section 1. Indeed in the latter case, the $1/c_i$ corrections to the large N limit of W/N are of order $1/N$ only, and to our knowledge, the resumation of all $1/N$ contributions is not known.

3. THE SU(N) CASE

For the purpose of completeness, we would like to present the application of the technique used above to study the weak coupling behaviour of the one link-integral on SU(N). It is defined according to

$$\widetilde{Z}(J, J^+) = \int_{\mathrm{SU}(N)} \mathrm{d}\widetilde{U} \exp \hbar(\widetilde{U}J^+ + \widetilde{U}^+ J). \tag{3.1}$$

In [1], the authors stressed that $\widetilde{Z}$ depends only on the eigenvalues of JJ^+ (see (2.2)) and on the determinant of J.

$$\widetilde{Z}(J, J^+) = \widetilde{Z}(z_i, \theta) \tag{3.2}$$

$$e^{2i N} = \frac{\det J}{\det J^+}. \tag{3.3}$$

Then, according to Brower *et al.*, the function $\widetilde{Z}$ takes the form

$$\widetilde{Z}(z_i, \theta) = \sum_{l=-\infty}^{\infty} e^{iNl\theta} \frac{W_l}{\Delta} \tag{3.4}$$

with

$$W_l = \det[Z_i^{j-1} \cdot I_{j-1-l}(z_i)]. \tag{3.5}$$

In the same spirit as in the preceding section, we have computed the weak coupling expansion (at least the first four terms) of W_l and we have found [12] (see the definitions in (2.8) and (2.11))

$$\frac{W_l}{\Delta} = e^{N\overline{W}}\left[(1-t)^{(1/2)(l^2-(1/4))} + \frac{(l^2-\frac{1}{4})(l^2-\frac{9}{4})}{24} t^{(3)}(1-t)^{(1/2)(l^2-(25/4))} + O\left(\frac{1}{N^4}, Z^{-5}\right)\right] \tag{3.6}$$

From (3.4) – (3.6) it is apparent that the large N limit of U(N) is factorized out of $\widetilde{Z}$ and that the remaining factor takes into account the finite N effects and the constraint due to the determinant [since we are in SU(N)]. In particular, the first term in the bracket of (3.6) (term of order N) can be summed in (3.4)

$$\sum_l (1-t)^{l^2/2} e^{iNl\theta} = \sum_l e^{-kl^2} e^{iNl\theta} \tag{3.7}$$

if $k = \frac{1}{2}\ln(1-t)$ and we can recognize the Villain action for a U(N) gauge theory in the continuum.

APPENDIX

Here we want to present a few exact results that we have obtained for the function (1.1) for some special value of N.

O(1) It is very easy to solve the ordinary differential Equation (1.8) leading to

$$Z = e^c. \tag{A1}$$

O(2) With some effort, one recognizes that Equation (1.8) is solved (with proper boundary conditions) by

$$Z = I_0(c_1 + c_2). \tag{A2}$$

Another soluble case occurs when the source matrix $C_{\alpha\beta}$ is a projector (only one eigenvalue is nonzero). Then Equation (1.8) reduces to

$$\lambda \frac{\partial^2}{\partial\lambda^2} Z + \frac{N}{2} \frac{\partial Z}{\partial\lambda} = \frac{1}{4} Z \tag{A3}$$

and the solution is

$$Z = \Gamma\left(\frac{N}{2}\right) \frac{I_{(N/2)-1}(c)}{(c/2)^{(N/2)-1}}, \quad c = \sqrt{\lambda}. \tag{A4}$$

O(3) This case is important because it corresponds to the integration for the adjoint representation of SO(2) [because of the SU(2) – SO(3) isomorphism]. We have a closed form when C is proportional to the identity

$$\int \mathrm{d}V \exp c \operatorname{tr} V = [I_0(2c) - I_1(2c)] \; \mathrm{e}^c. \tag{A5}$$

For general values of c, the integral has been studied in [13]. Here we present its weak coupling expansion which has been obtained independently. The result is:

$$Z(c_i) = \frac{\exp(c_1 + c_2 + c_3)}{\sqrt{(c_1 + c_2)(c_1 + c_3)(c_2 + c_3)}} \left[1 + \sum_{n=1}^{\infty} \delta_n I_n\right] \tag{A6}$$

$$\delta_n = (-1)^n 2^{1-n} \frac{1}{n!(2n+1)!!} \frac{\Gamma(n + \frac{3}{2})}{\Gamma(\;n + \frac{1}{2})} \tag{A7}$$

$$I_n = \sum_{p=0}^{n} \sum_{q=0}^{p} \frac{C_p^n C_q^p (2q-1)!!(2p-2q-1)!!(2n-2p-1)!!}{(c_2 + c_3)^q (c_1 + c_3)^{p-q} (c_1 + c_2)^{n-p}}. \tag{A8}$$

This result can be checked to be in agreement with (A5) for $c_1 = c_2 = c_3 = c$.

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